# Bootstrapping gauge theories (QCD)

Yifei He

LPENS CNRS

based on 2309.12402 and *to appear* with Martin Kruczenski

# Low energy physics of asymptotically free gauge theory

asymptotically free gauge theory $SU(N_c)$

*chiral symmetry breaking and confinement*

$N_f$ massive quarks $m_q \ll \Lambda_{\rm QCD}$ fundamental representation of gauge group

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$$\mathcal{L} = i\sum_j^{N_f} \bar{q}_j \not{D} q_j - \sum_j^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \text{gauge fixing} + \text{ghost}$$

gauge theory parameters: $N_c$ $N_f$ $m_q$ $\Lambda_{\rm QCD}$

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gauge theory parameters: $N_c$ $N_f$ $m_q$ $\Lambda_{\text{QCD}}$

***What is the low energy physics?***

# Physics of Goldstone bosons

chiral symmetry breaking

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$$

**(approximate) Goldstone bosons dominate the low energy physics**

*e.g.* $N_f = 2$ pions $\pi_0 = \pi^3 \quad \pi_\pm = \frac{1}{\sqrt{2}}\left(\pi^1 \pm i\pi^2\right)$

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very low energy effective Lagrangian (lowest order):

$$\mathcal{L} = \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + m_\pi^2 \mathrm{Tr}\left(U + U^\dagger\right)\right\} \qquad U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}}$$

$$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}m_\pi^2\vec{\pi}^2 \qquad \mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2 \quad \ldots$$

# The problem of strongly coupled physics

*energy*

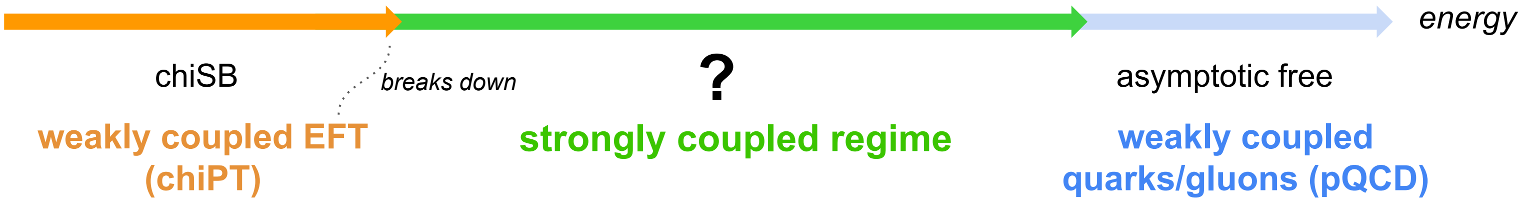
chiSB

**weakly coupled EFT (chiPT)**

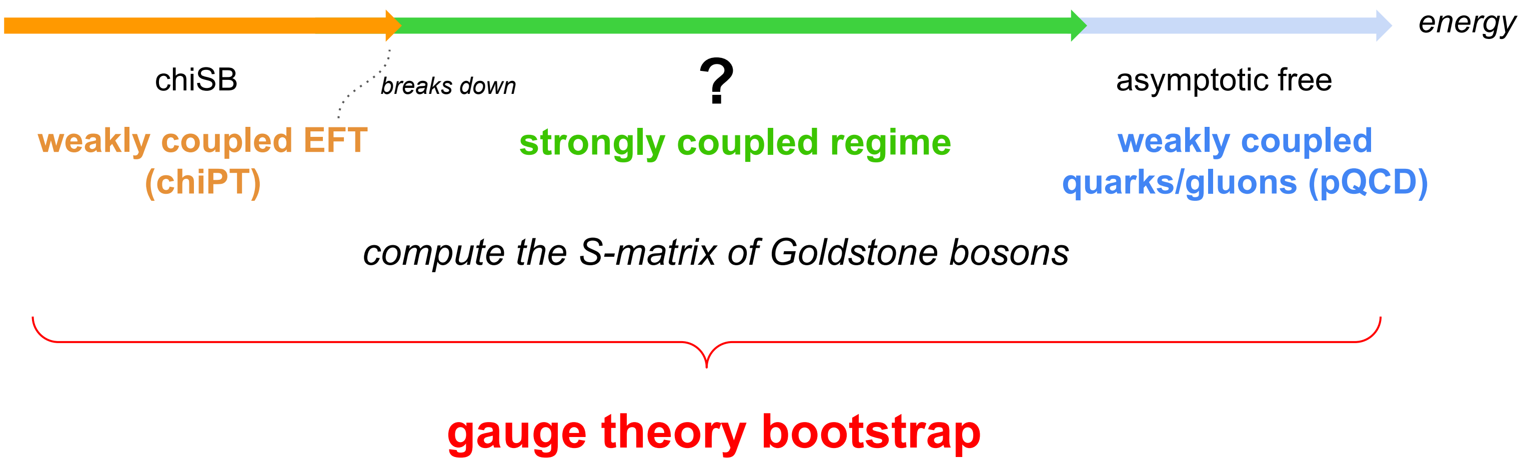
asymptotic free

**weakly coupled quarks/gluons (pQCD)**

# The problem of strongly coupled physics



# The problem of strongly coupled physics



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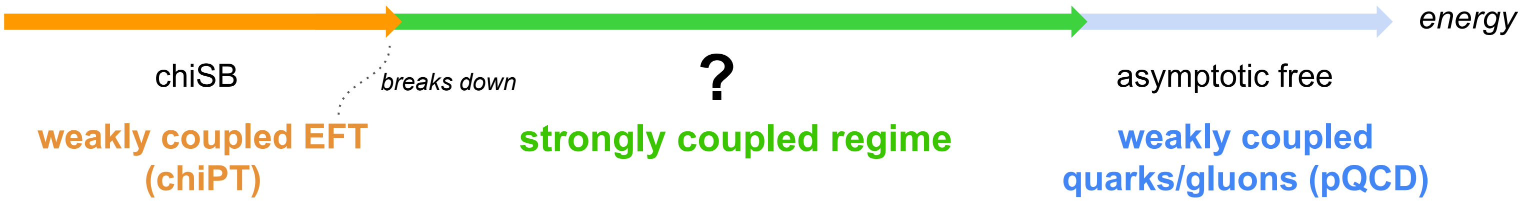


*compute the S-matrix of Goldstone bosons*

**gauge theory bootstrap**

***rules:***

assume — chiral symmetry breaking & confinement

input — gauge theory parameters – define the theory

as few as possible (universal) low energy parameters

# Gauge theory bootstrap

- **Pure S-matrix bootstrap:**

symmetry, analyticity, crossing, unitarity

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- **Form factor bootstrap + SVZ sum rules:**

gauge theory information

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symmetry, analyticity, crossing, unitarity ⟵ $SU(N_f)_V$

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general very low energy behavior ⟵ $f_\pi \quad m_\pi$

- **Form factor bootstrap + SVZ sum rules:**

gauge theory information ⟵ $N_c \quad m_q \quad \Lambda_{\mathrm{QCD}}$

# Gauge theory bootstrap

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symmetry, analyticity, crossing, unitarity ← $SU(N_f)_V$

- **Chiral symmetry breaking:**

general very low energy behavior ← $f_\pi \quad m_\pi$

- **Form factor bootstrap + SVZ sum rules:**

gauge theory information ← $N_c \quad m_q \quad \Lambda_{\text{QCD}}$

Numerical test of the method: $N_f = 2 \quad N_c = 3$

can be compared with experimental data

***for general gauge theories — compare with lattice data***

# Gauge theory bootstrap

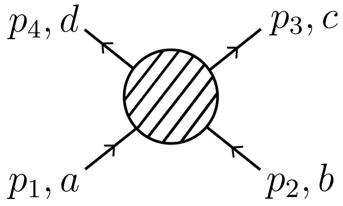
- **Pure S-matrix bootstrap:**

symmetry, analyticity, crossing, unitarity

$SU(N_f)_V$

# Pure S-matrix bootstrap

modern S-matrix bootstrap: [Paulos, Penedones, Toledo, van Rees, Vieira, 2016 & 2017]




2-to-2 pion scattering: $\pi_a(p_1) + \pi_b(p_2) \to \pi_c(p_3) + \pi_d(p_4)$

$$\langle p_1, a; p_2, b|\mathbf{T}|p_3, c; p_4, d\rangle = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$$

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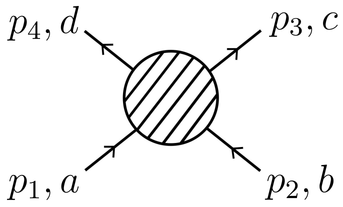


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*constrain amplitudes using generic consistency conditions*

***crossing*** $A(s,t,u) = A(s,u,t)$

***analyticity*** cuts $s,t,u > 4$

$m_\pi = 1$

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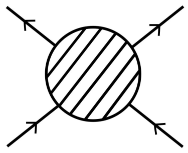


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*constrain amplitudes using generic consistency conditions*

***crossing*** $A(s,t,u) = A(s,u,t)$

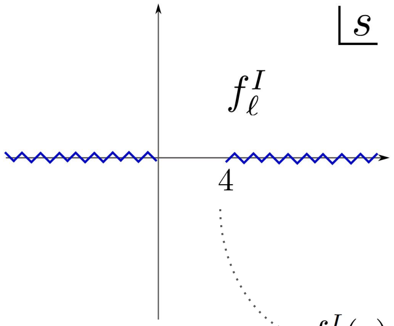
***analyticity*** cuts $s,t,u > 4$

$m_\pi = 1$

$$A(s,t,u) = \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \left[\frac{\rho_1(x,y)}{(x-s)(y-t)} + \frac{\rho_1(x,y)}{(x-s)(y-u)} + \frac{\rho_2(x,y)}{(x-t)(y-u)}\right]$$ +subtraction terms

parameters: $\{\rho_{\alpha=1,2}(x,y), \dots\}$ numerics: discretize $\{\rho_{\alpha,ij}, \dots\}$ bootstrap variables

# Pure S-matrix bootstrap




$$
\begin{aligned}
T^{I=0}(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s) \\
T^{I=1}(s,t,u) &= A(t,s,u) - A(u,t,s) \\
T^{I=2}(s,t,u) &= A(t,s,u) + A(u,t,s)
\end{aligned}
$$

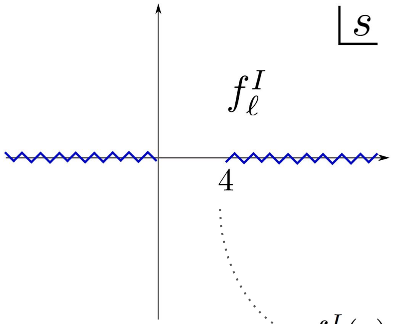
$SU(2)_V$ isospin

***symmetry***

⇓

$$f_\ell^I(s) = \frac{1}{4}\int_{-1}^{1} d\cos\theta \, P_\ell(\cos\theta) T^I(s,t)$$

# Pure S-matrix bootstrap

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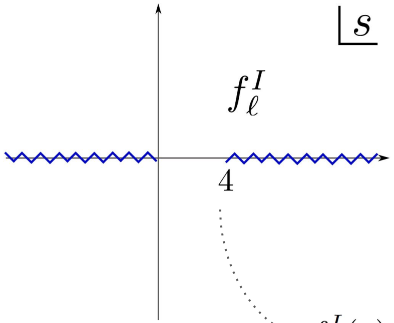
$$f_\ell^I(s) = \frac{1}{4}\int_{-1}^{1} d\cos\theta P_\ell(\cos\theta) T^I(s,t)$$

analytic function of s

$f_\ell^I(0 < s < 4)$ *real linear functionals of bootstrap variables*

unphysical region

# Pure S-matrix bootstrap

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***symmetry***

physical kinematic region $s > 4$

$$f_\ell^I(s) = \frac{1}{4}\int_{-1}^{1} d\cos\theta P_\ell(\cos\theta) T^I(s,t)$$

$$S_\ell^I(s) = 1 + i\pi\sqrt{\frac{s-4}{s}} f_\ell^I(s) = \eta_\ell^I(s)\, e^{2i\delta_\ell^I(s)}$$

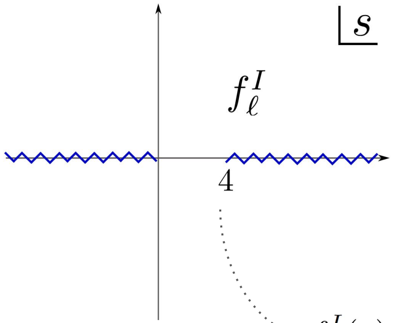
analytic function of s

$f_\ell^I(0 < s < 4)$ *real linear functionals of bootstrap variables*

phase shift

unphysical region

# Pure S-matrix bootstrap

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***unitarity***

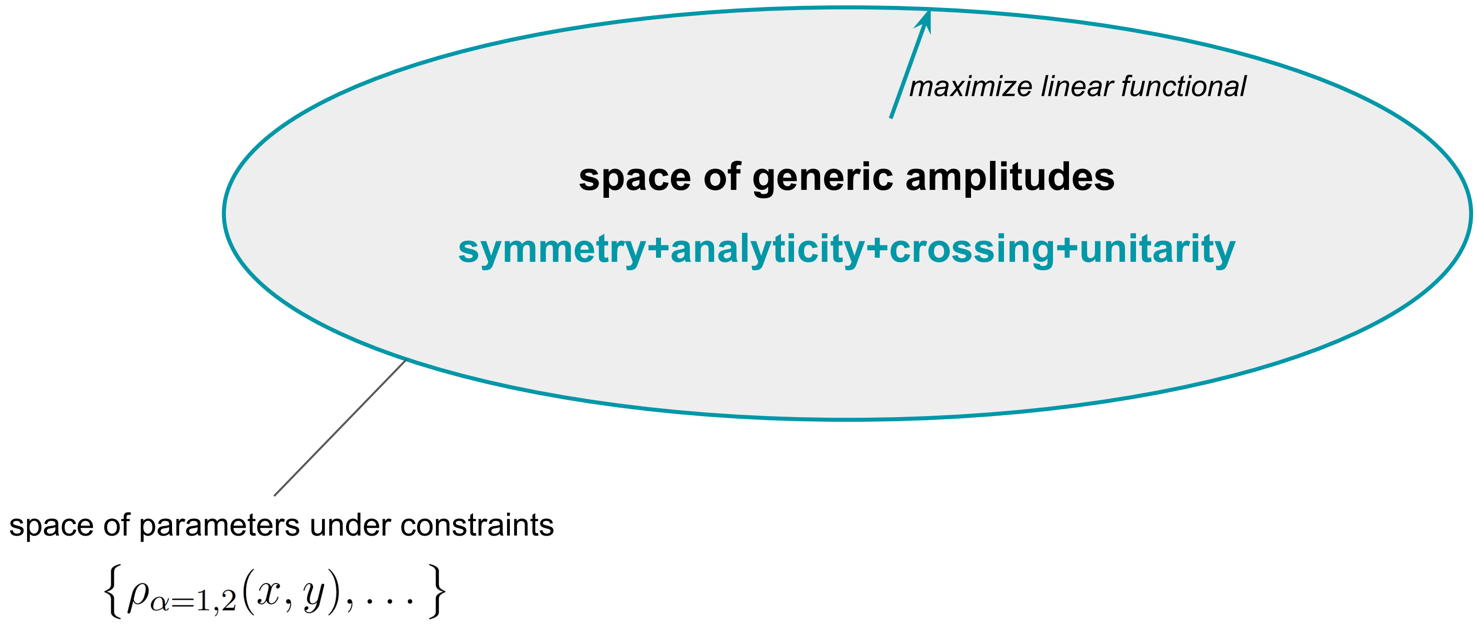
$$|S_\ell^I(s^+)| \leq 1, \; s > 4 \quad \forall \ell, I$$

