Amplitudes and bootstrap

Rencontres de Physique des Particules
Janvier 2024, Sorbonne Université

Revue séminaire

Piotr Tourkine,
Laboratoire d'Annecy-le-Vieux de Physique Théorique,
CNRS & USMB

- Mostly a review talk, but by the end I’ll flash some directions of research of mine, with collaborators:

Alexander Zhiboedov
CERN

Felipe Figueroa
LAPTh

Christopher Eckner
Nova Gorica

Damien Leflot
LAPTh

Mehmet Gumus
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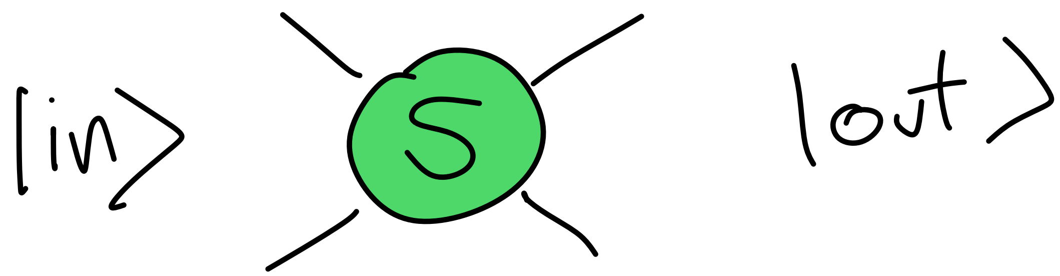
The S-matrix



- Essential for precision physics in colliders + gravitational waves
- Also a beautiful *theoretical laboratory* where you can ask questions regarding phenomena not accessible to experiments yet

Perturbative

“Scattering amplitudes’” methods

QCD background in colliders
EW sector
Gravitational waves

Non-perturbative

S-matrix bootstrap

strongly coupled phenomena:
super-transplanckian scattering in gravity
confining gauge theories,
…

Amplitudes' methods

1-slide lightning review

- "On-shell revolution", Lagrangians and gauge-invariance are a useless complication, geometric methods

talk by Quentin Bonnefoy

- Massive progress on Feynman integrals, methods from algebraic geometry in particular

- *Double copy*:

gravity ~ (gauge)^2,

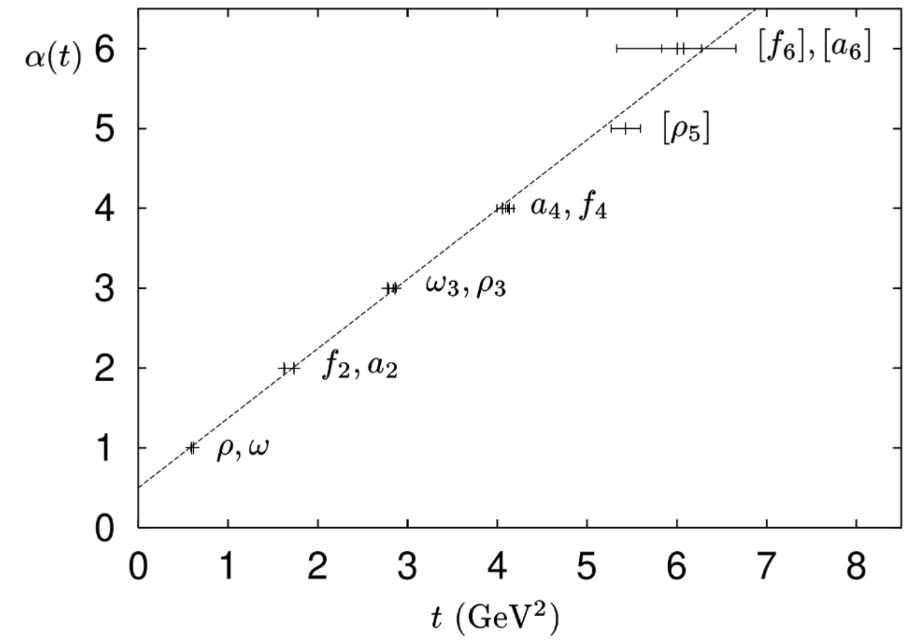
↳ applications to gravitational wave from binary mergers

Outline

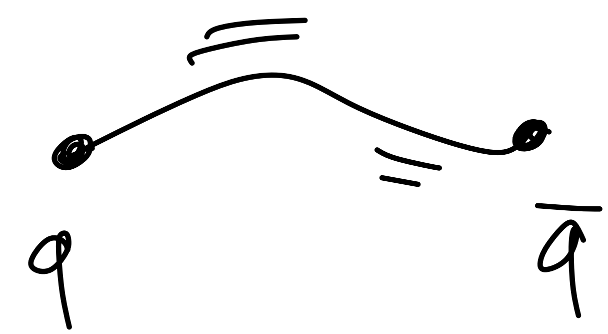
- Bootstraps: old and new
- Teaser
- Bootstrap axioms
- Dual approach; large N QCD
- Primal approach; scalar theories, quantum gravity
- Future prospects

Old bootstrap

Soviet + Western groups



Regge trajectories



- People trying to understand strong interactions in the 60s and were confused by abundance of particles. Not at all like QED.
- Idea of bootstrap: use physical constraints plus a little something to solve theory (exp. input, other principle, etc.)
- Beautiful program, left aside when QCD came out. <u>Important results</u>: Froissart bound, crossing symmetry, Regge theory, etc. **String theory.**
- Main issues / unsolved questions:
  - lack of usable computational tools
  - not a super clear definition of principles to use (analyticity in particular);
  - multi-particle processes

$$A(s,t) \sim_{s\to\infty} s^{\alpha(t)}$$

$$J = \alpha(m_n^2)$$

Modern developments

S-matrix bootstrap

- Re-starts in 2016 with: **The S-matrix Bootstrap I, II, III, IV**
  M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, P. Vieira

- Builds on numerical success from CFT bootstrap

- New ingredients:

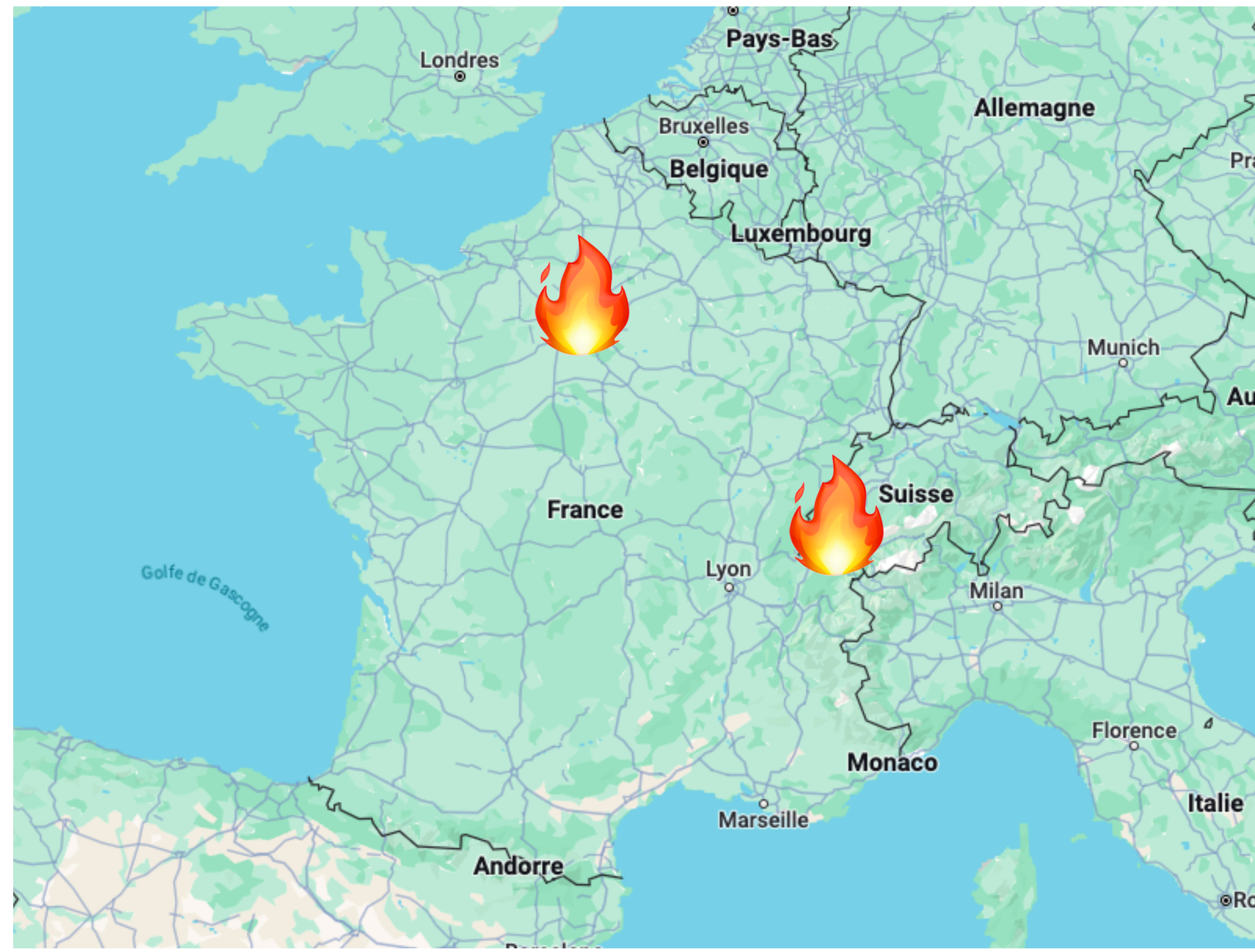
  Paradigm change: explore **space** of theories, rather than solve one theory in particular

  At boundaries of space of theories must lie special objects which one can hope to solve (e.g. 3d Ising model).

  Right now, proposition that large-N Yang-Mills might lie at such a place, but other exotic phenomena as well

- Paris area and Geneva+Lausanne+Annecy are big hotspot for S-matrix bootstrap research in Europe

- S-matrix bootstrap series of workshops going strong (5 editions so far)



A maths exercise

A maths exercise

- Consider a sequence of real numbers and $(s_n)_{n \geq 0}$, $0 \leq s_n < s_{n+1}$
- Q: Do there exist functions of two complex variables, *(s,t)*:

$$A(s,t) = \sum_{n=0}^{N} \frac{\sum_{j=0}^{n} a_{n,j}\, t^j}{s - s_n}$$

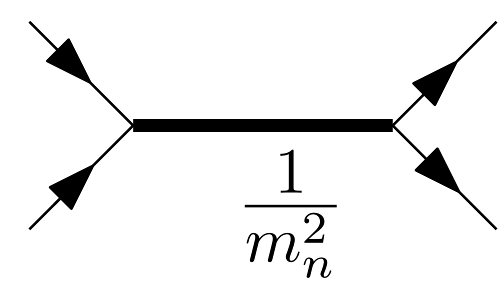
the unknowns ($a_{n,j}$)

given (s_n)

- Such that $A(s,t) = A(t,s)$ and $\forall n, j, \;\; a_{n,j} \geq 0$?