positive semidefinite → convex space of amplitudes

$$\begin{pmatrix} 1 & S_\ell^I(s) \\ S_\ell^{I*}(s) & 1 \end{pmatrix} \succeq 0$$

convex optimization

# Pure S-matrix bootstrap



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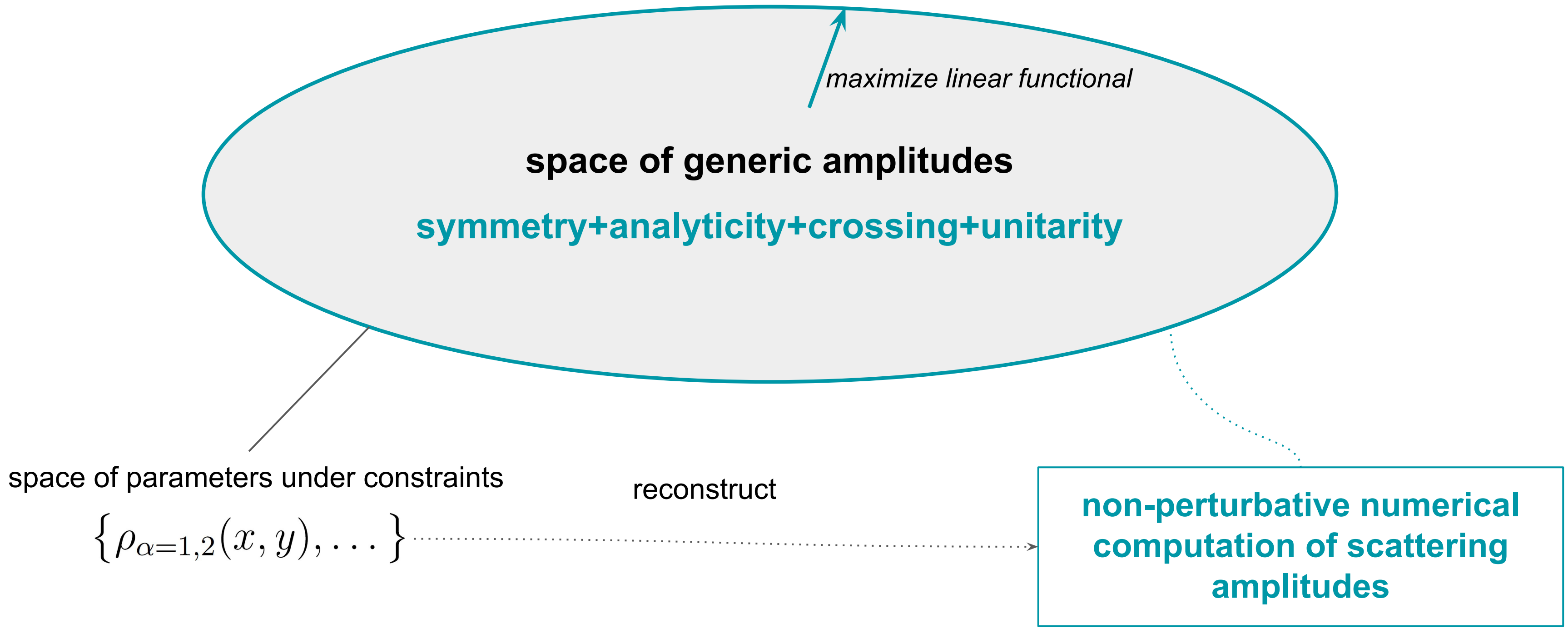
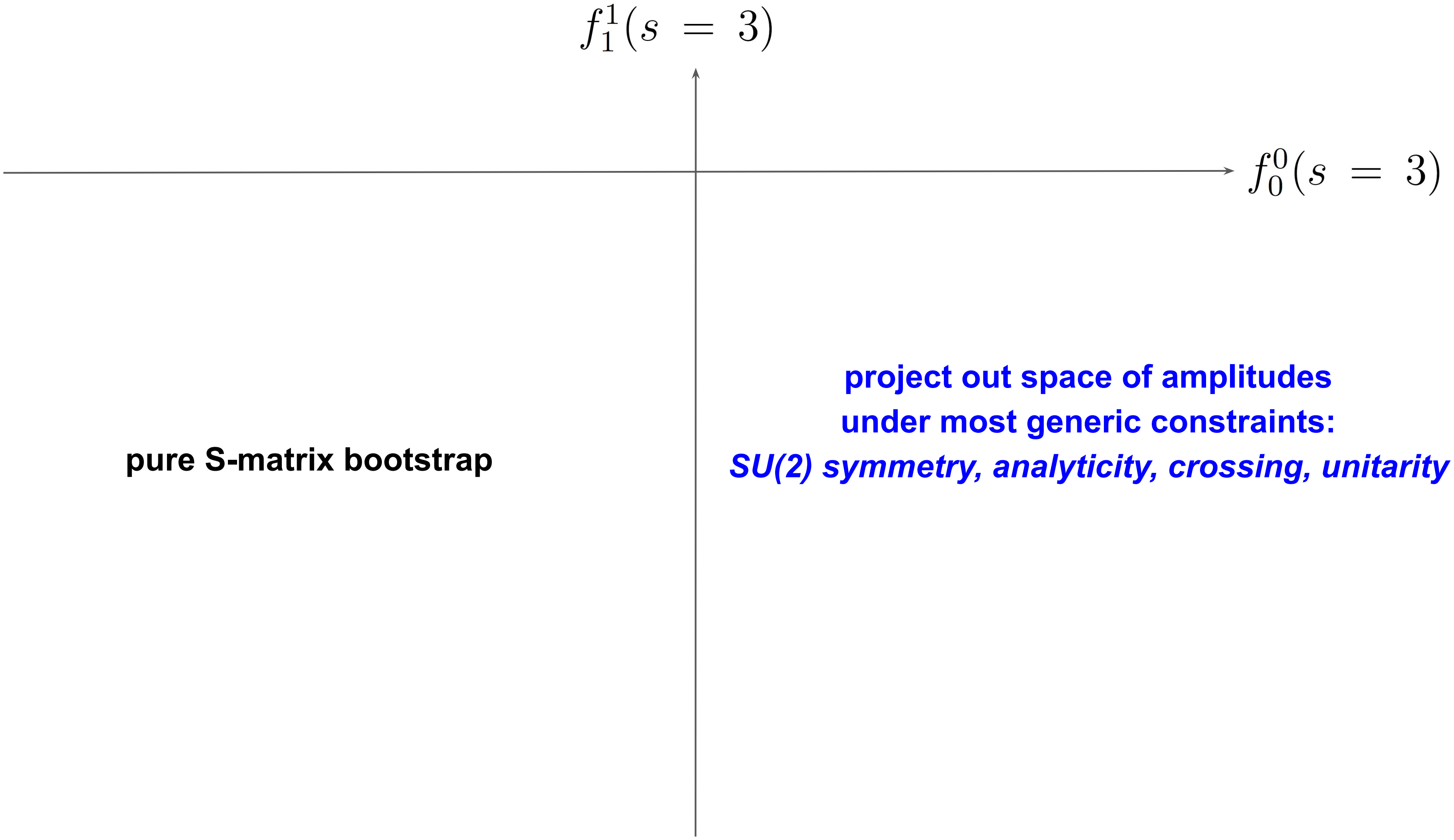
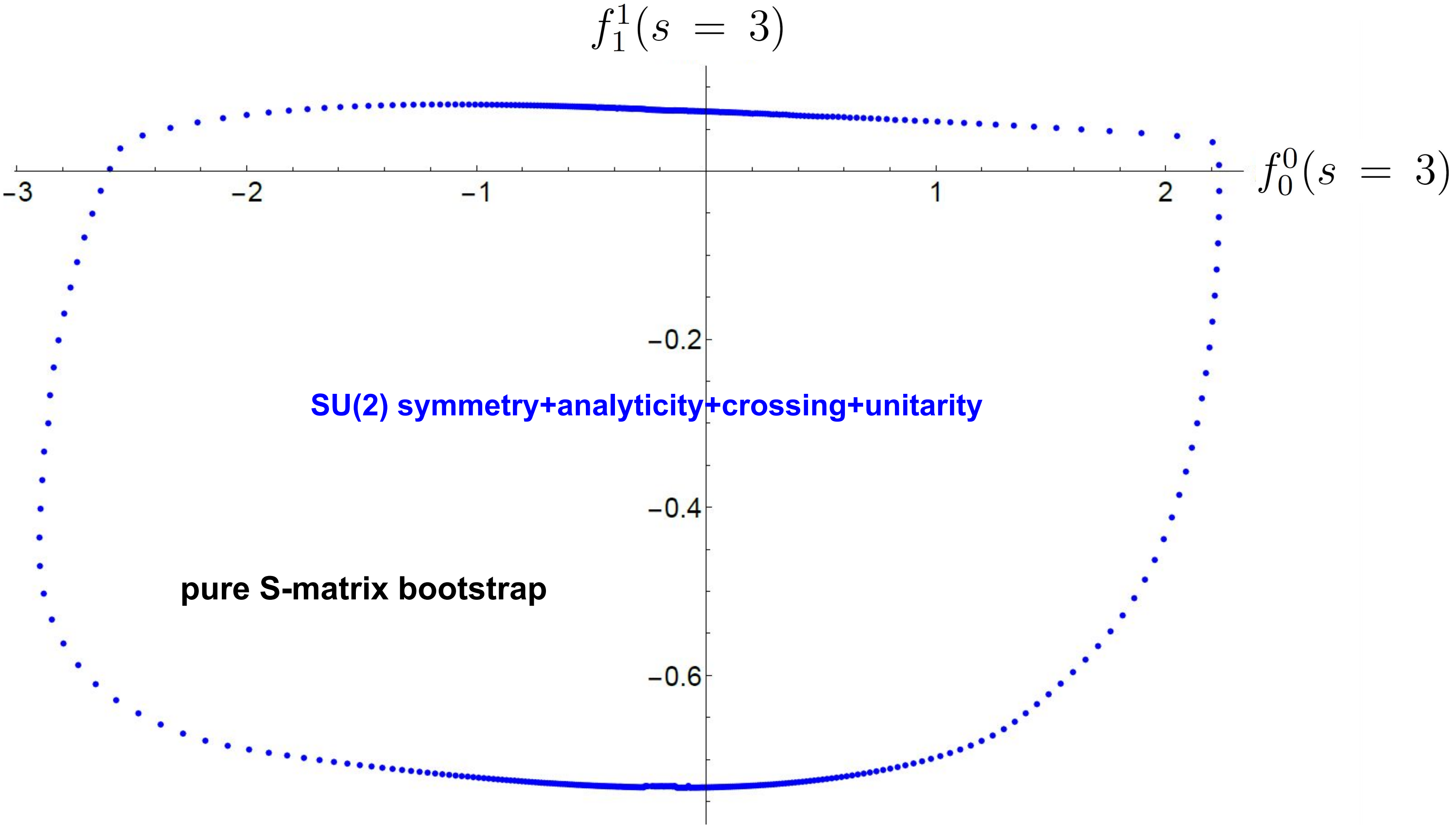