Meromorphic amp

$$\sum_{n=0}^{\infty}$$



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the unknowns

given

- Such that $A(s,t) = A(t,s)$ and $\forall n, j, \quad a_{n,j} \geq 0$?
- Firstly, $N = \infty$, otherwise a finite polynomial in t cannot have singularities at $t = s_n$.
- $A(s,t) = A(t,s)$ gives:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} a_{n,j} \left(\frac{t^j}{s - x_n} - \frac{s^j}{t - x_n} \right) = 0$$

(crossing)

A maths exercise

- $A(s,t) = A(t,s)$ gives:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} a_{n,j} \left(\frac{t^j}{s - x_n} - \frac{s^j}{t - x_n} \right) = 0 \qquad \text{(}\textit{crossing}\text{)}$$

$$\sum_{n,j}^{\infty} a_{n,j} A_{n,j}(t,s) = 0$$

A maths exercise

$$\sum_{n,j}^{\infty} a_{n,j} A_{n,j}(t,s) = 0 \qquad (*)$$

A maths exercise

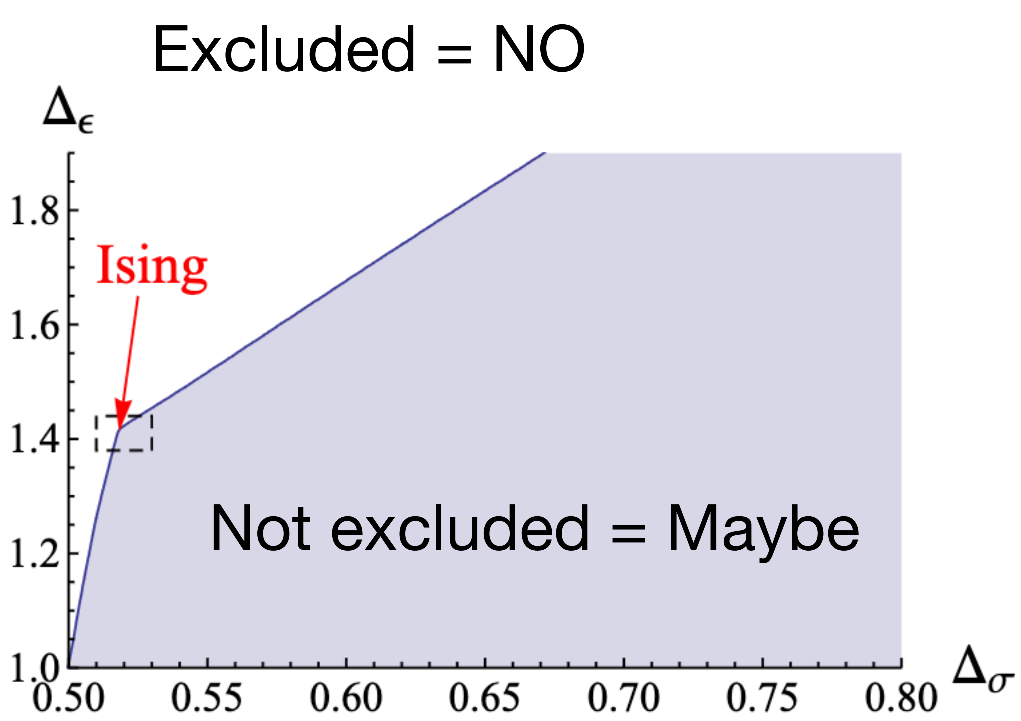
$$\sum_{n,j}^{\infty} a_{n,j} A_{n,j}(t,s) = 0 \qquad (\ast)$$

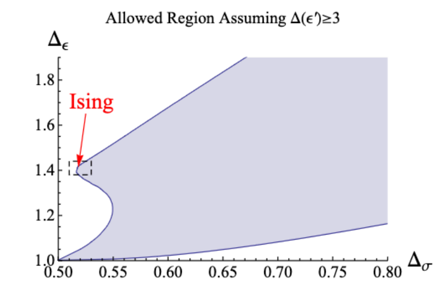
- If you can find a *linear* functional F

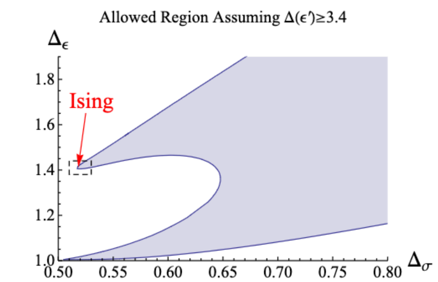
$$\forall\ n, j, \quad F[A_{n,j}] \geq 0$$

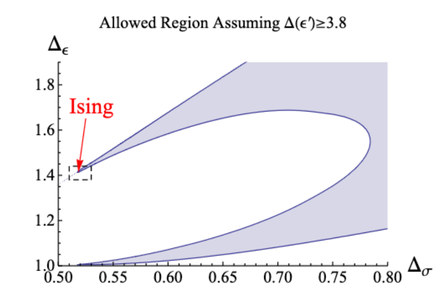
- Then you know that (✱) does not have a solution, *without having to search the values of the a's*.
- E.g. : $F(A) := A(s_0, t_0)$, or $F(A) = \sum f_{i,j} (\partial_s)^i (\partial_t)^j A(s,t) \Big|_{s=0,t=0}$, …
- scan the space of $f_{i,j}$: if no solution, can't say anything. If one is found: excluded.

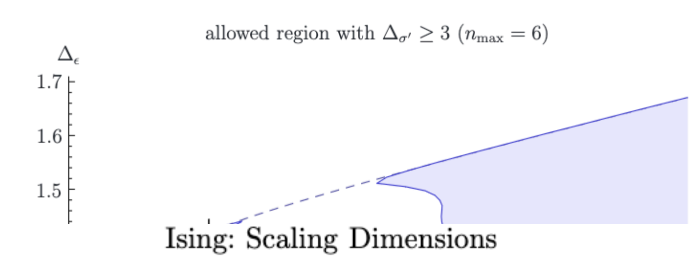
Bootstrap of 3D Ising

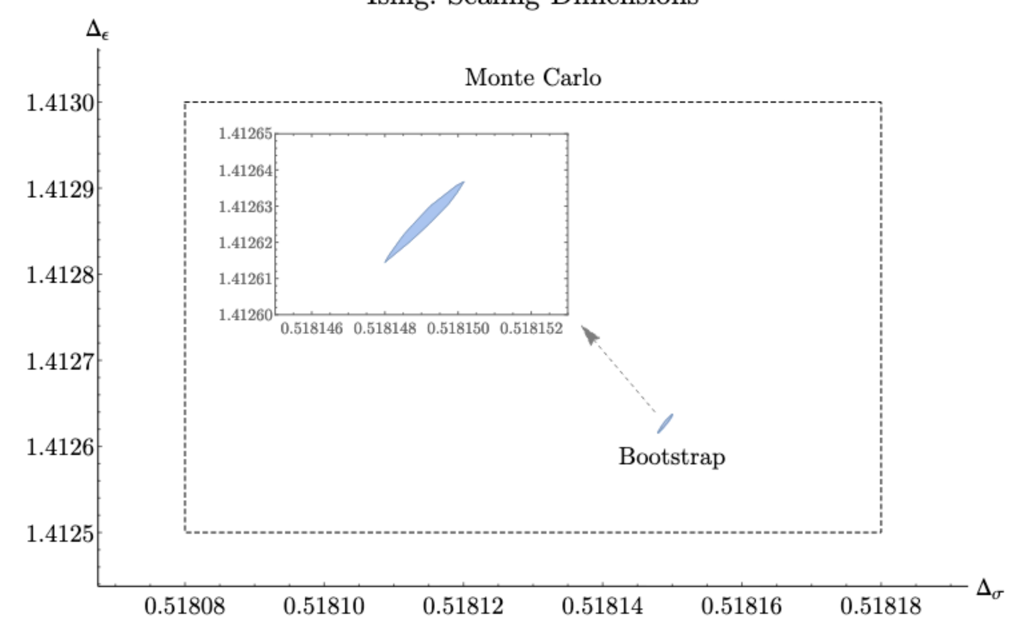












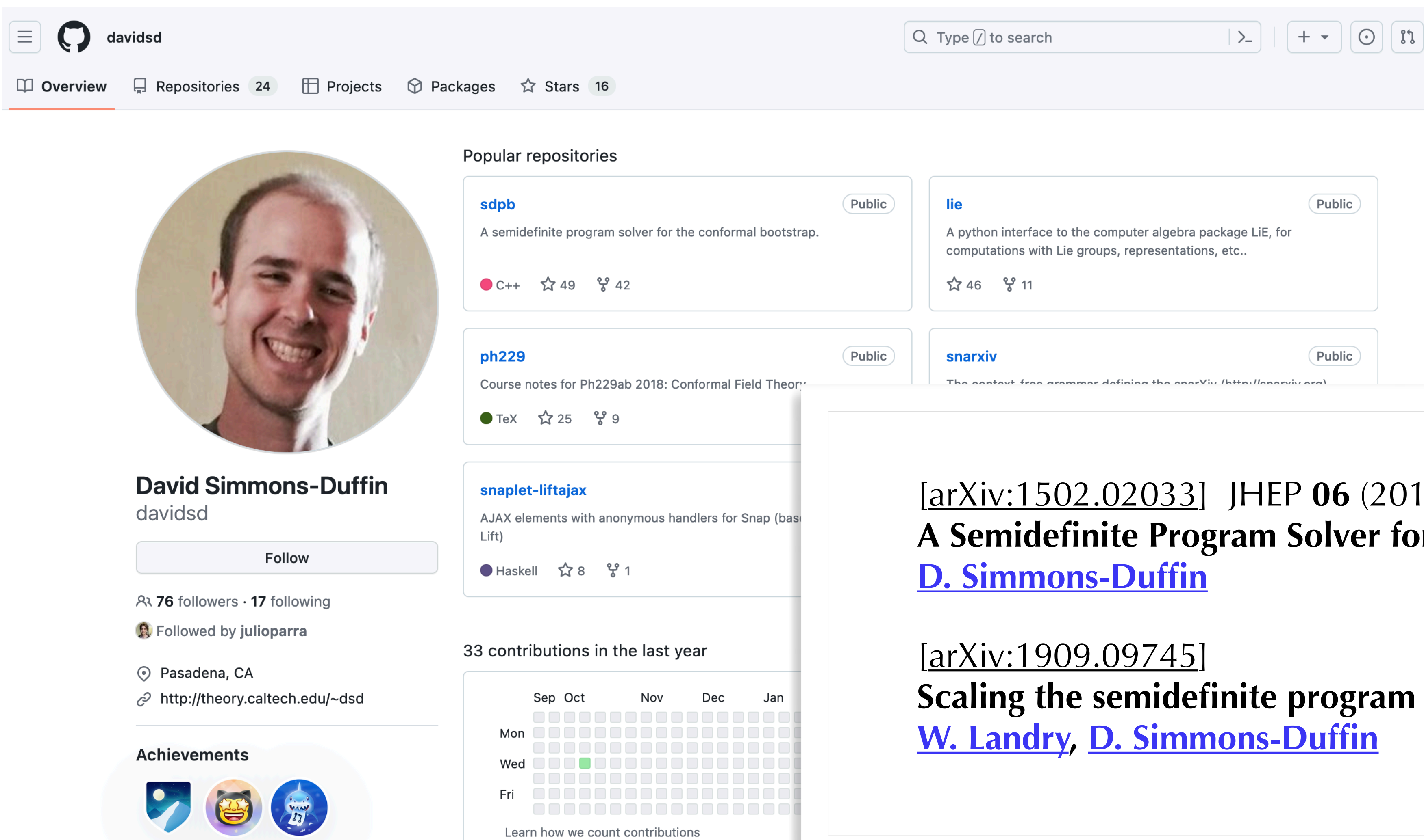
[arXiv:0807.0004] JHEP **12** (2008) 031
Bounding scalar operator dimensions in 4D CFT
R. Rattazzi, V. S. Rychkov, E. Tonni, A. Vichi

[arXiv:1203.6064] Phys.Rev. **D86** (2012) 025022
Solving the 3D Ising Model with the Conformal Bootstrap
S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, A. Vichi

SDPB

Semi-definite programming adapted to conformal bootstrap

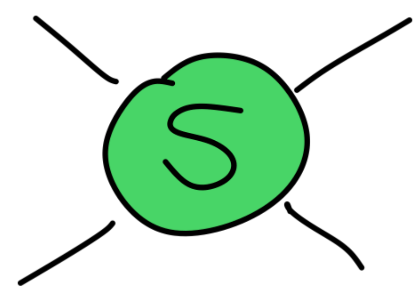
Main feature : arbitrary precision



[arXiv:1502.02033] JHEP **06** (2015) 174
A Semidefinite Program Solver for the Conformal Bootstrap
D. Simmons-Duffin

[arXiv:1909.09745]
Scaling the semidefinite program solver SDPB
W. Landry, D. Simmons-Duffin

The bootstrap axioms



from now on: mostly 2 to 2 scattering

most of the time gapped theory

$$s = 4E^2$$

$$t = -4E^2(1 - \cos(\theta))$$

Bootstrap axioms

A - C - U

- Analyticity — Causality, locality; Dispersion relations
- Crossing — Relativistic invariance
- Unitarity — Quantum Mechanics

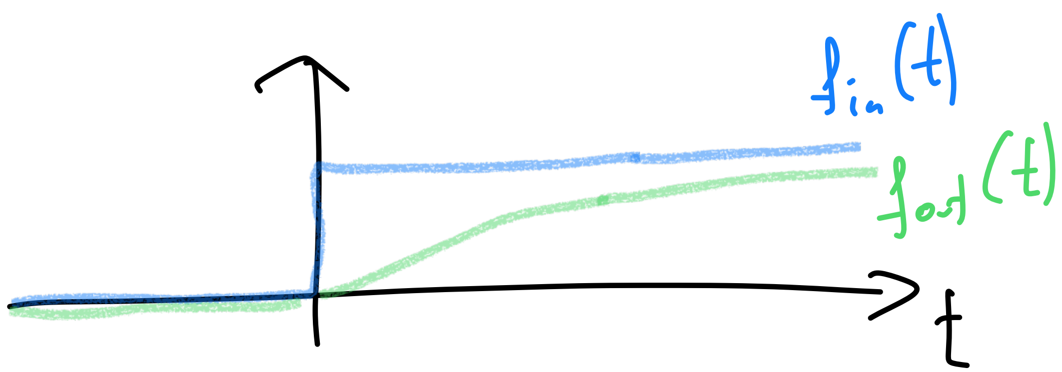
The S-matrix bootstrap : “what is the theory of relativistic quantum mechanical interactions ?”