**pure S-matrix bootstrap**

**project out space of amplitudes**
**under most generic constraints:**
***SU(2) symmetry, analyticity, crossing, unitarity***




each boundary point: an extremal numerical amplitude

# Gauge theory bootstrap

- **Pure S-matrix bootstrap:**

symmetry, analyticity, crossing, unitarity

$SU(N_f)_V$

+

- **Chiral symmetry breaking:**

general very low energy behavior

$f_\pi \quad m_\pi$

# Weakly coupled Goldstone bosons

*chiral symmetry breaking: weakly coupled Goldstone bosons at very low energy*

interaction: $$\mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2$$

tree-level amplitude: $$A_{\text{tree}}(s,t,u) = \frac{4}{\pi}\frac{s-m_\pi^2}{32\pi f_\pi^2}$$ *linear in s* [Weinberg, 1966]

good in the unphysical region (very low energy) $0 < s,t,u < 4m_\pi^2$

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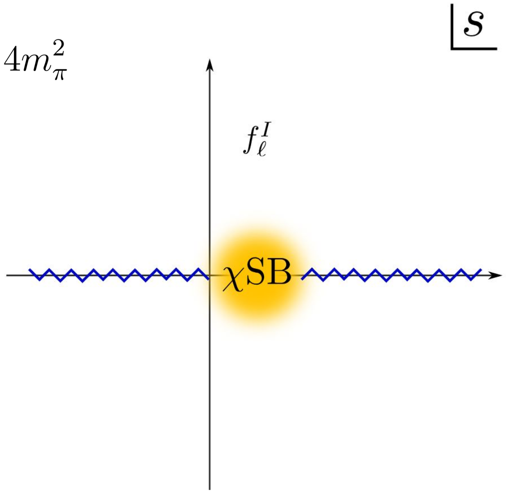
corresponding partial waves

good in the unphysical region (very low energy) $0 < s,t,u < 4m_\pi^2$

S0: $f^0_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2s-m_\pi^2}{32\pi f_\pi^2}$ P1: $f^1_{1,\text{tree}}(s) = \frac{2}{\pi}\frac{s-4m_\pi^2}{96\pi f_\pi^2}$ S2: $f^2_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2m_\pi^2-s}{32\pi f_\pi^2}$

good in unphysical region (very low energy) $0 < s < 4m_\pi^2$



# Chiral symmetry breaking input

**approximate linear behavior at very low energy: input in gauge theory bootstrap**

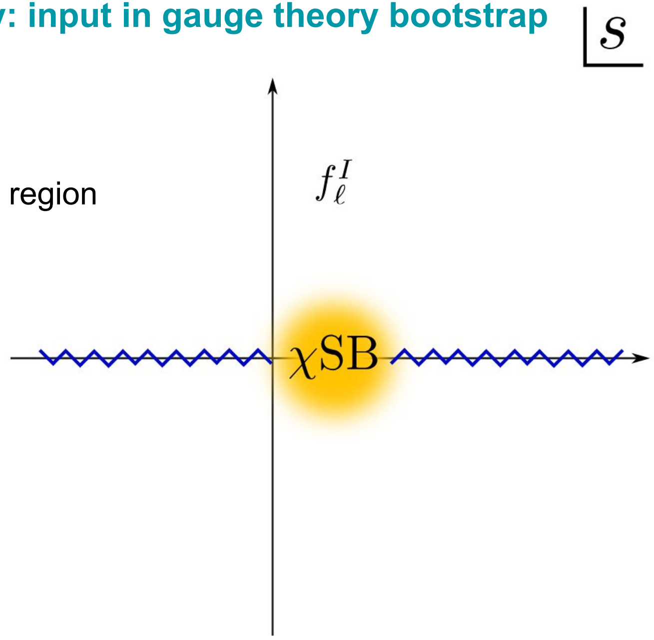
**numerically**

requires p.w. in the bootstrap match the tree level p.w. in unphysical region

$$f_0^0(s) \simeq f_{0,\text{tree}}^0(s)$$

$$f_1^1(s) \simeq f_{1,\text{tree}}^1(s) \qquad 0 < s < 4m_\pi^2$$

$$f_0^2(s) \simeq f_{0,\text{tree}}^2(s)$$



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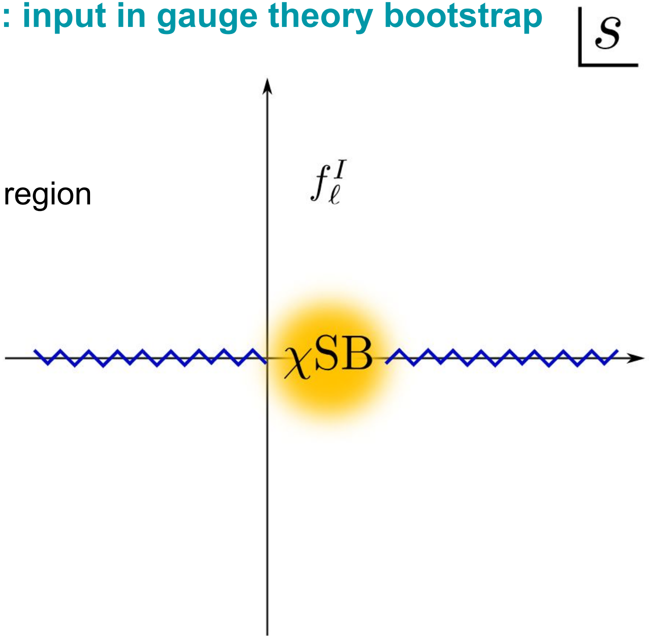
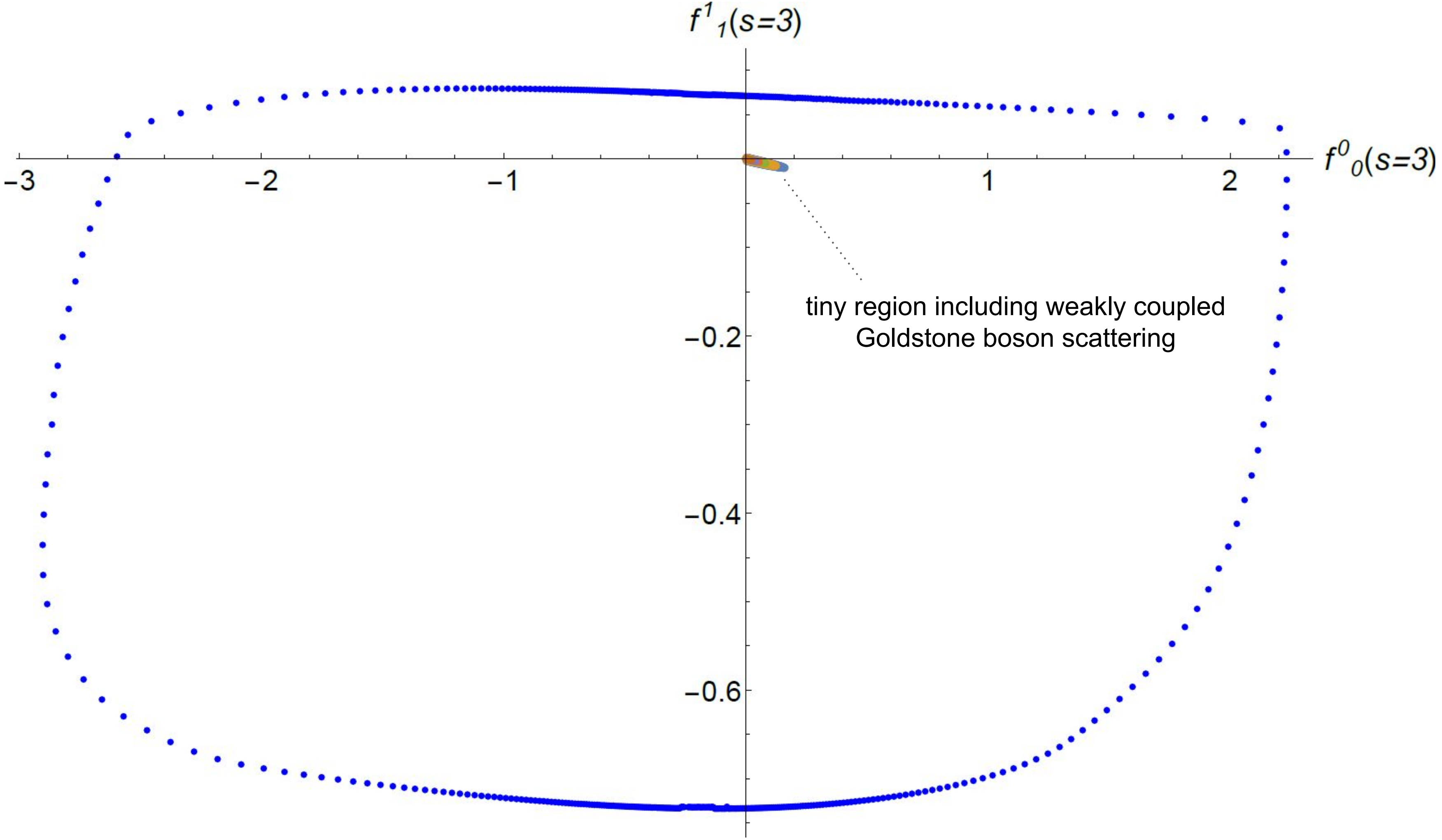



$\epsilon^\chi$ {
*too loose: large deviation from chiSB prediction*
*too tight: exclude the desired theory*

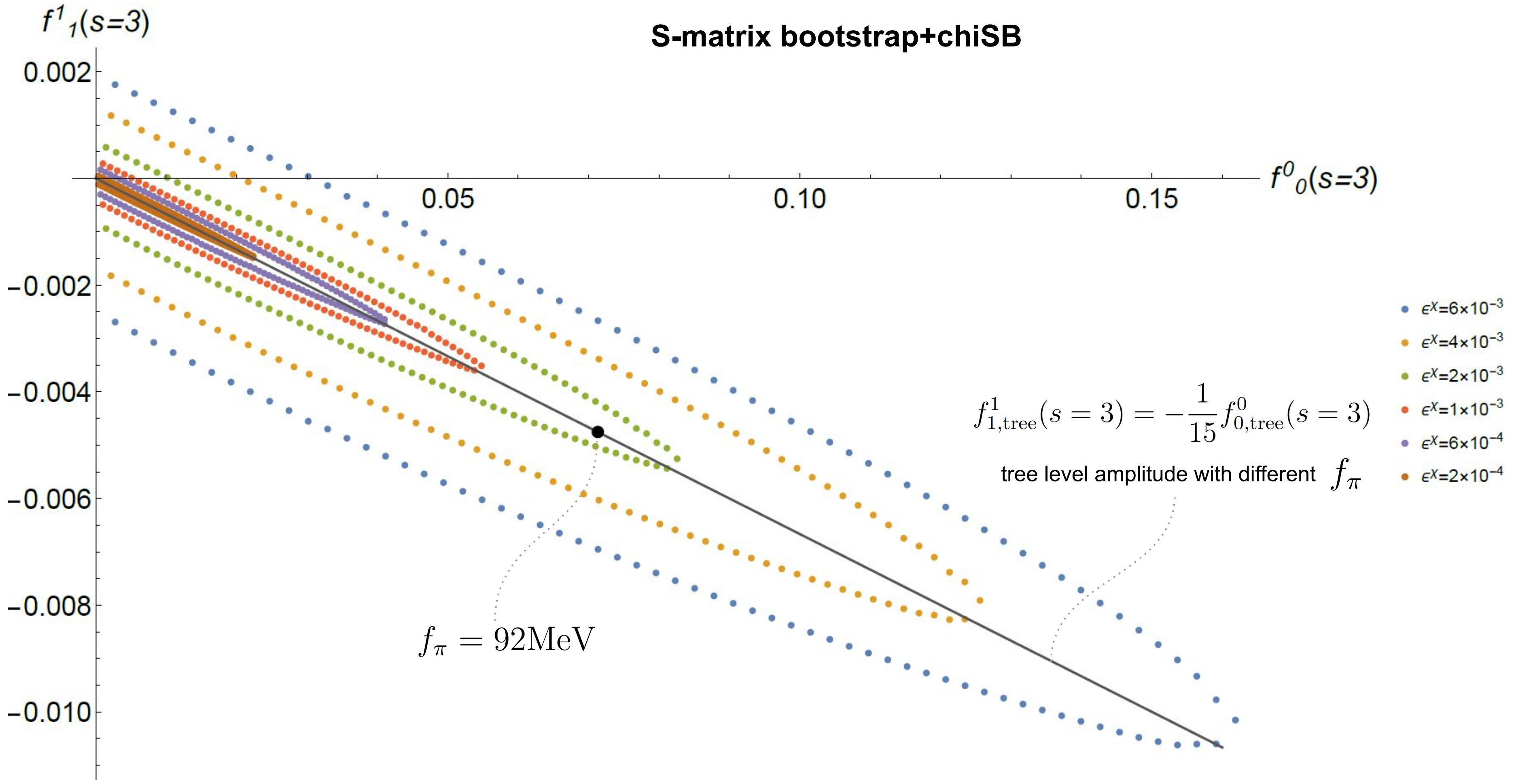
numerics with a series of tolerance

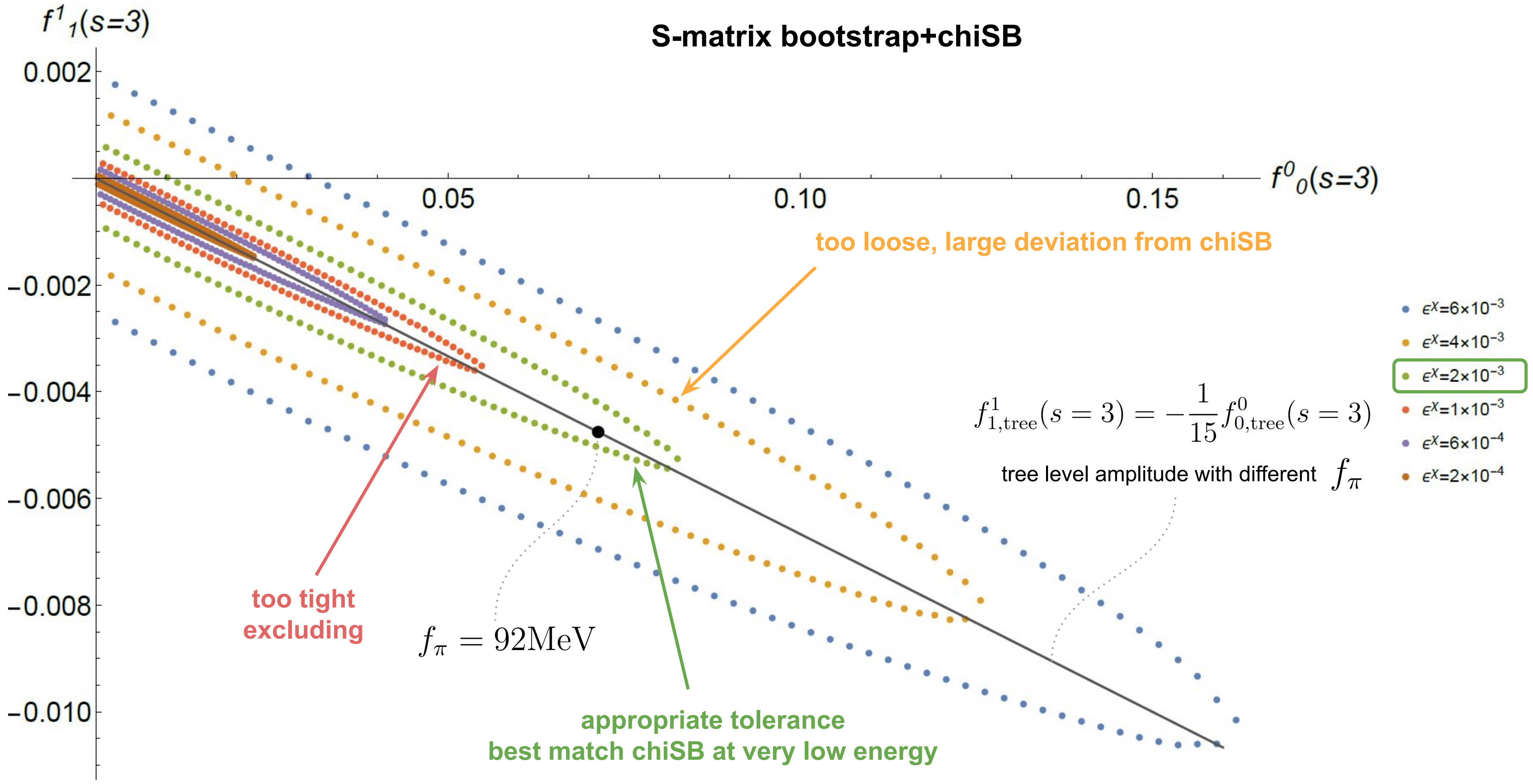
**use** $f_\pi = 92\text{MeV}$ **to select appropriate tolerance**



f¹₁(s=3)
f⁰₀(s=3)
−3
−2
−1
1
2
−0.2
−0.4
−0.6
tiny region including weakly coupled
Goldstone boson scattering

# S-matrix bootstrap+chiSB





S-matrix bootstrap+chiSB
$f^1_1(s=3)$
$f^0_0(s=3)$
0.002
−0.002
−0.004
−0.006
−0.008
−0.010
0.05
0.10
0.15
too loose, large deviation from chiSB
too tight excluding
appropriate tolerance
best match chiSB at very low energy
$f_\pi = 92\text{MeV}$
$f^1_{1,\text{tree}}(s=3) = -\frac{1}{15} f^0_{0,\text{tree}}(s=3)$
tree level amplitude with different $f_\pi$
$\epsilon^\chi=6\times10^{-3}$
$\epsilon^\chi=4\times10^{-3}$
$\epsilon^\chi=2\times10^{-3}$
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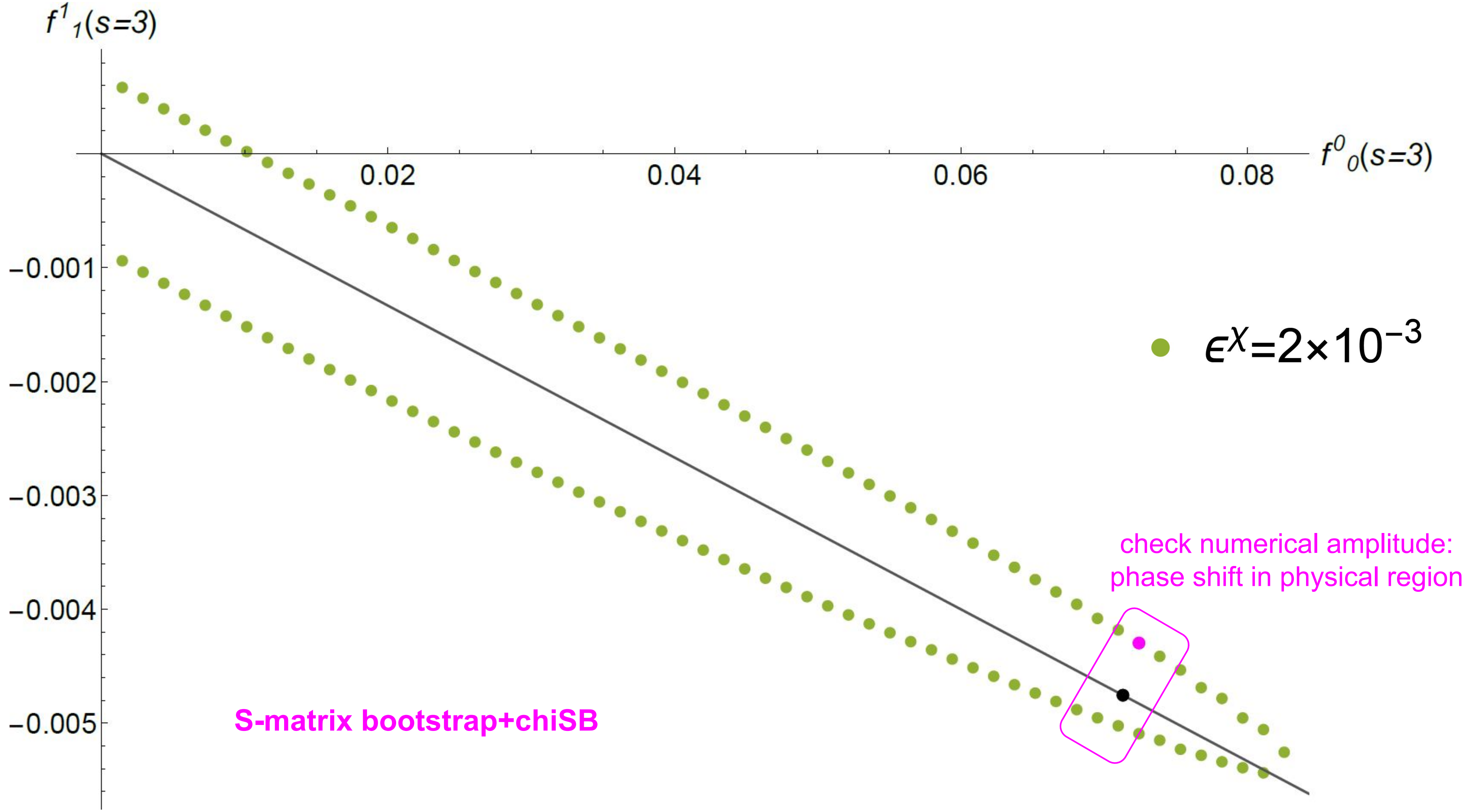
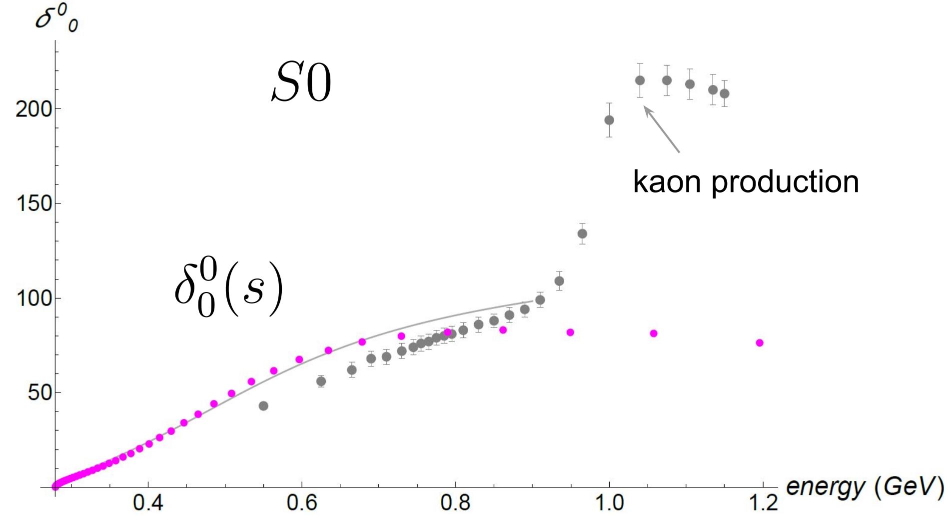
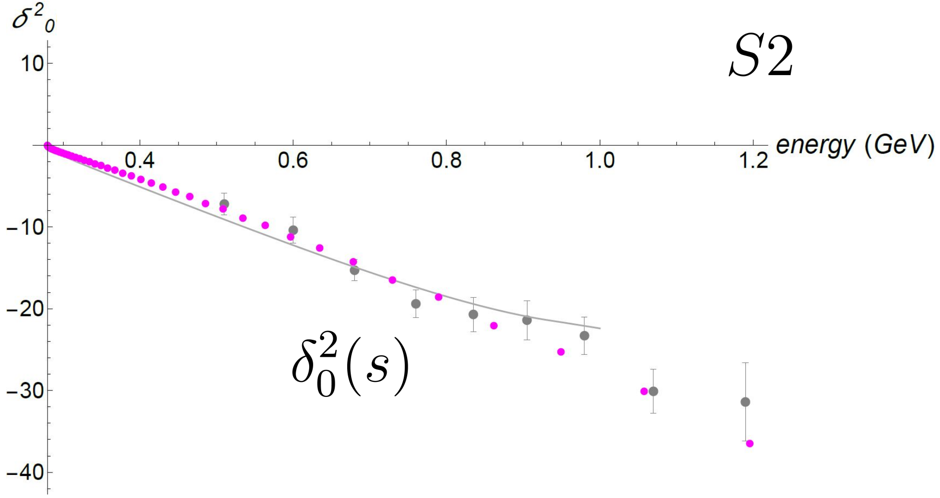


$f^1_1(s=3)$
$f^0_0(s=3)$
0.02
0.04
0.06
0.08
−0.001
−0.002
−0.003
−0.004
−0.005
$\epsilon^\chi=2\times10^{-3}$
check numerical amplitude:
phase shift in physical region
S-matrix bootstrap+chiSB

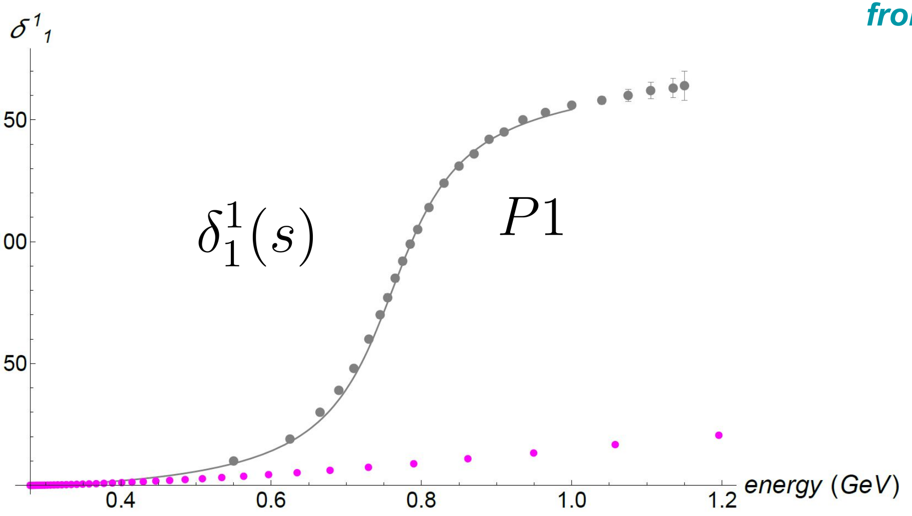


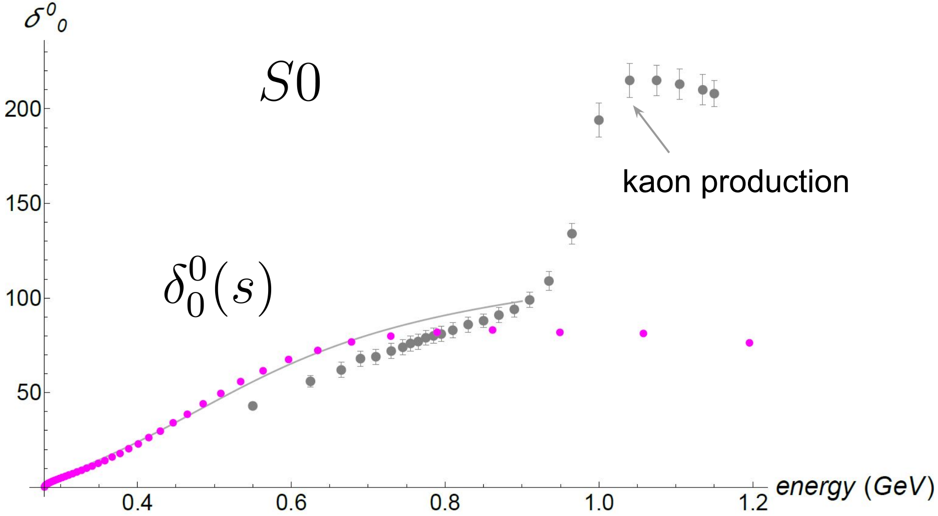



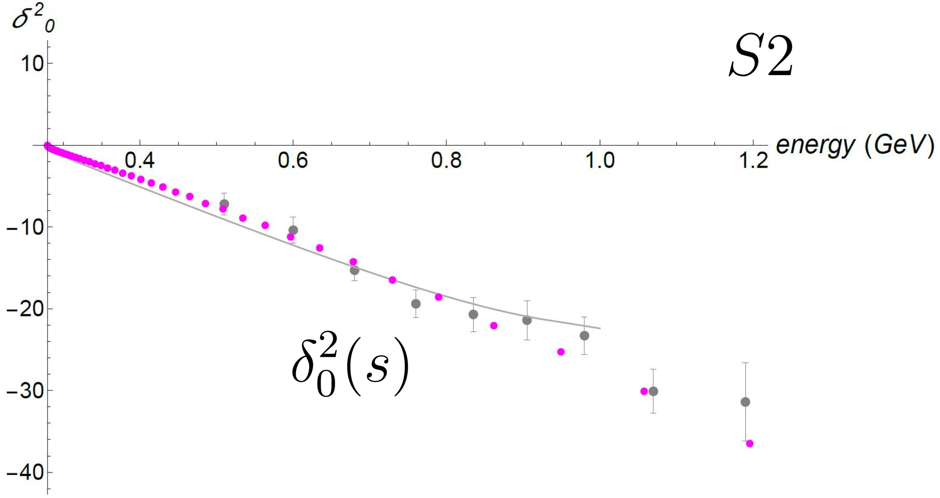

***reasonable S0 S2 waves from weakly coupled EFT***

**phase shift with only chiSB (EFT) input**

experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit (gray line)
[Pelaez, Yndurain, 2005]






reasonable S0 S2 waves
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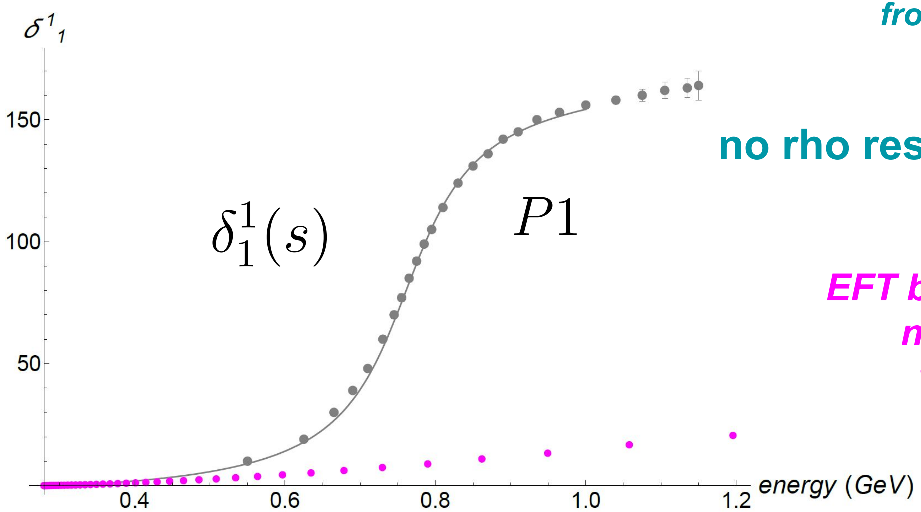


no rho resonance without UV info

experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit (gray line)
[Pelaez, Yndurain, 2005]

makes sense
EFT breaks down at rho mass
not expecting get rho
with only EFT input