[arXiv:2006.08221] JHEP **03** (2021) (null)
An Analytical Toolkit for the S-matrix Bootstrap
M. Correia, A. Sever, A. Zhiboedov

Analyticity

- Reflects causality.
- Signal model: Fourier transform of causal system can be analytically extended in upper half-plane.

$$f_{out}(t) = \int_{-\infty}^{\infty} S(t'-t) f_{in} dt'(t')$$



$$S(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} S(t) dt \xrightarrow[\text{causality}]{} \int_{0}^{\infty} e^{i\omega t} S(t) dt$$

$$e^{i(\omega_1 + i\omega_2)t} = e^{i\omega_1 t} e^{-\omega_2 t}$$

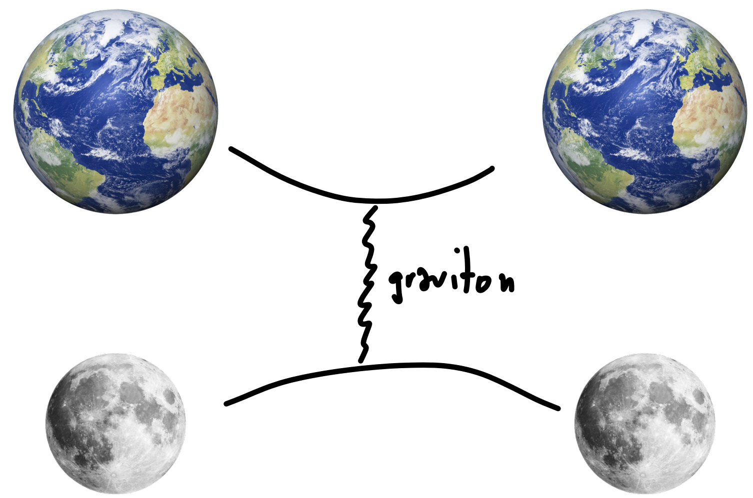
convergence is improved if $\Im \omega > 0$

Analyticity

- Reflects causality.
- Various levels of proven/unproven analyticity
- Maximal analyticity: the only singularities of the S-matrix are those dictated by unitarity. Unproven but believed to apply for the scattering of lightest states (with mass gap).
- Dispersion relations
- If scattering of non lightest-states, anomalous thresholds

[Correia; Mizera, Hannesdottir, …]

Crossing



- $A(s,t) = A(t,s)$
- Singularities in *t*-channel ~ potential
- 2 to 2 and 2 to 3 *proven*, 60's (Bros Epstein Glaser; Bros)
- Crossing-symmetric dispersion relations
- Recent developments in *n* to *m*

[arXiv:2012.04877] Phys.Rev.Lett. **126** (2021) 181601
Crossing Symmetric Dispersion Relations in QFTs
A. Sinha, A. Zahed

[arXiv:2310.12199]
Crossing beyond scattering amplitudes
S. Caron-Huot, M. Giroux, H. S. Hannesdottir, S. Mizera

Unitarity

Partial wave unitarity

$$T(s,t) = 16\pi \sum_{J=0}^{\infty} (2J+1) P_J(\cos(\theta)) f_J(s)$$

Legendre polynomials

$$S_J(s) = 1 + i\rho(s) f_J(s)$$

$$S_J(s) = 1 + i\frac{(s-4)^{(d-3)/2}}{\sqrt{s}} f_J(s)$$

↳ partial waves diagonalise unitarity:

Full, NP unitarity:

- $|S_J(s)|^2 \leq 1$

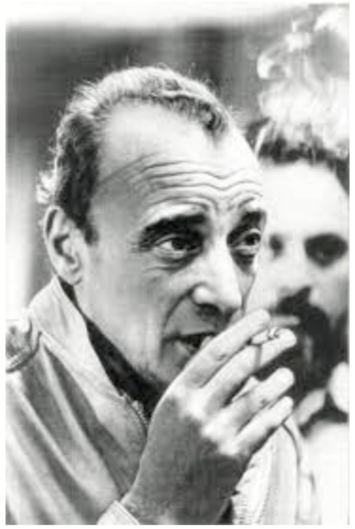
Positivity:

- $\Im f_J(s) \geq 0$

Straightforward to check, difficult to combine with crossing

Regge theory

- Needed to constrain high energy limit of amplitudes
- a.k.a. : Theory of complex angular momentum
- Beautiful formalism, could be improved and understood better
- E.g.: Can amplitudes with only finitely many trajectories exist?
  [Eckner, Figueroa, T., *wip*]



Gribov

Regge

Mandelstam

Two broad classes of techniques

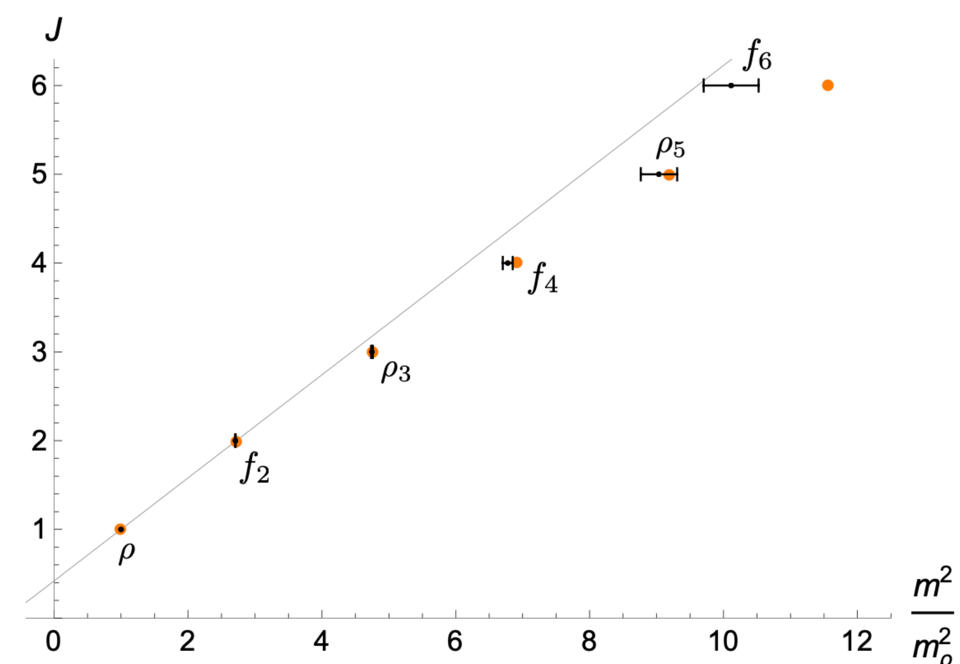
Primal; Dual

- Primal: Constructive
  - rho-Ansatz
  - Atkinson-Mandelstam iteration
- Dual: Functionals
  - Null constraints; “EFT-hedron”, large N QCD

Dual approaches, Fuctionals

Functionals

- Take a function, associate to it a scalar.
- E.g. : $\Phi[f] = f(x), \Phi[f] = \frac{\partial f}{\partial x}\Big|_{x=0} \ \ldots$



Null constraints

Space of consistent EFTs

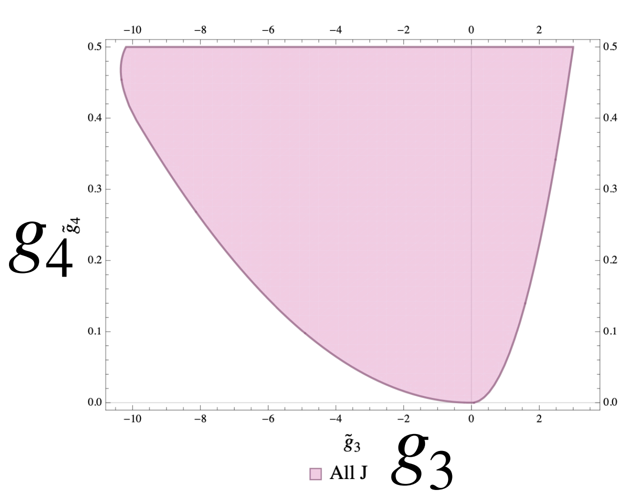
[arXiv:2011.02957] JHEP **05** (2021) 280
Extremal Effective Field Theories
S. Caron-Huot, V. van Duong

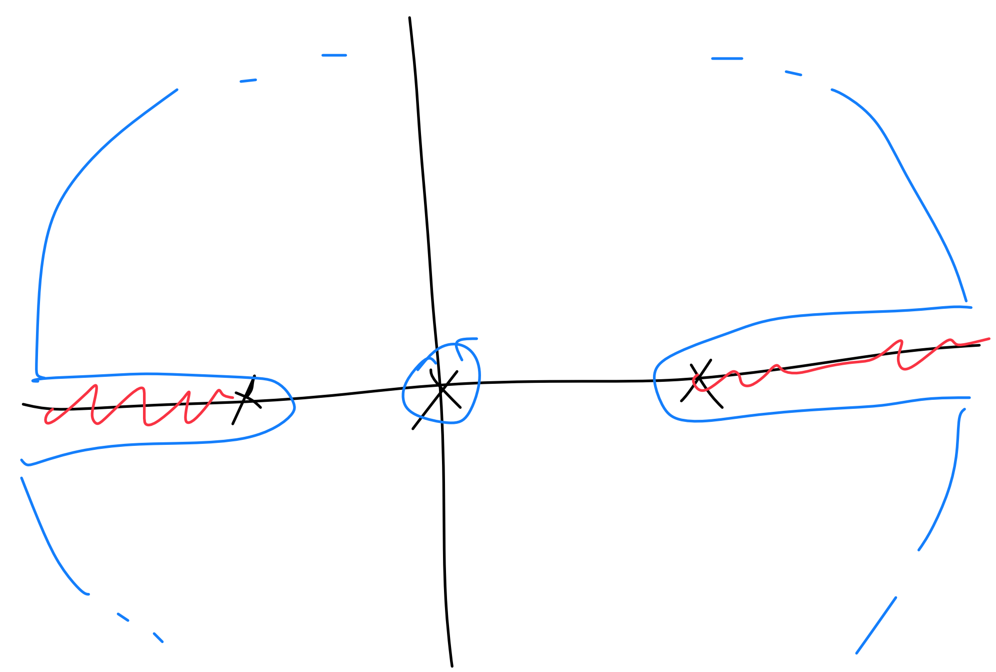
scalar EFT:

$$\mathcal{M}_{\text{low}}(s,t) = -g^2\left[\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right]-\lambda$$
$$+g_2(s^2+t^2+u^2)+g_3(stu)+g_4(s^2+t^2+u^2)^2+g_5(s^2+t^2+u^2)(stu) \quad (2.3)$$