# Gauge theory bootstrap

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symmetry, analyticity, crossing, unitarity

$SU(N_f)_V$

+

- **Chiral symmetry breaking:**

general very low energy behavior

$f_\pi \quad m_\pi$

+

- **Form factor bootstrap + SVZ sum rules:**

gauge theory information

$N_c \quad m_q \quad \Lambda_{\text{QCD}}$

# Form factor bootstrap

[Karateev, Kuhn, Penedones, 2019]

an important development: $|\psi_1\rangle = |p_1, p_2\rangle_{in}\,,\qquad |\psi_2\rangle = |p_1, p_2\rangle_{out}\,,\qquad |\psi_3\rangle = \int dx e^{-i(p_1+p_2)\cdot x}\mathcal{O}(x)|0\rangle$

*state created by UV local operator*

positive semidefinite matrix

$$\langle\psi_a|\psi_b\rangle = \begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0$$

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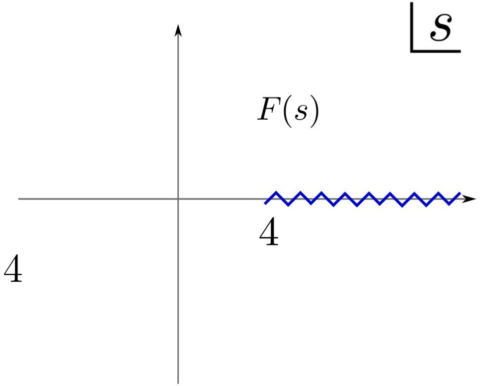
2-particle form factor: $${}_{\text{out}}\langle p_1, p_2|\mathcal{O}(0)|0\rangle = F(s)$$ *analytic function of s*

$$F(s) = \frac{1}{\pi}\int_4^\infty dx \frac{\text{Im}F(x)}{x-s}$$ +subtractions

spectral density: $$\int \frac{d^4x}{(2\pi)^4} e^{iPx}\langle 0|\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle = \rho(s)$$ *supported at* $s > 4$



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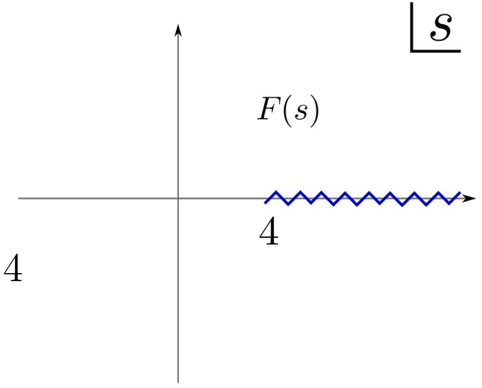


**bootstrap variables:** $\{\rho_{1,2}(x,y), \ldots, \text{Im}F(x), \rho(x)\}$

***allow connection with UV theory***

# Current correlators from the UV gauge theory

***to connect with UV gauge theory***

$$
\begin{array}{c} \\ \langle \text{in}|_{P',I,\ell} \\ \langle \text{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^{\dagger}_{P',I,\ell} \end{array}
\begin{array}{c} \begin{array}{ccc} |\text{in}\rangle_{P,I,\ell} & |\text{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \end{array} \\ \begin{pmatrix} 1 & S^I_\ell(s) & \mathcal{F}^I_\ell \\ S^{I*}_\ell(s) & 1 & \mathcal{F}^{I*}_\ell \\ \mathcal{F}^{I*}_\ell & \mathcal{F}^I_\ell & \rho^I_\ell(s) \end{pmatrix} \end{array} \succeq 0 \qquad s > 4 \quad \forall \ell, I
$$

***construct operators from gauge theory with desired quantum numbers***

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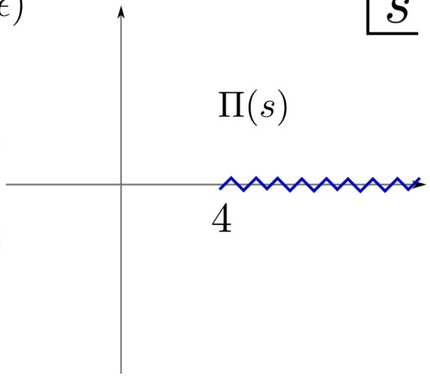
$$\begin{array}{c} \\ \langle \mathrm{in}|_{P',I,\ell} \\ \langle \mathrm{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^{\dagger}_{P',I,\ell} \end{array} \begin{array}{c} \begin{array}{ccc} |\mathrm{in}\rangle_{P,I,\ell} & |\mathrm{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \end{array} \\ \begin{pmatrix} 1 & S^I_\ell(s) & \mathcal{F}^I_\ell \\ S^{I*}_\ell(s) & 1 & \mathcal{F}^{I*}_\ell \\ \mathcal{F}^{I*}_\ell & \mathcal{F}^I_\ell & \rho^I_\ell(s) \end{pmatrix} \succeq 0 \end{array} \qquad s > 4 \quad \forall \ell, I$$

***construct operators from gauge theory with desired quantum numbers***

$$\rho^I_\ell(s) = 2\,\mathrm{Im}\Pi^I_\ell(x + i\epsilon)$$




*e.g.*

$$S0 \quad : \quad j_S(x) = m_q(\bar{u}u + \bar{d}d) \qquad \Pi^0_0(s) \; = \; i\int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{ j_S(x) j_S(0) \right\}|0\rangle$$

$$P1 \quad : \quad j^\mu_V(x) = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d) \qquad \Pi^1_1(s) \; = \; i\int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{ j^\dagger_\sigma(x) j_\sigma(0) \right\}|0\rangle$$

# Current correlators from the UV gauge theory

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\begin{array}{c} \\ \langle \text{in}|_{P',I,\ell} \\ \langle \text{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^{\dagger}_{P',I,\ell} \end{array}
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$$

***construct operators from gauge theory with desired quantum numbers***

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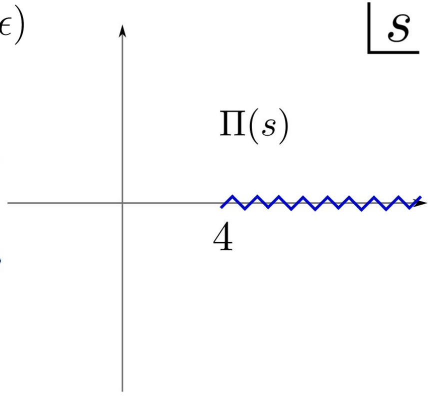


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$$S0 \quad : \quad j_S(x) = m_q(\bar{u}u + \bar{d}d) \qquad \Pi_0^0(s) \;=\; i\int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{ j_S(x) j_S(0)\right\}|0\rangle$$

$$P1 \quad : \quad j_V^\mu(x) = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d) \qquad \Pi_1^1(s) \;=\; i\int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{ j_\sigma^\dagger(x) j_\sigma(0)\right\}|0\rangle$$

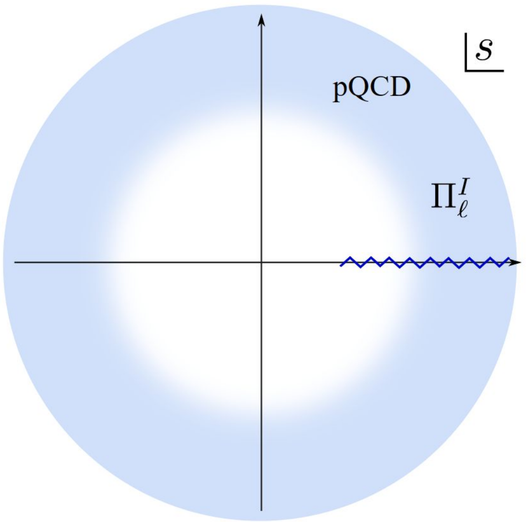
*large spacelike momenta — asymptotic free region with pQCD computation*

# SVZ expansion

[Shifman, Vainshtein, Zakharov, 1979]

OPE: $$T\{j(x)j(0)\} = C_{\mathbb{1}}(x)\ \mathbb{1} + \sum_{\mathcal{O}} C_{\mathcal{O}}(x)\ \mathcal{O}(0)$$

$$\langle 0|T\{j(x)j(0)\}|0\rangle = C_{\mathbb{1}}(x) + C_{\bar{q}q}(x)\ \langle 0|j_S(0)|0\rangle + C_{G^2}(x)\ \langle 0|\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{a\,\mu\nu}|0\rangle + \ldots$$



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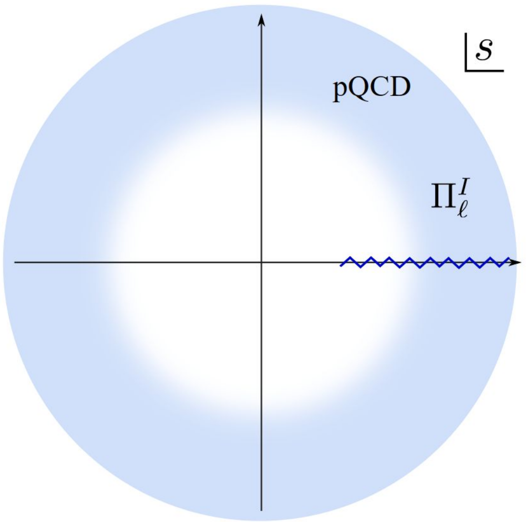
SB vacuum

*quark condensate*

*gluon condensate*

**pQCD**



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SB vacuum

quark condensate

gluon condensate

pQCD

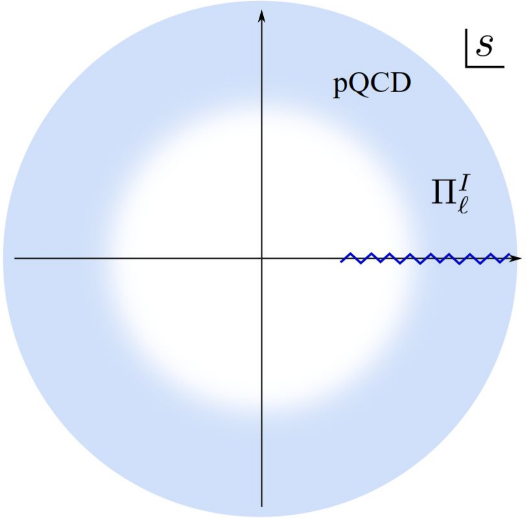
Fourier transform

$$\Pi^0_0(s) = \frac{N_f m_q^2}{(2\pi)^4}\left\{-\frac{3}{8\pi^2}\left(1+\frac{13}{3}\frac{\alpha_s}{\pi}\right)s\ln\!\left(-\frac{s}{\mu^2}\right) - \frac{1}{8s}\langle\frac{\alpha_s}{\pi}G^2\rangle - \frac{3}{2s}\langle j_S\rangle + \dots\right\}$$

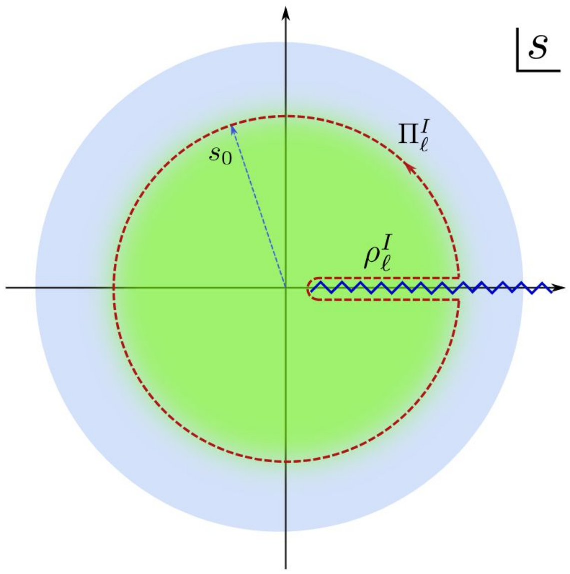
$$\Pi^1_1(s) = \frac{1}{2}\frac{1}{(2\pi)^4}\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)s\ln\!\left(-\frac{s}{\mu^2}\right) + \frac{1}{12s}\langle\frac{\alpha_s}{\pi}G^2\rangle + \frac{1}{s}\langle j_S\rangle + \dots\right\}$$

$\vdots$

$$N_c = 3$$



# Finite energy sum rule



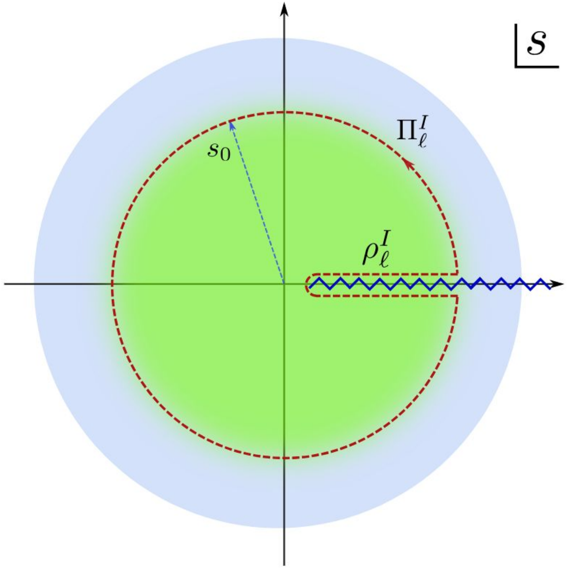

**connect with pQCD with bootstrap at s0**

contour integral $s^n \Pi(s)$ vanishes

SVZ

$$\int_4^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

# Finite energy sum rule

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**bootstrap variables** ← linear constraints **gauge theory information**

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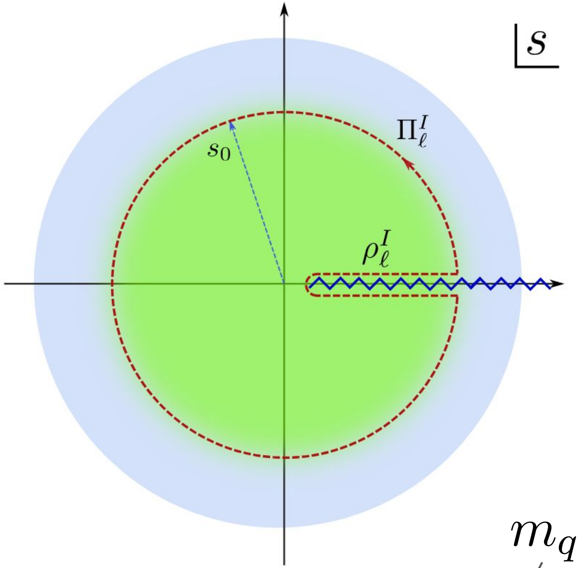


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**bootstrap variables** ← linear constraints — **gauge theory information**

$m_q$ $N_c$ $\Lambda_{\text{QCD}}$ additional IR parameters

$$S0 \;\; : \int_4^{s_0} \rho_0^0(x) x^n dx = \frac{s_0^{n+1} N_f m_q^2}{(2\pi)^4} \left\{ \frac{3 s_0}{4\pi(n+2)} \left(1 + \frac{13}{3}\frac{\alpha_s}{\pi}\right) + \delta_n \frac{\pi}{4 s_0} \langle \frac{\alpha_s}{\pi} G^2 \rangle + \delta_n \frac{3\pi}{s_0} \langle j_S \rangle + \dots \right\}, \quad n \geq 0$$

$$P1 \;\; : \int_4^{s_0} \rho_1^1(x) x^n dx = -\frac{s_0^{n+1}}{(2\pi)^4} \frac{1}{2} \left\{ -\frac{s_0}{2\pi(n+2)} \left(1 + \frac{\alpha_s}{\pi}\right) + \delta_n \frac{\pi}{6 s_0} \langle \frac{\alpha_s}{\pi} G^2 \rangle + \delta_n \frac{2\pi}{s_0} \langle j_S \rangle + \dots \right\}, \quad n \geq -1$$

# Gauge theory parameters: numerical input

gauge theory info:

$$N_f = 2 \qquad N_c = 3$$

*for comparison with experiments*

$$s_0 = (1.2\,\mathrm{GeV})^2, \quad \alpha_s \simeq 0.41, \quad m_u \simeq 4\,\mathrm{MeV} \quad m_d \simeq 7.3\,\mathrm{MeV}$$

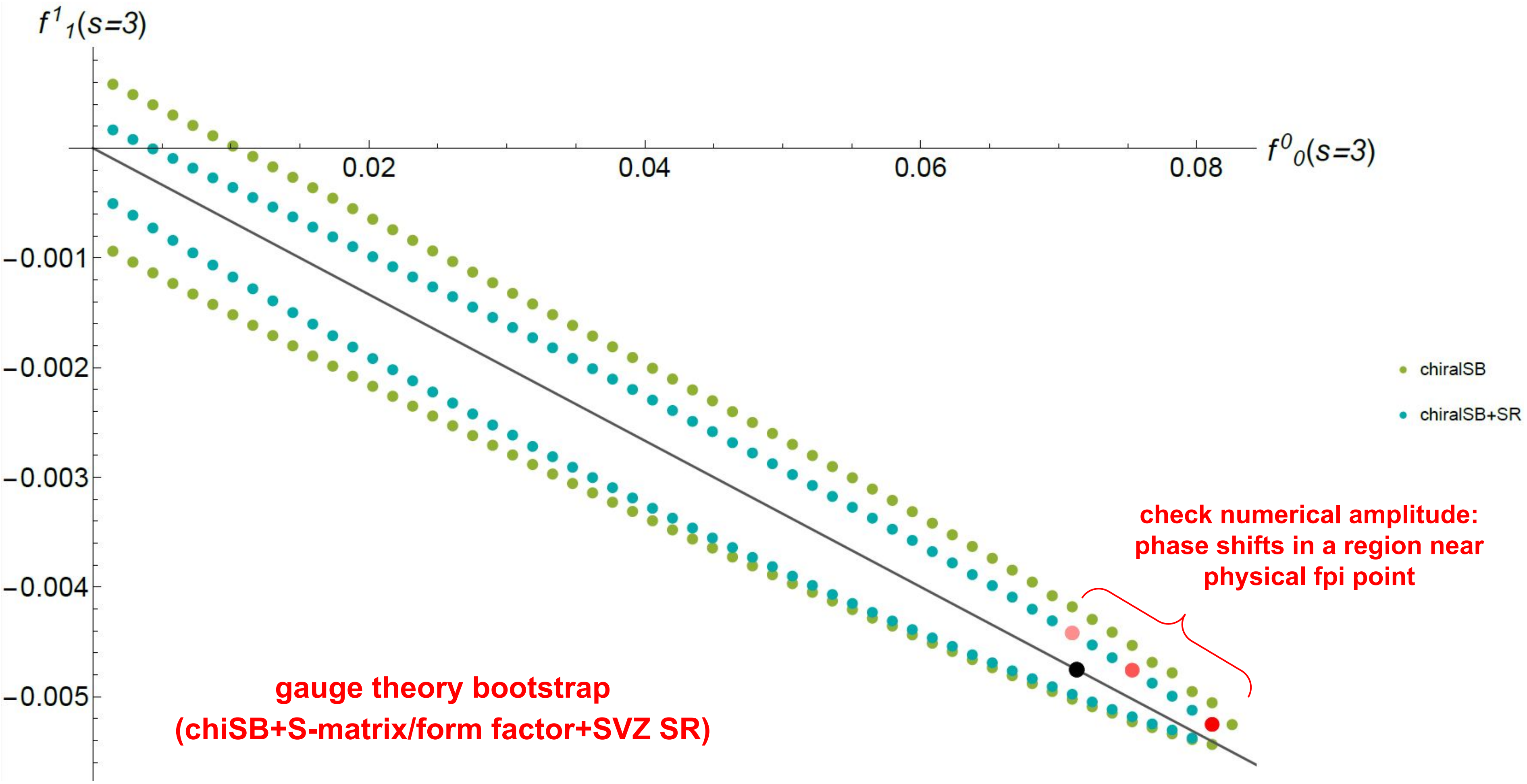
***more recently (to appear):***

$$s_0 = (2\,\mathrm{GeV})^2, \quad \alpha_s \simeq 0.31, \quad m_u \simeq 3.6\,\mathrm{MeV} \quad m_d \simeq 6.5\,\mathrm{MeV}$$

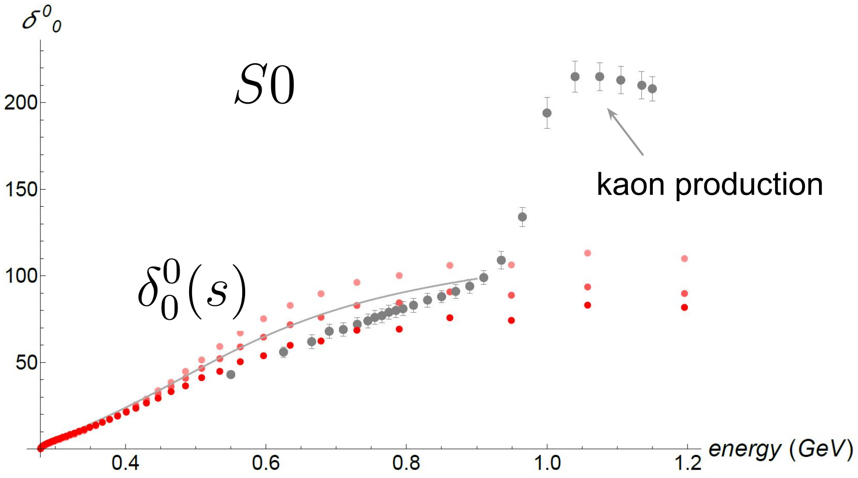
IR parameters

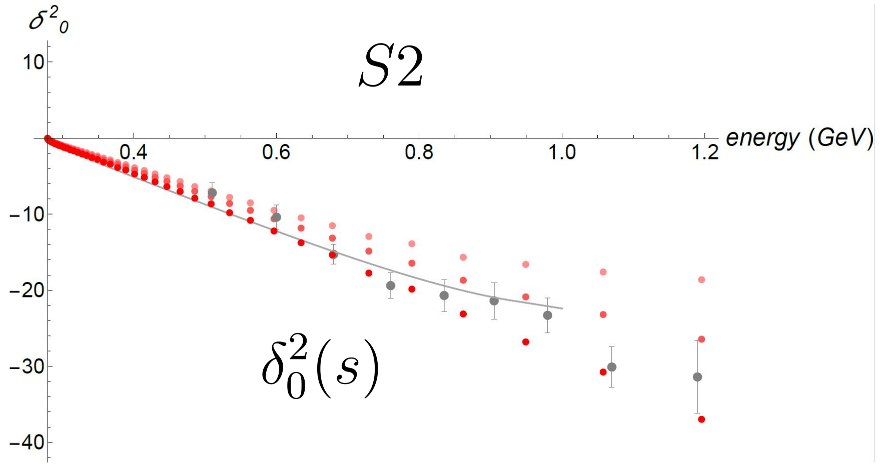
$$\langle \frac{\alpha_s}{\pi} G^2 \rangle \simeq 0.023\,\mathrm{GeV}^4, \quad \langle j_S(0) \rangle = m_q \langle \bar{u}u + \bar{d}d \rangle \simeq -(0.1\,\mathrm{GeV})^4$$

**numerically not significant in our working precision**



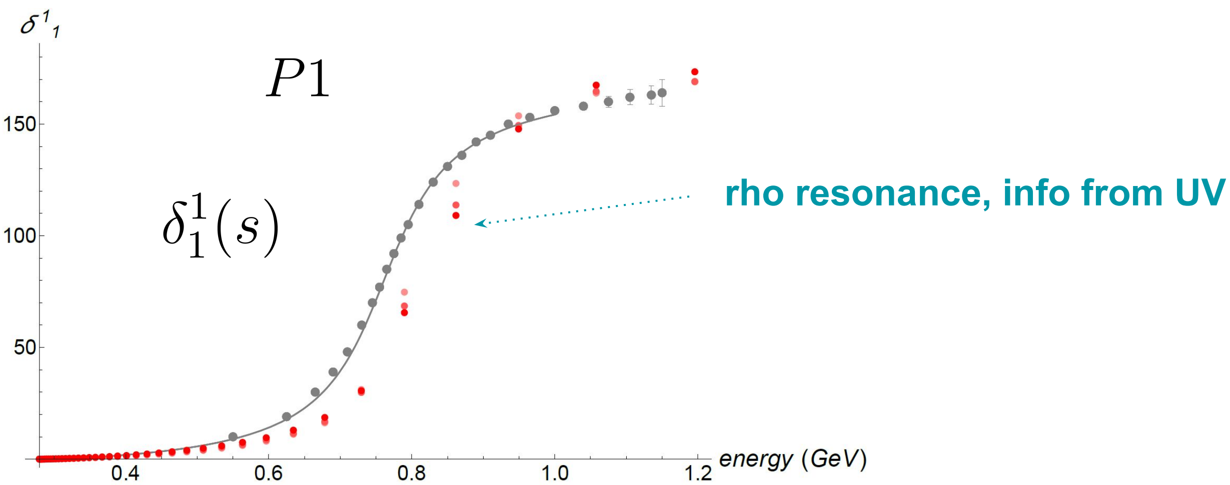
$f^1_1(s=3)$
$f^0_0(s=3)$
0.02
0.04
0.06
0.08
−0.001
−0.002
−0.003
−0.004
−0.005
chiralSB
chiralSB+SR
check numerical amplitude: phase shifts in a region near physical fpi point
gauge theory bootstrap
(chiSB+S-matrix/form factor+SVZ SR)




**gauge theory bootstrap**

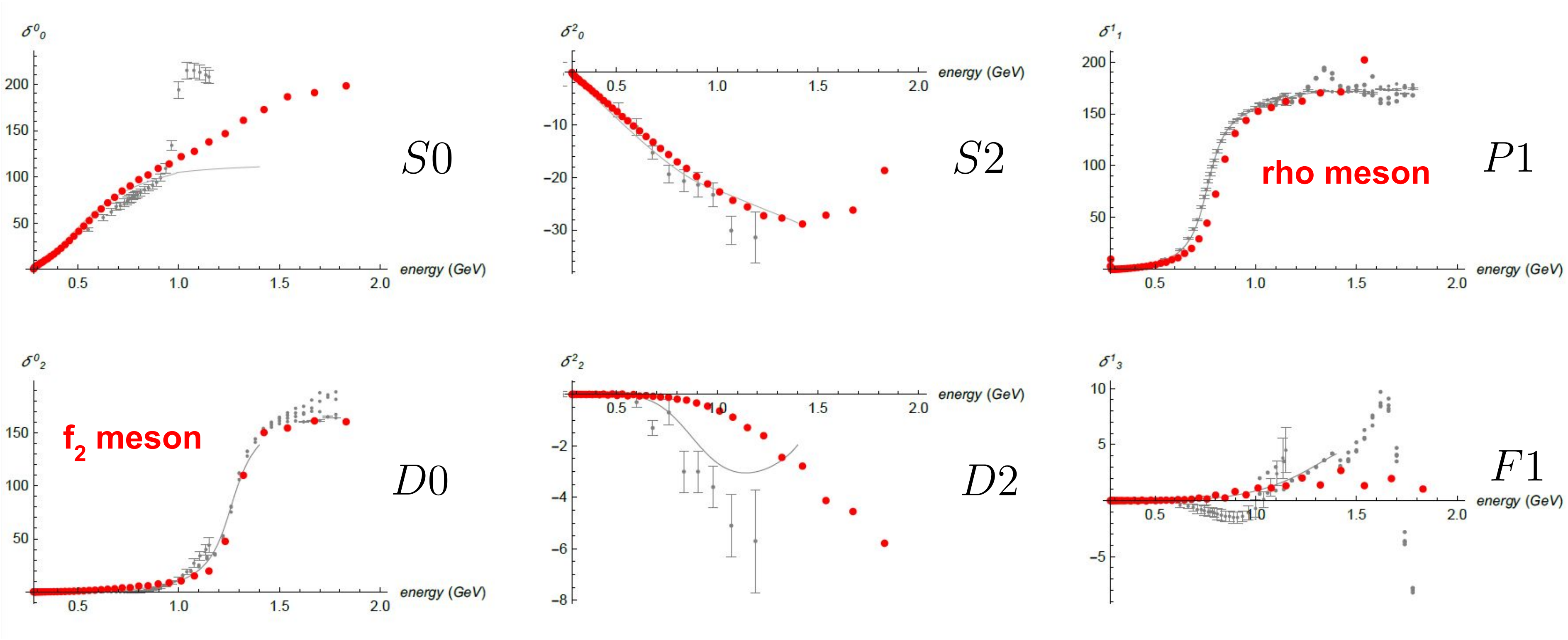
experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit (gray line)
[Pelaez, Yndurain, 2005]