Crossing symmetry implies relation between

coefficient of s^4 and s^2t^2





Null constraints, positivity

Many groups working on different aspects of the problem

only a very partial list

see Marc Riembau's talk

[arXiv:2011.02957] JHEP **05** (2021) 280
Extremal Effective Field Theories
S. Caron-Huot, V. van Duong

[arXiv:2012.15849] JHEP **05** (2021) 259
The EFT-Hedron
N. Arkani-Hamed, T. Huang, Y. Huang

[arXiv:2011.00037] Phys.Rev.D **104** (2021) 036006
Positive Moments for Scattering Amplitudes
B. Bellazzini, J. Elias Miro, R. Rattazzi, M. Riembau, F. Riva

[arXiv:2102.08951] JHEP **07** (2021) 110
Sharp Boundaries for the Swampland
S. Caron-Huot, D. Mazac, L. Rastelli, D. Simmons-Duffin

[arXiv:2011.02400] JHEP **05** (2021) 255
New positivity bounds from full crossing symmetry
A. J. Tolley, Z. Wang, S. Zhou

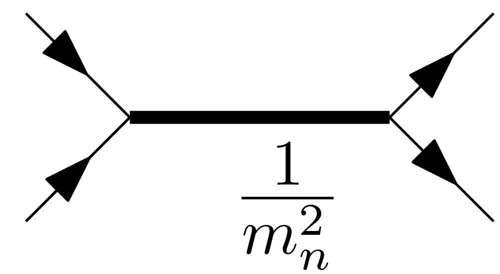
[arXiv:2103.12728] J.Phys.A **54** (2021) 344002
Gravitational Effective Field Theory Islands, Low-Spin Dominance, and the Four-Graviton Amplitude
Z. Bern, D. Kosmopoulos, A. Zhiboedov

Large N QCD

- At large N, mesons become exactly stable ['t Hooft, Witten, '70s]
- Loops are suppressed, amplitudes become meromorphic

$$A(s,t) = \sum_{n=0}^{N} \frac{\sum_{j=0}^{n} a_{n,j}\, t^j}{s - s_n}$$

$$A(s,t) = \sum_{n=0}^{\infty}$$



The simplest, non-trivial, non-perturbative, S-matrices

(yet: only example - the Veneziano amplitude)

[arXiv:2203.11950] JHEP **08** (2022) 151
Bootstrapping Pions at Large N
J. Albert, L. Rastelli

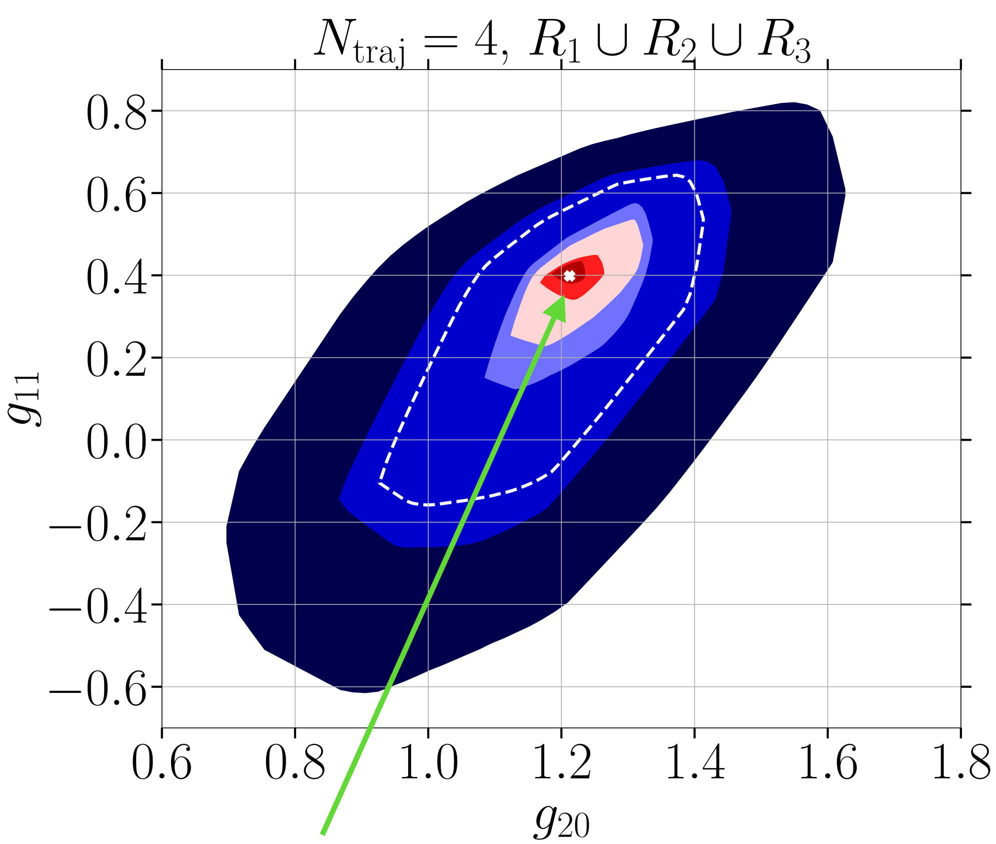
[arXiv:2312.15013]
Bootstrapping mesons at large N: Regge trajectory from spin-two maximization
J. Albert, J. Henriksson, L. Rastelli, A. Vichi

[arXiv:2401.08736]
The Regge bootstrap, from linear to non-linear trajectories
C. Eckner, F. Figueroa, P. Tourkine

Dual models

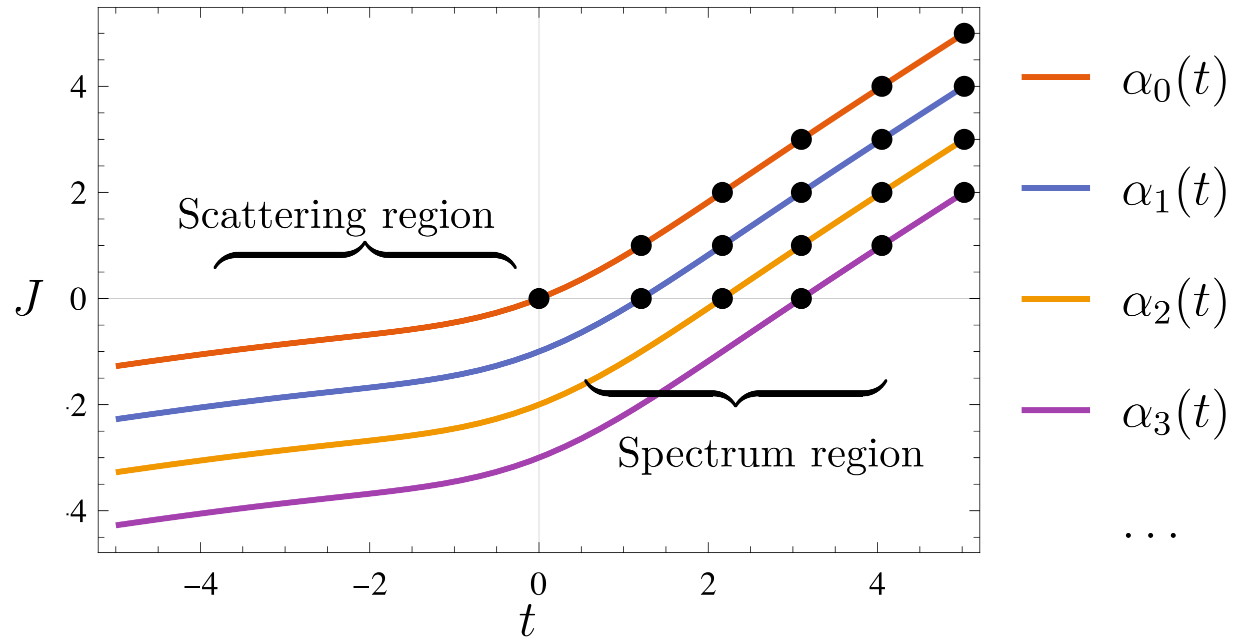
[arXiv:2401.08736]
The Regge bootstrap, from linear to non-linear trajectories
C. Eckner, F. Figueroa, P. Tourkine

- Found a way to do primal bootstrap for meromorphic amplitudes



String theory

Can deform linear, Veneziano-like trajectory to more pheno-looking:



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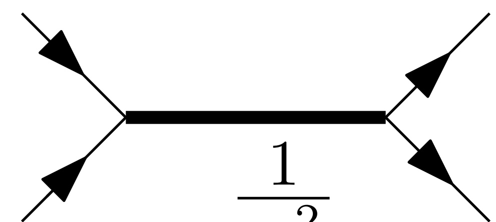

$$\frac{1}{m_n^2}$$

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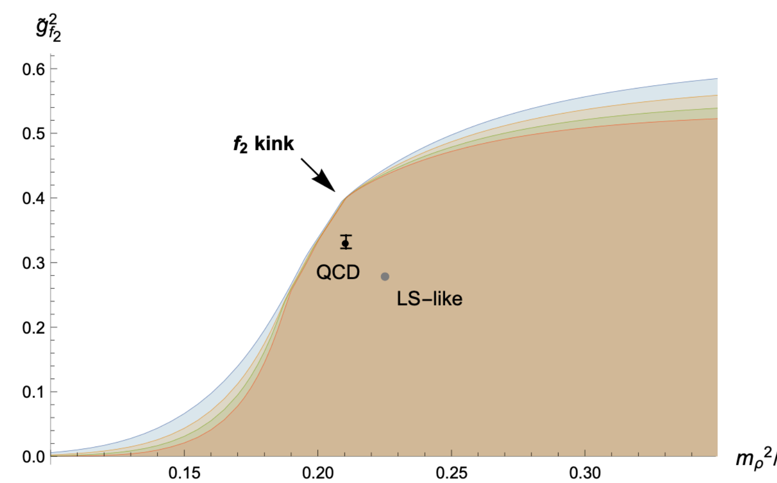
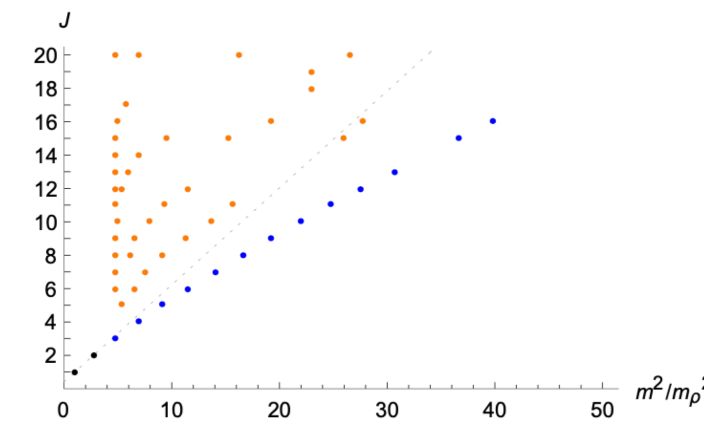
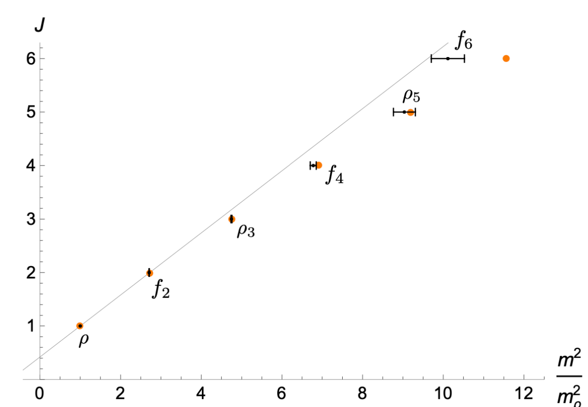
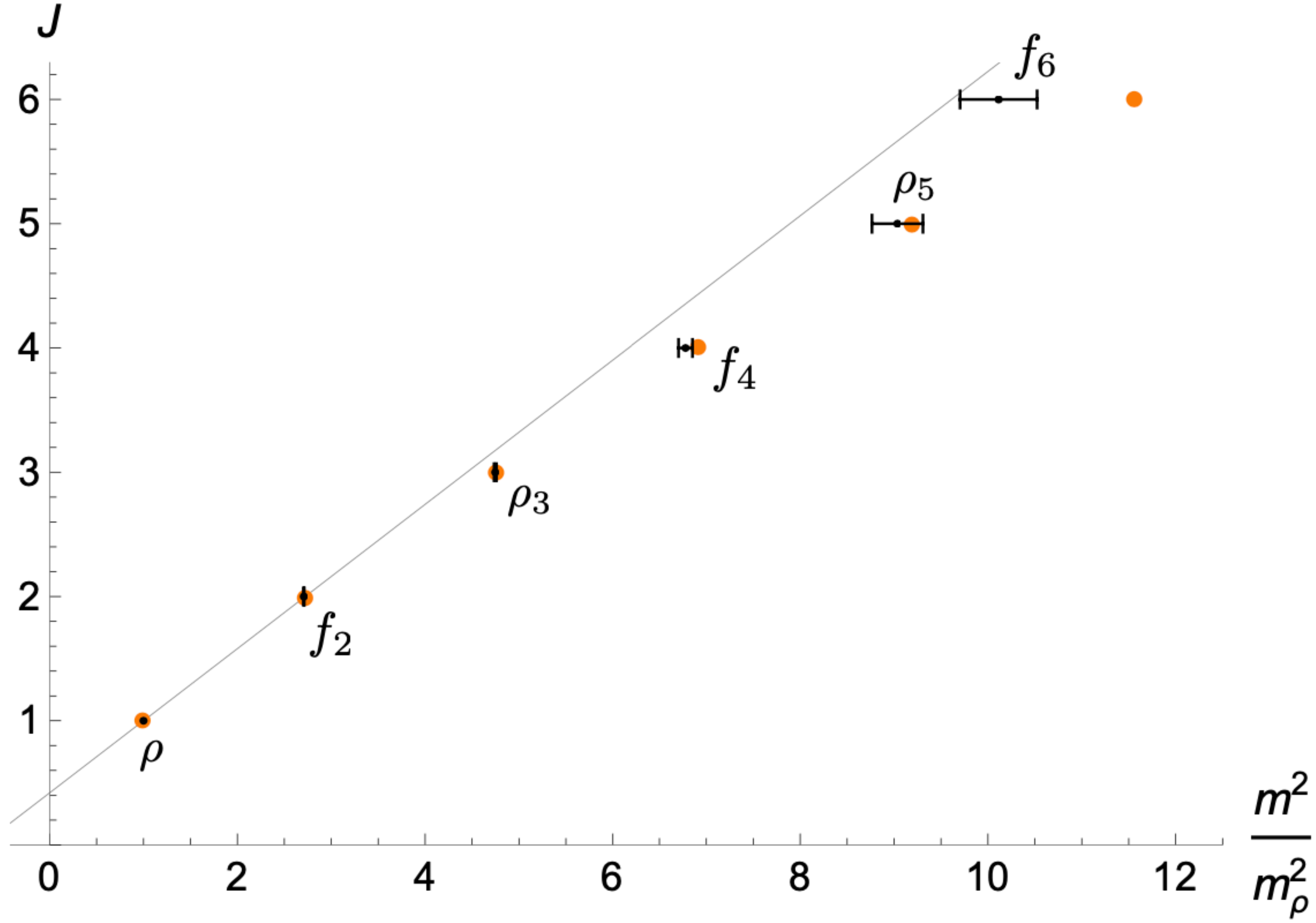
extra assumption of massive spin 2

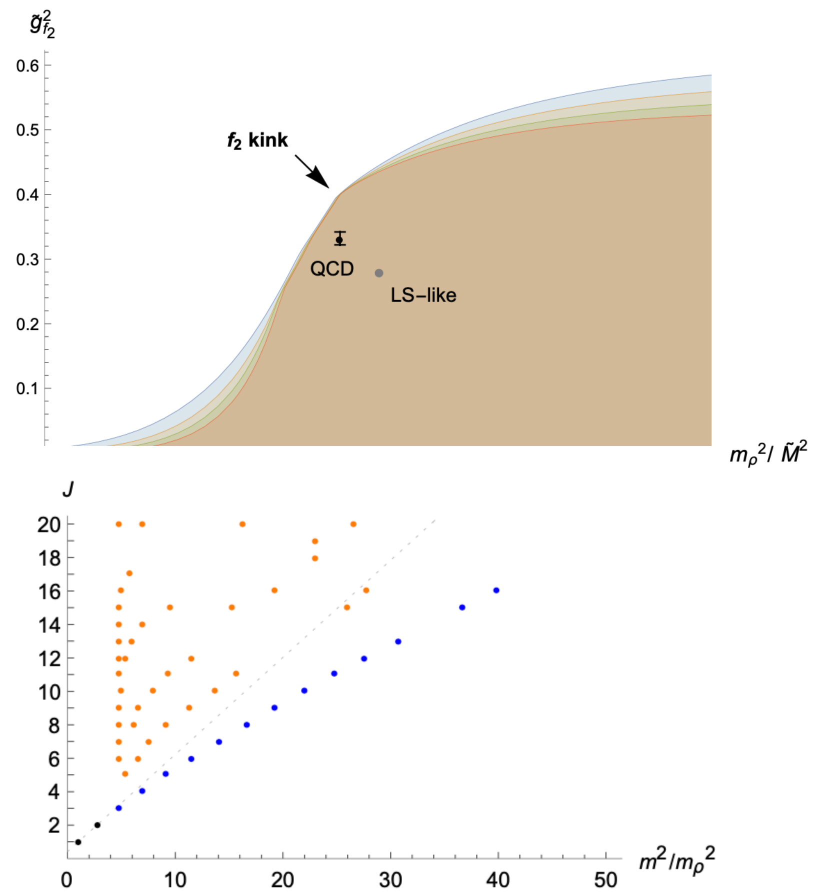




(a)







(a)

[arXiv:2312.15013]

Bootstrapping mesons at large N: Regge trajectory from spin-two maximization

J. Albert, J. Henriksson, L. Rastelli, A. Vichi

Primal approaches

not that

Toy model primal amplitude

Motivation: QFT in (1+1) dimensions

t=0: $S(s,t) \to S(s)$

- Take: $S(z) = -\dfrac{g^2}{z - z_0} + \sum_{n=0}^{\infty} a_n z^n, \quad |z| \leq 1$
- Q: Given unitarity $|S(z)|^2 \leq 1$ for $|z| = 1$, what is the max/min value of g^2 given any value of a_n ?
- Theorem : $g^2 = \pm (1 - z_0^2)$, with $a_0 = z_0$ and $a_n = 0, \ \forall \, n \geq 1$

The scalar space of amplitudes

d=4

Parametrise the amplitude as: [PPTvRV '17]

$$M^{\text{ans}}(s,t,u) = \sum_{a,b,c=0}^{N} \alpha_{(abc)} \rho_s^a \rho_t^b \rho_u^c, \qquad s \mapsto \rho_s = \frac{\sqrt{4-s_0} - \sqrt{4-s}}{\sqrt{4-s_0} + \sqrt{4-s}}$$

Compute partial waves, as functions of the alpha's

Define observables:

$$c_0 = M(s_0, t_0, u_0)$$

$$c_2 = \partial_s^2 M(s, t_0, u_0)\Big|_{s=s_0}$$

...

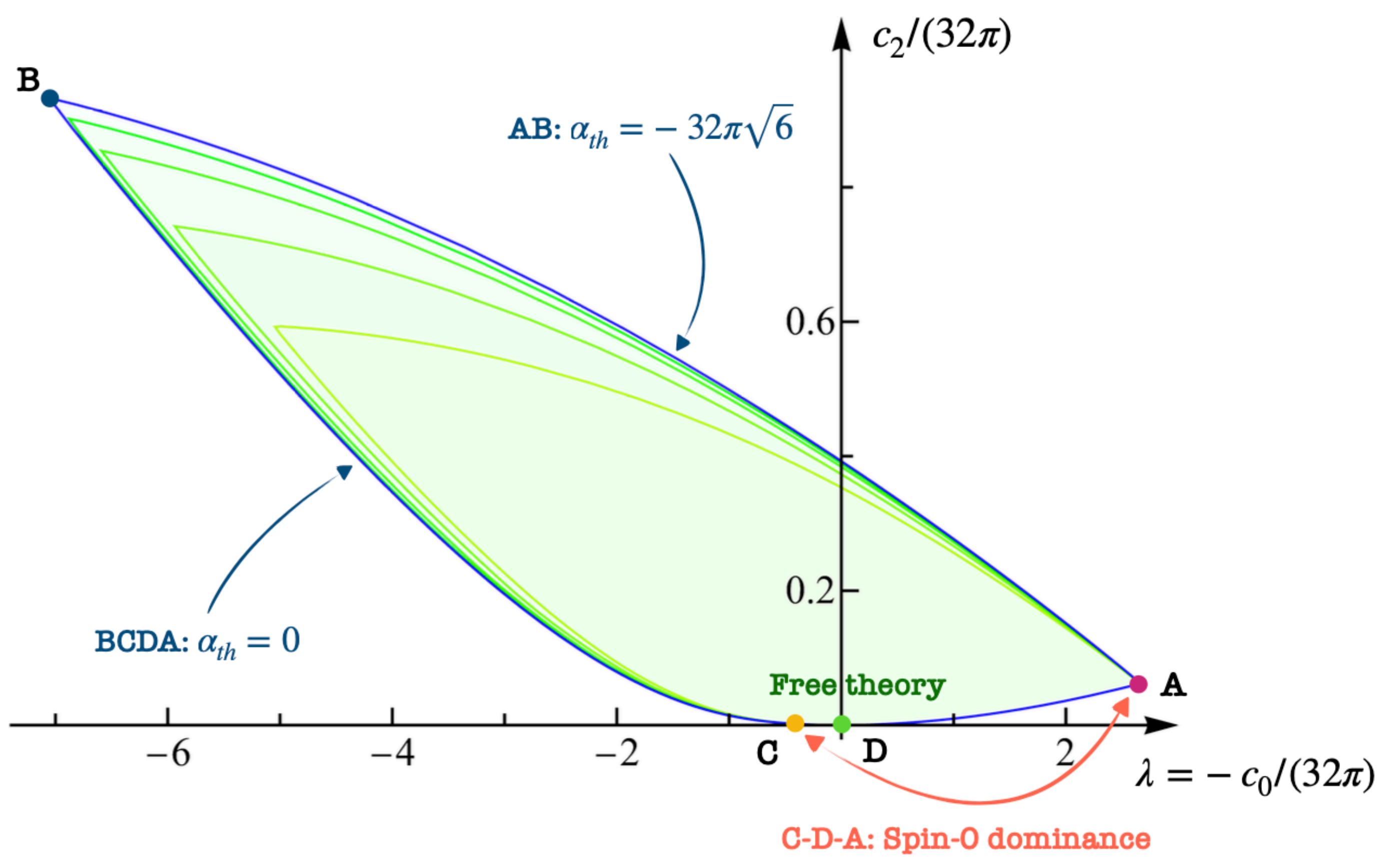
```
NMaximize[observable,|SJ|²≤1,alpha's,Method->"SDPB"]
```

The scalar space of amplitudes

amplitudes with $\mathbb{Z}_2$symmetry (no ϕ^3 term)

[arXiv:2210.01502] JHEP **05** (2023) 001
Bridging Positivity and S-matrix Bootstrap Bounds
J. Elias Miro, A. Guerrieri, M. A. Gumus

[arXiv:2207.12448] JHEP **12** (2022) 092
Nonperturbative Bounds on Scattering of Massive Scalar Particles in d ≥ 2
H. Chen, A. L. Fitzpatrick, D. Karateev