**gauge theory bootstrap**

# phase shifts up to 2GeV

[YH, Kruczenski, *to appear*]




experiments (gray dots) [Protopopescu et al, 1973][Losty et al, 1974][Hyams et al, 1975] scattering lengths and effective range parameters

$$\mathrm{Re} f_\ell^I(s) \overset{k\to 0}{\simeq} \frac{2m_\pi}{\pi} k^{2\ell}\left(a_\ell^I + b_\ell^I k^2 + \dots\right) \qquad k = \frac{\sqrt{s-4m_\pi^2}}{2}$$

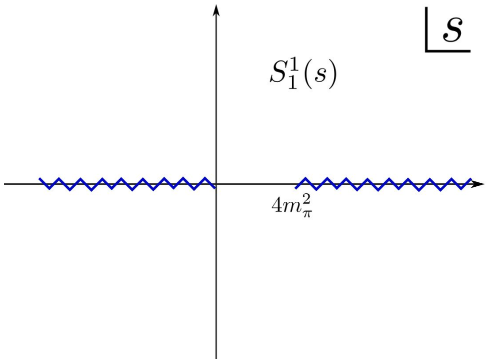
| | DFGS | ACGL | CGL | PY | **gauge theory bootstrap** | | |
|---|---|---|---|---|---|---|---|
| $a_0^{(0)}$ | $0.228 \pm 0.012$ | $0.240 \pm 0.060$ | $0.220 \pm 0.005$ | $0.230 \pm 0.010$ | 0.178 | 0.188 | 0.201 |
| $a_0^{(2)}$ | $-0.0382 \pm 0.0038$ | $-0.036 \pm 0.013$ | $-0.0444 \pm 0.0010$ | $-0.0422 \pm 0.0022$ | -0.0362 | -0.0388 | -0.0425 |
| $b_0^{(0)}$ | | $0.276 \pm 0.006$ | $0.280 \pm 0.001$ | $0.268 \pm 0.010$ | 0.31 | 0.307 | 0.297 |
| $b_0^{(2)}$ | | $-0.076 \pm 0.002$ | $-0.080 \pm 0.001$ | $-0.071 \pm 0.004$ | -0.0629 | -0.0681 | -0.075 |

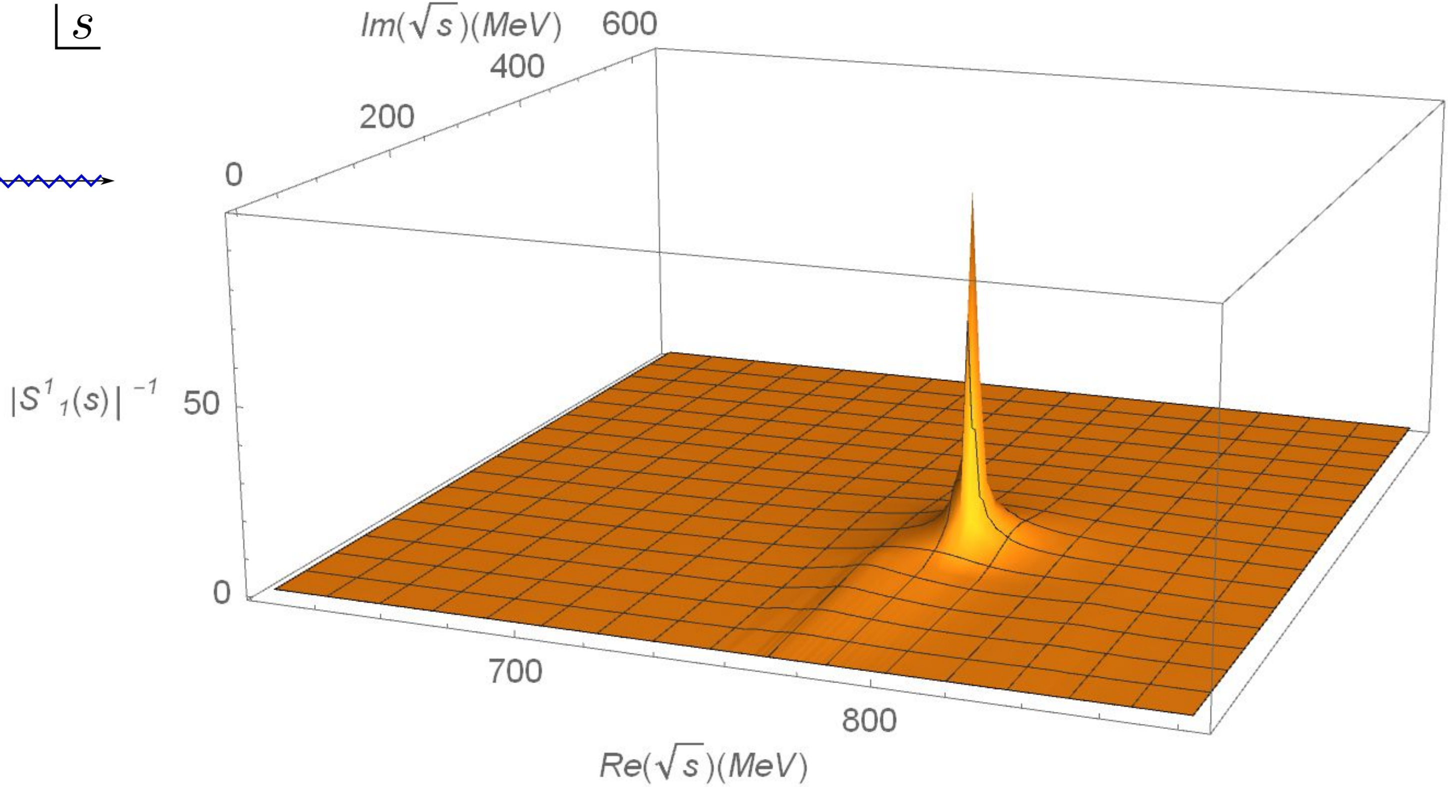
| | Nagel | PSGY | CGL | PY | | | |
|---|---|---|---|---|---|---|---|
| $a_1$ | $38 \pm 2$ | $38.5 \pm 0.6$ | $37.0 \pm 0.13$ $[37.9 \pm 0.5]$ [a] | $38.1 \pm 1.4$ $[38.6 \pm 1.2]$ [b] $\times 10^{-3}$ | 0.0281 | 0.0304 | 0.0343 |

[Nagel et al, 1979][Descotes et al, 2002][Ananthanarayan et al, 2001]
[Colangelo, Gasser, Leutwyler, 2001][Pelaez, Yndurain, 2003]

[YH, Kruczenski, *to appear*]

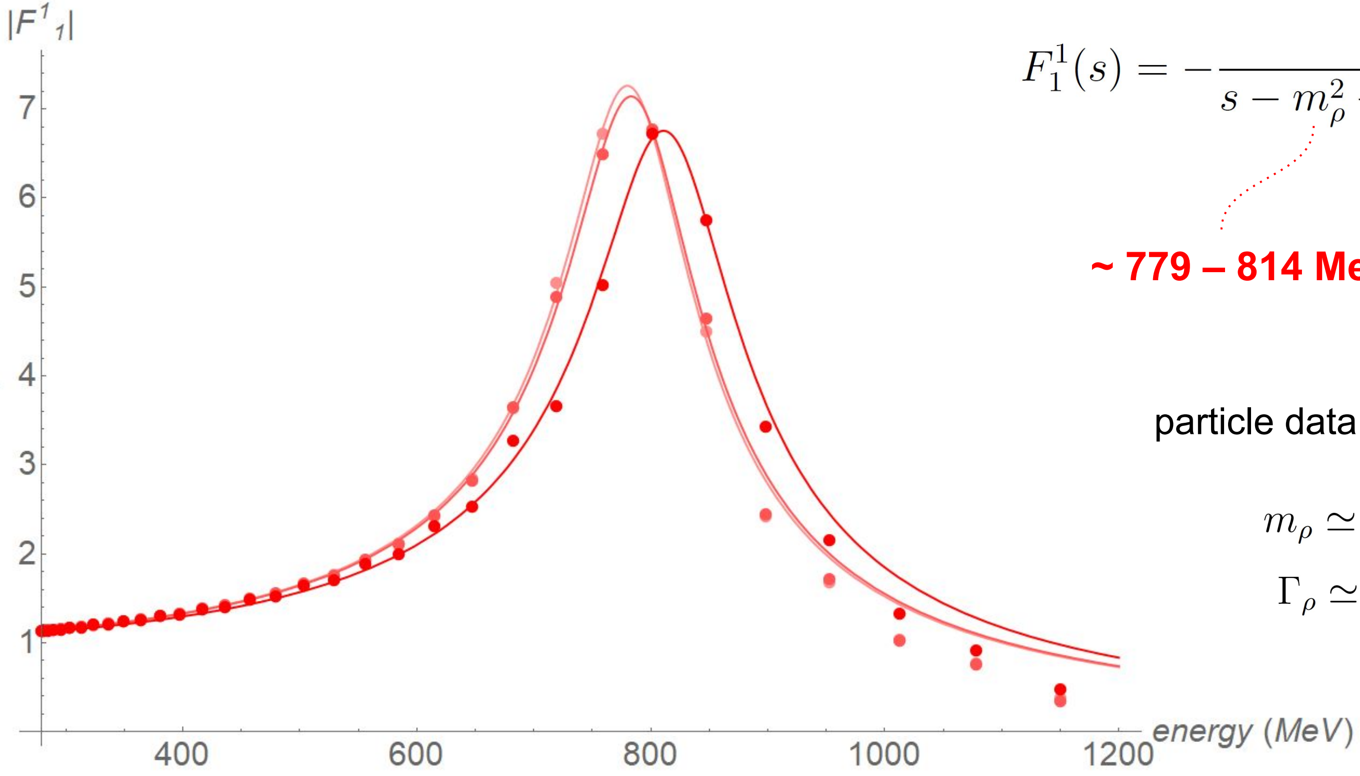
***rho meson as pole on the second sheet of*** $S_1^1(s)$




[YH, Kruczenski, *to appear*]

## fit P1 form factor with Breit-Wigner form




$$F_1^1(s) = -\frac{m_\rho^2}{s - m_\rho^2 + i m_\rho \Gamma_\rho \theta(s - 4m_\pi^2)}$$

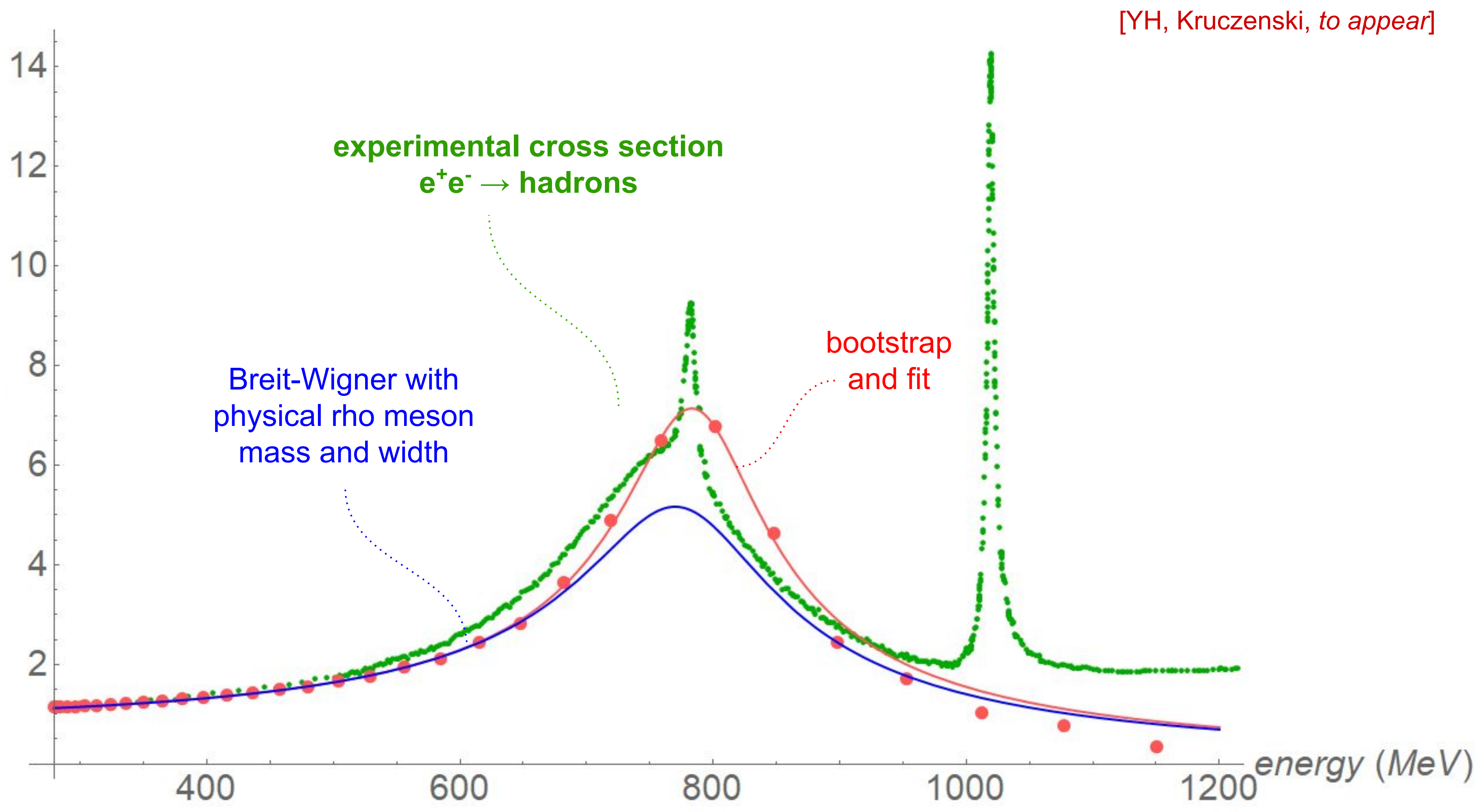
**~ 779 – 814 MeV** **~ 107 – 128 MeV**

particle data group (T-matrix pole):