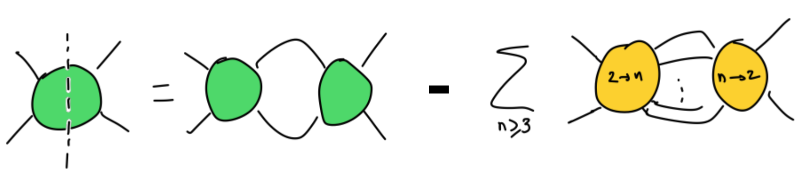


The scalar space of amplitudes

- Drawback: numerical precision not adapted, amplitudes seem *too unitarity*
- Aks theorem ('64): in d>2, S-matrices cannot be purely elastic.
- In [Tourkine, Zhiboedov '21; '23], rigorous construction method

[Mandelstam; Atkinson '60s]

Scattering from production



input

$$X = X^2 + f$$

Solve by iteration

$$X_n = X_{n-1}^2 + f$$

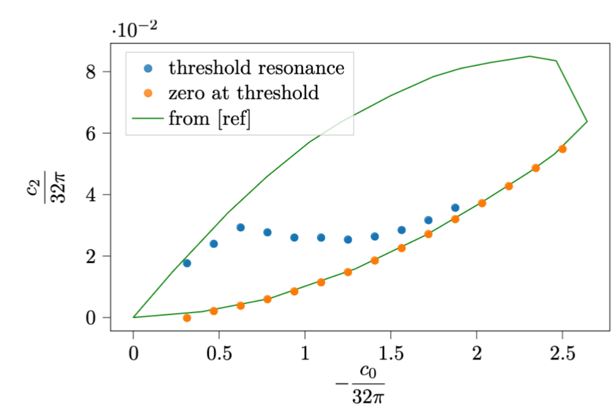
Recent results w/ L. Leflot: use machine learning for toy-model (only J=0 scalar sector)

[Submitted on 22 Dec 2014 (v1), last revised 30 Jan 2017 (this version, v9)]

Adam: A Method for Stochastic Optimization

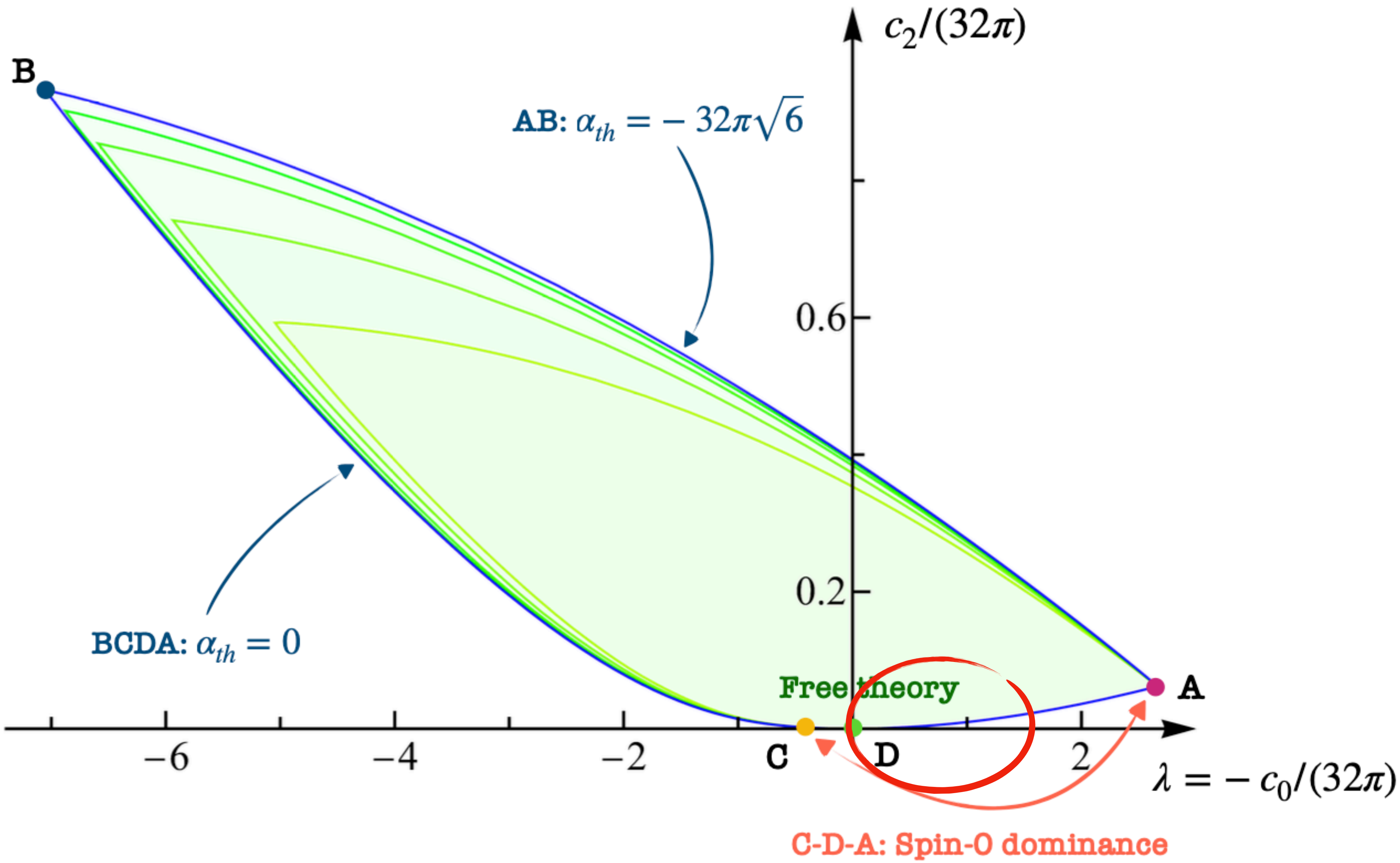
Diederik P. Kingma, Jimmy Ba

Extend applicability of rigorous construction in a much larger domain



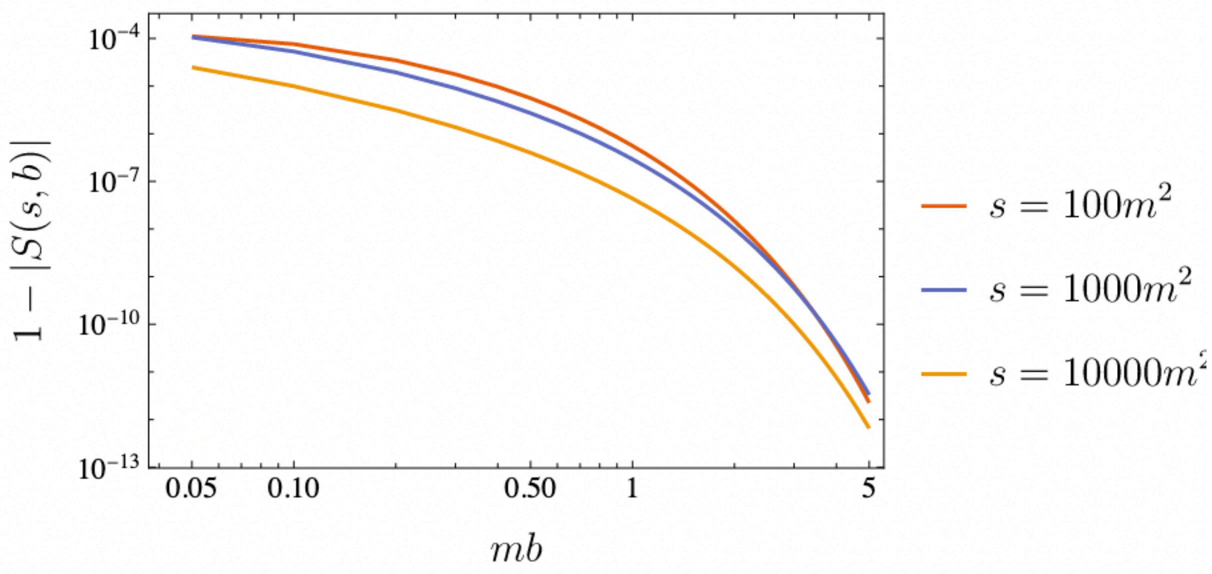
Drawback

convergence in smaller parameter space

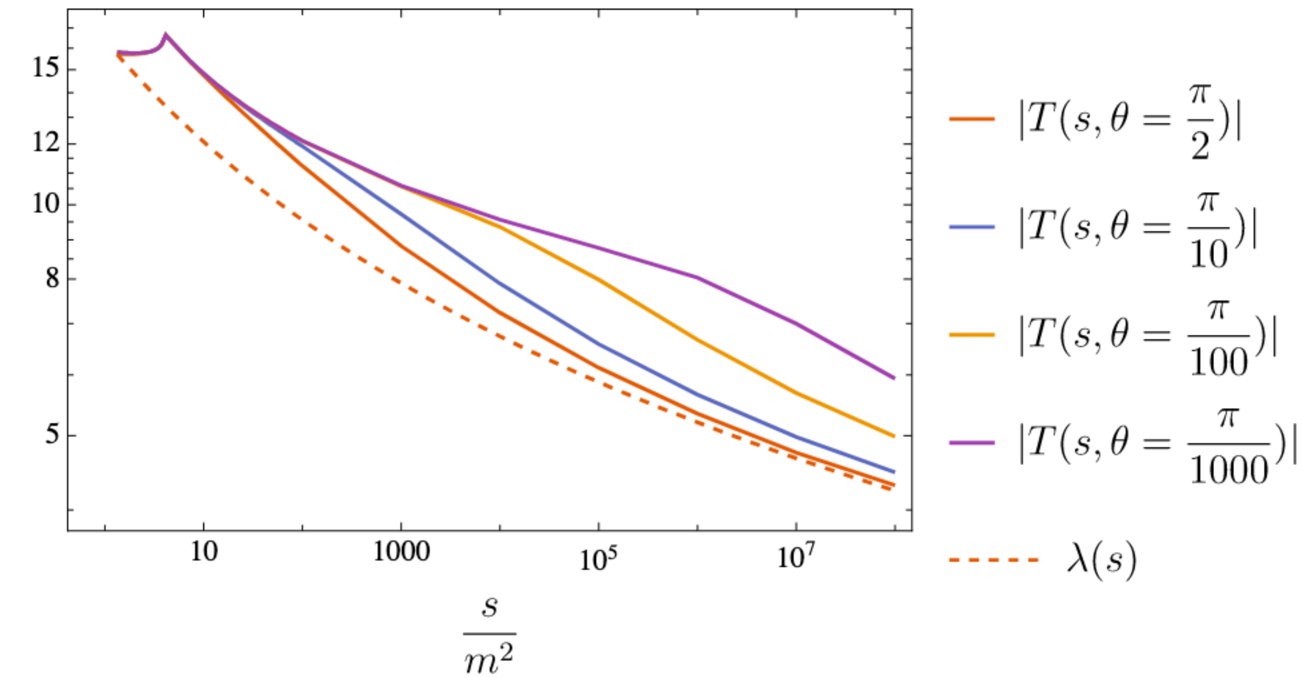


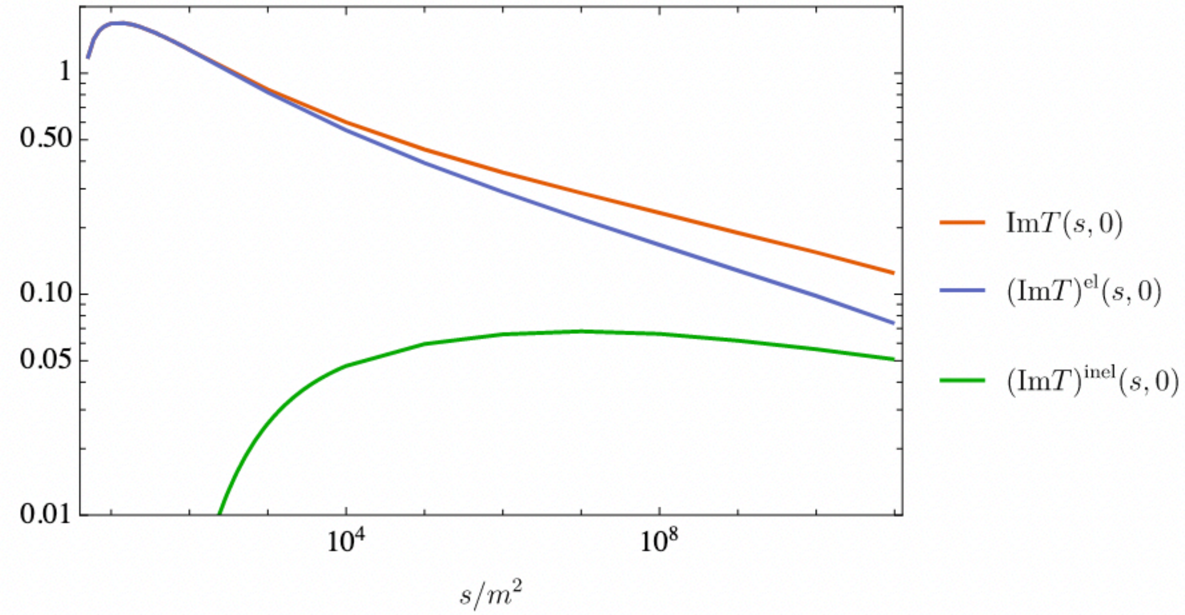
Results in d=4

Amplitude in impact parameter space



Amplitude at fixed angle

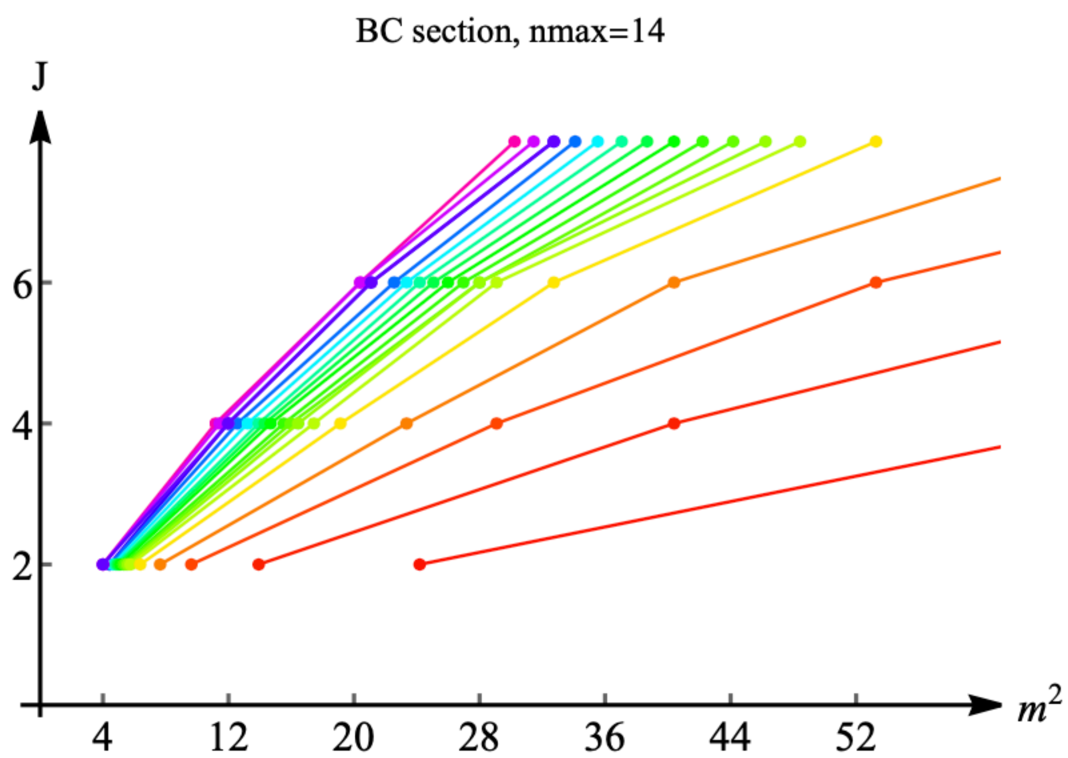


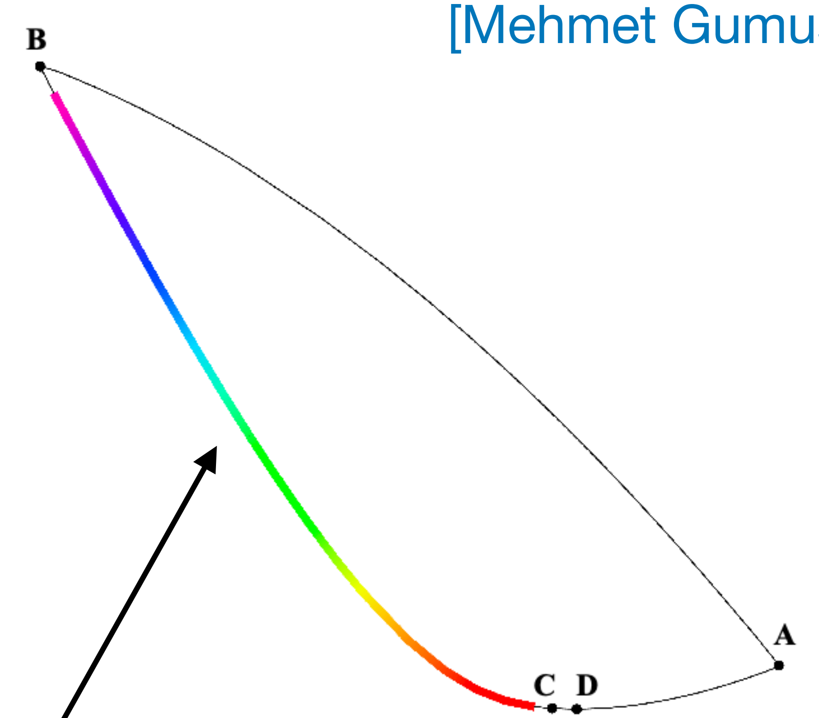


Cross-section

The scalar space of amplitudes

[Mehmet Gumus, PhD thesis]





A tower of

Regge resonances!

Completely non-perturbative

Pion amplitude

- Recent progress by [He, Krusczenski] (see `Yifei's talk` later)

[arXiv:2309.12402]
Bootstrapping gauge theories
Y. He, M. Kruczenski

Pion amplitude

- Recent progress by [He, Krusczenski] (see Yifei's talk later)
- Emergence of rho meson, unitarising the scattering

[arXiv:2309.12402]
Bootstrapping gauge theories
Y. He, M. Kruczenski

Quantum gravity

- One of the big questions: how is gravity unitarised?
- Looked at 9-10-11 dim’ supergravity

[arXiv:2102.02847] Phys.Rev.Lett. **127** (2021) 081601
Where is String Theory in the space of scattering amplitudes?
A. Guerrieri, J. Penedones, P. Vieira

[arXiv:2212.00151] JHEP **06** (2023) 064
Where is M-theory in the space of scattering amplitudes?
A. Guerrieri, H. Murali, J. Penedones, P. Vieira

$$\frac{T(s,t,u)}{8\pi G_N} = s^4\left(\frac{1}{stu} + \boxed{\alpha}\,\ell_P^6 + \dots\right)$$

First correction to (super) gravity

Dimension	Bootstrap	String/M–Theory
9	0.223 ± 0.002	0.241752
10	0.124 ± 0.003	0.138949
11	0.101 ± 0.005	0.102808

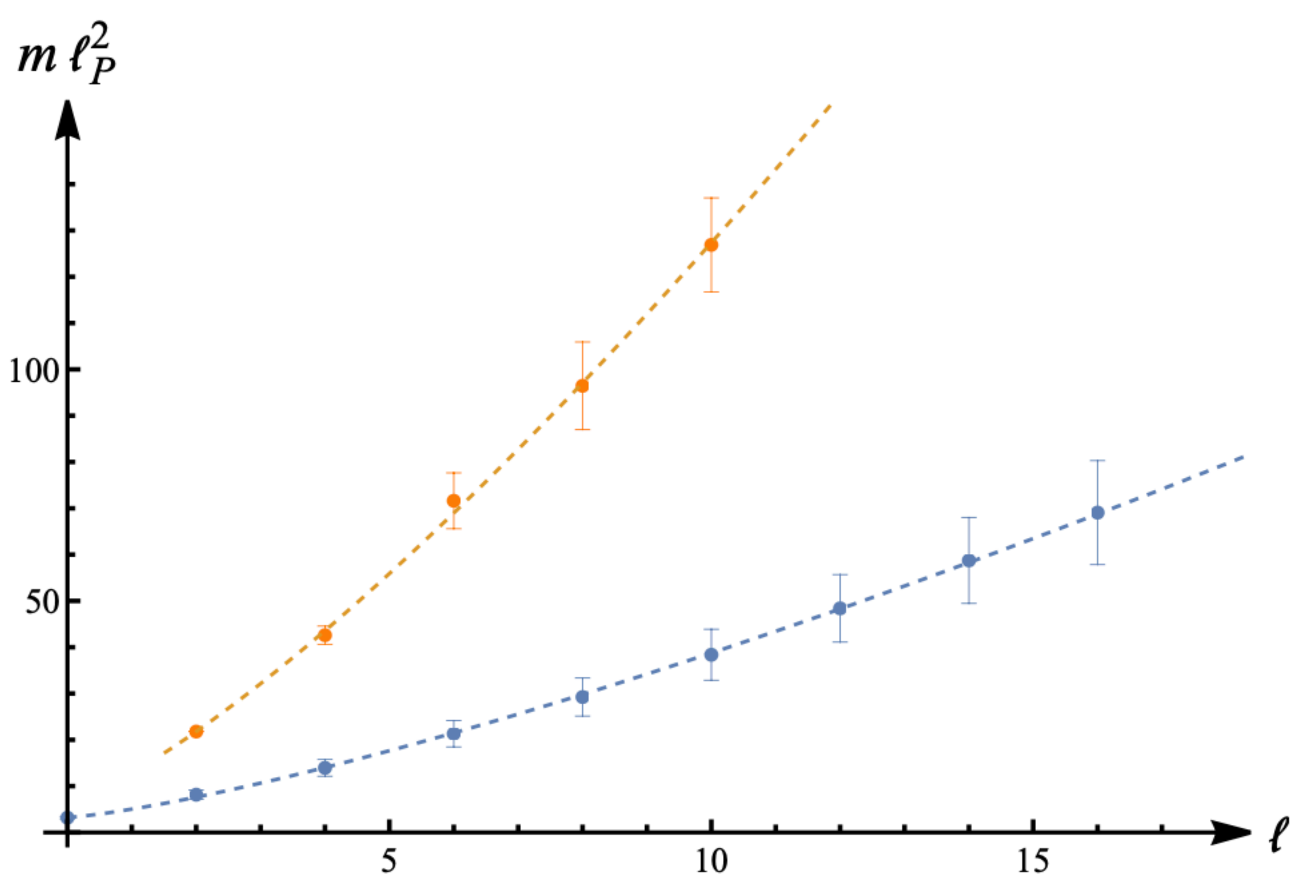
In string theory, this number is bounded (from below), can be calculated non-perturbatively