$$m_\rho \simeq 761 - 765 \text{ MeV}$$

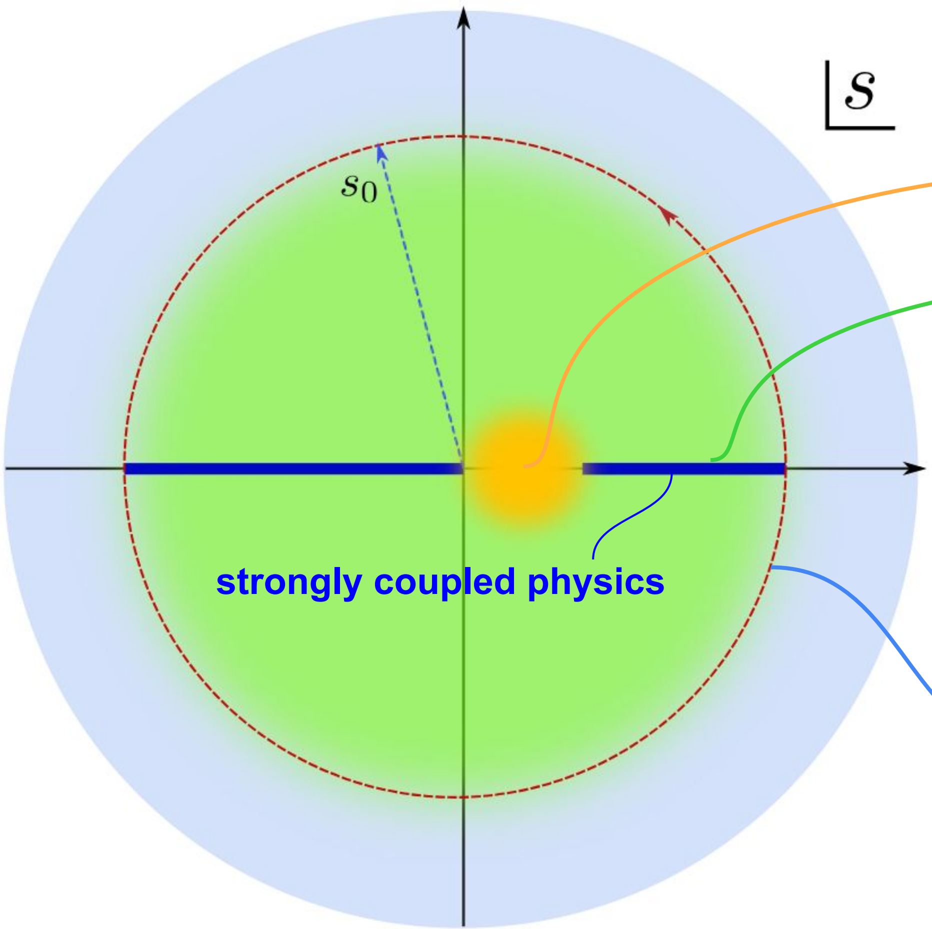
$$\Gamma_\rho \simeq 142 - 148 \text{ MeV}$$

[YH, Kruczenski, *to appear*]



[YH, Kruczenski, to appear]
experimental cross section
e+e- → hadrons
bootstrap
and fit
Breit-Wigner with
physical rho meson
mass and width
14
12
10
8
6
4
2
400
600
800
1000
1200
energy (MeV)

# gauge theory bootstrap – summary




**bootstrap variables:**

$$\{\rho_{1,2}(x,y),\ldots,\mathrm{Im}F_\ell^I(x),\rho_\ell^I(x)\}$$

**chiSB** $\quad A(s,t,u)\simeq\frac{4}{\pi}\frac{s-m_\pi^2}{32\pi f_\pi^2}$

**S-matrix/form factor**

$$\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\ \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s) \end{pmatrix}\succeq 0$$

**pQCD (SVZ, asymptotics)**

$$\frac{1}{s_0^{n+2}}\int_4^{s_0}dx\,\rho_0^0(x)x^n,\ \frac{1}{s_0^{n+2}}\int_4^{s_0}dx\,\rho_1^1(x)x^n,\ \ldots$$

$$F_\pi(s)\simeq-\frac{16\pi\alpha_s(s)f_\pi^2}{s}$$

# Conclusions

- Gauge theory bootstrap:

combining old (SVZ sum rules) and new (S-matrix/form factor bootstrap) techniques

using only $\underbrace{N_c \; N_f \; m_q \; \Lambda_{\mathrm{QCD}}}_{\textit{gauge theory parameters}}$ $\qquad \underbrace{f_\pi \quad m_\pi}_{\textit{universal low energy parameters}}$

strongly coupled low energy physics of asymptotically free gauge theories

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strongly coupled low energy physics of asymptotically free gauge theories

- Numerical test with $N_f = 2 \;\; N_c = 3$ find good agreement with experiments

results show: strongly coupled QCD physics is computable

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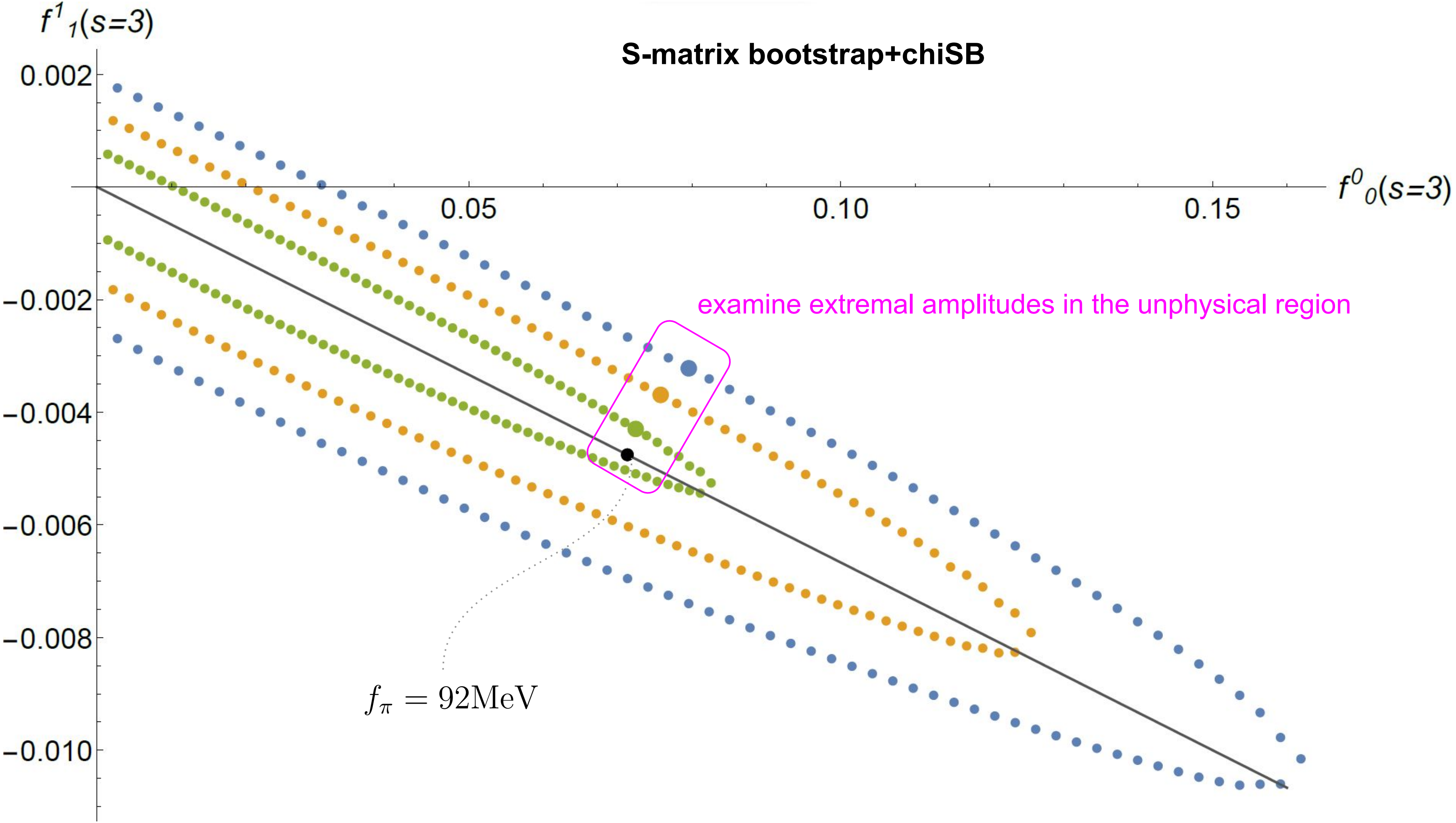
strongly coupled low energy physics of asymptotically free gauge theories

- Numerical test with $N_f = 2 \;\; N_c = 3$ find good agreement with experiments

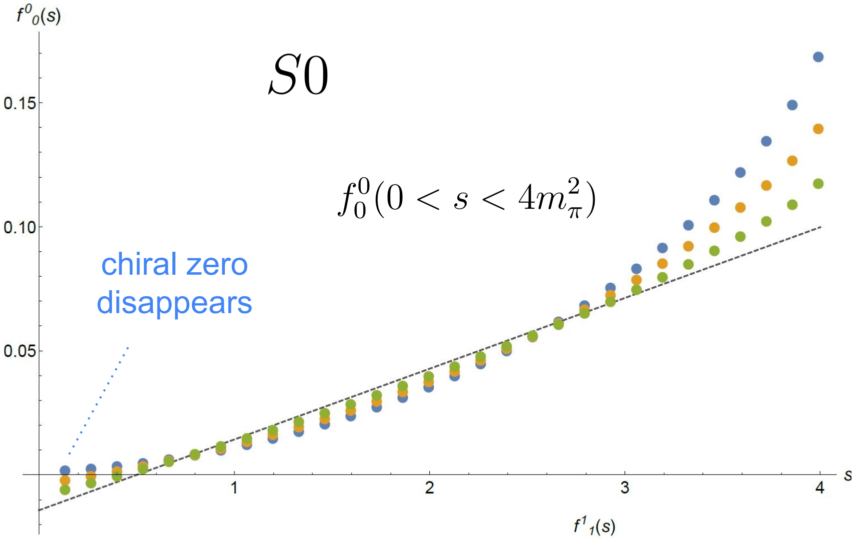
results show: strongly coupled QCD physics is computable

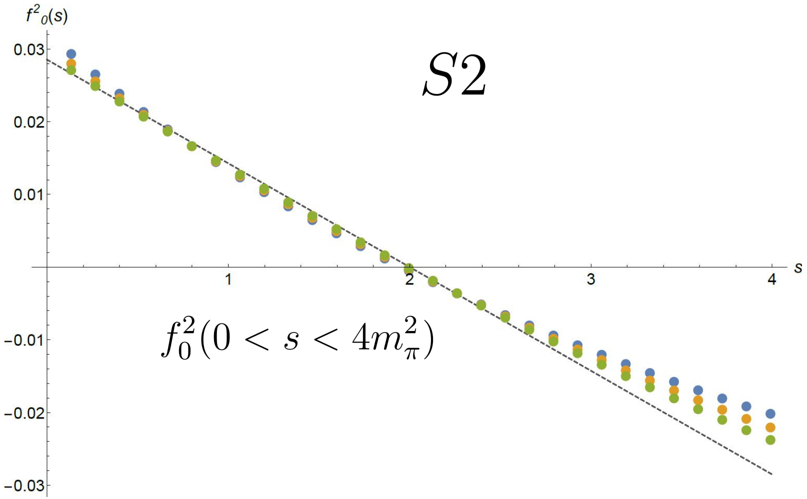
- Further developments — deep understanding of gauge theories

Thank you!



$f^1{}_1(s=3)$
S-matrix bootstrap+chiSB
0.002
−0.002
−0.004
−0.006
−0.008
−0.010
0.05
0.10
0.15
$f^0{}_0(s=3)$
examine extremal amplitudes in the unphysical region
$f_\pi = 92\text{MeV}$

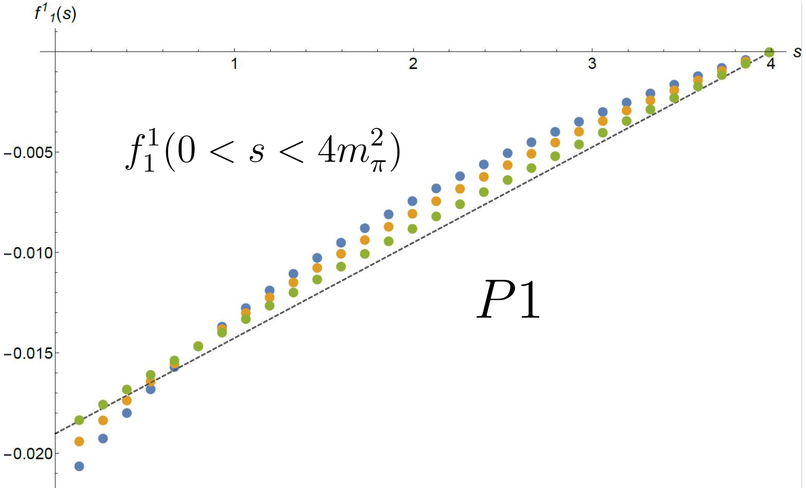



dashed lines: tree-level p.w.

$f^0_{0,\text{tree}}(s) \quad f^1_{1,\text{tree}}(s) \quad f^2_{0,\text{tree}}(s)$

$f_\pi = 92\text{MeV}$




$\epsilon^\chi = 6\times10^{-3}$

$\epsilon^\chi = 4\times10^{-3}$

$\epsilon^\chi = 2\times10^{-3}$

$0 < s < 4m_\pi^2$