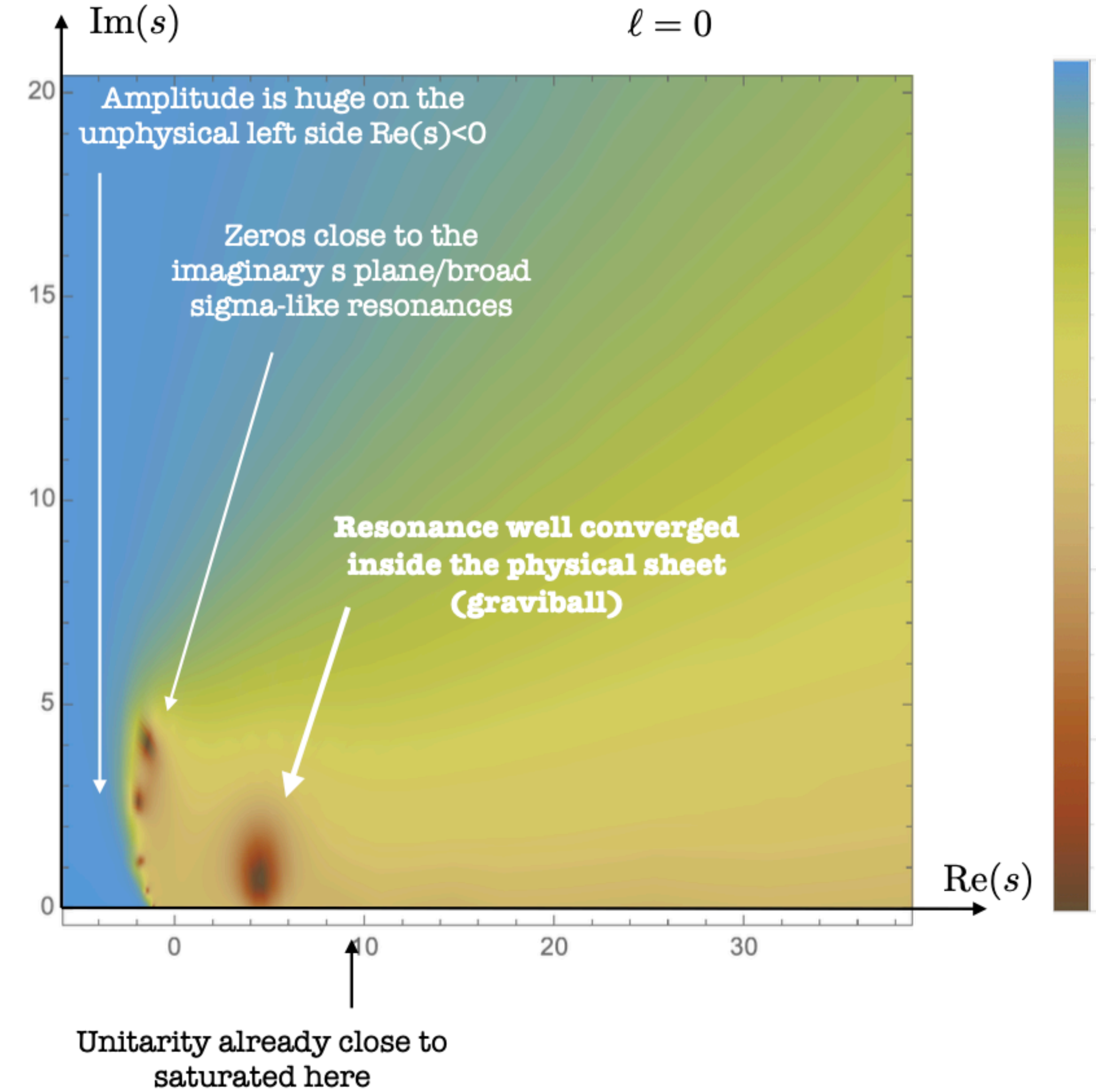
Where does the small discrepancy come from?

Quantum gravity

[arXiv:2212.00151] JHEP **06** (2023) 064
Where is M-theory in the space of scattering amplitudes?
A. Guerrieri, H. Murali, J. Penedones, P. Vieira

- A whole zoo of Regge resonances !

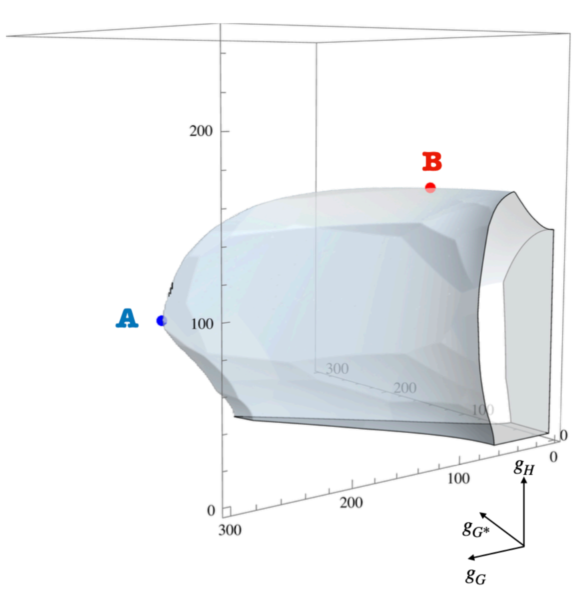


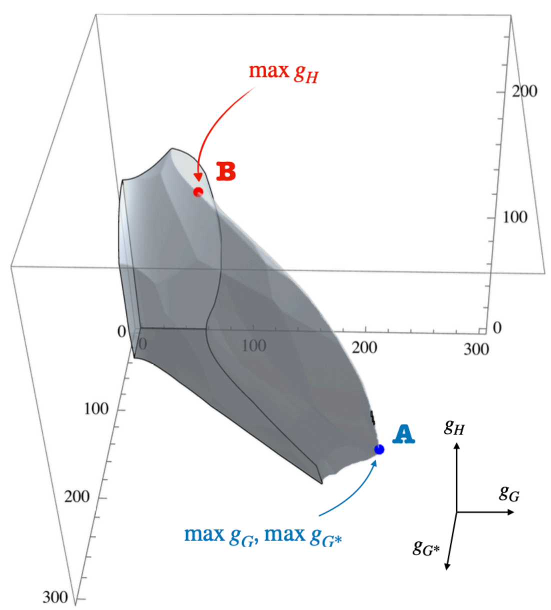


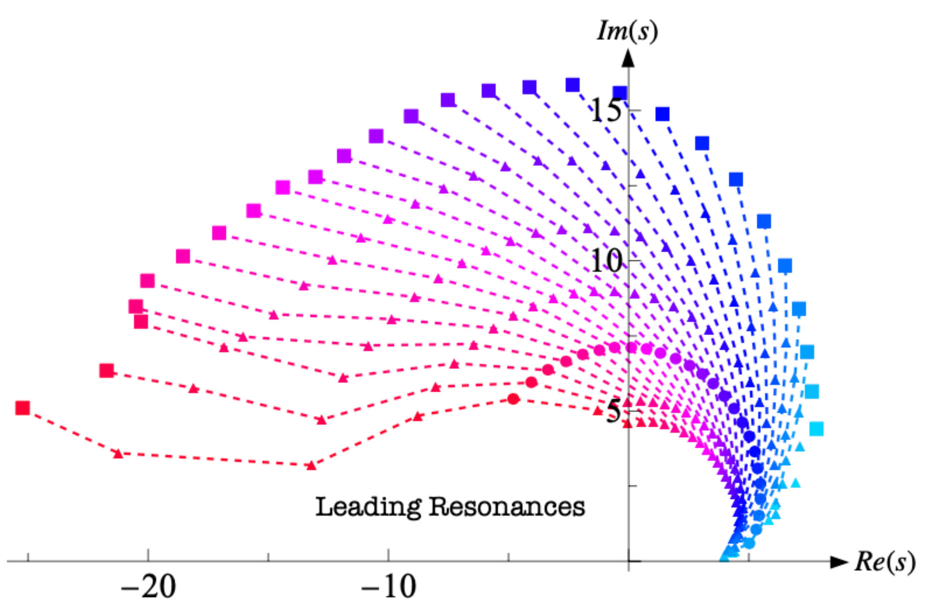
The gluehedron*

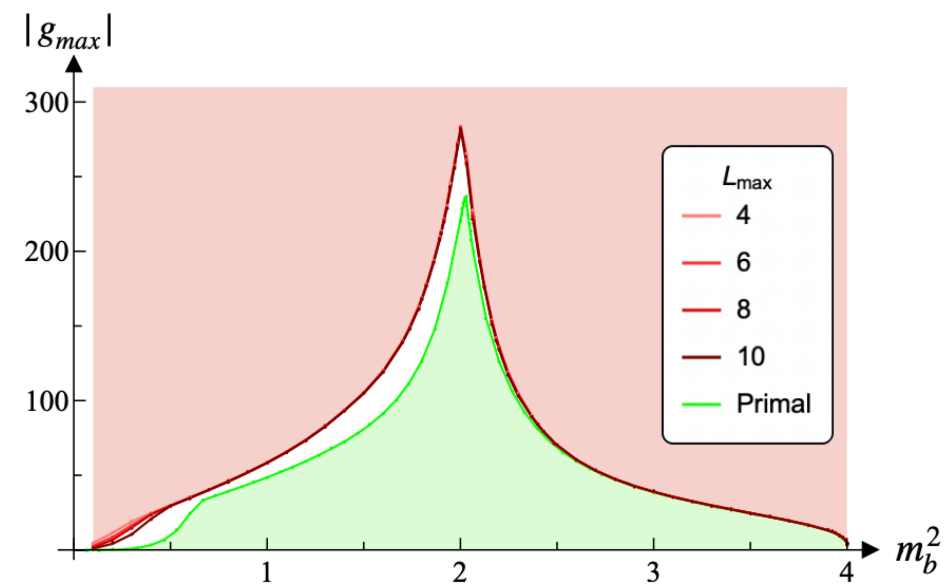
[arXiv:2312.00127]
Constraining Glueball Couplings
A. L. Guerrieri, A. Hebbar, B. C. van Rees

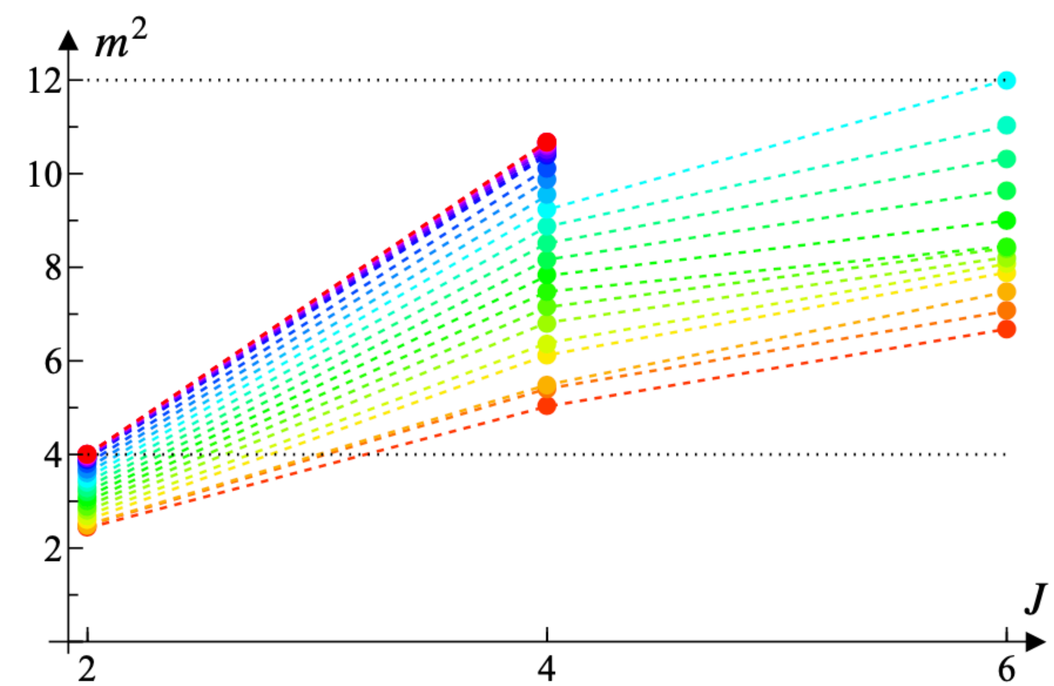
Beautiful set of Regge trajectories









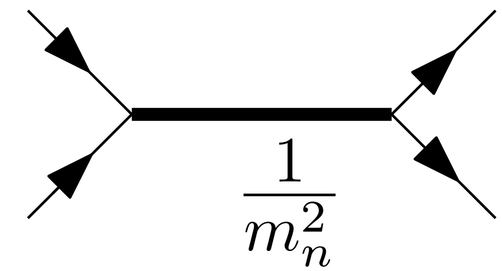


*Slide added a posteriori, for reference purposes, as another beautiful recent example of remarkable Regge trajectories in primal (& dual) approach

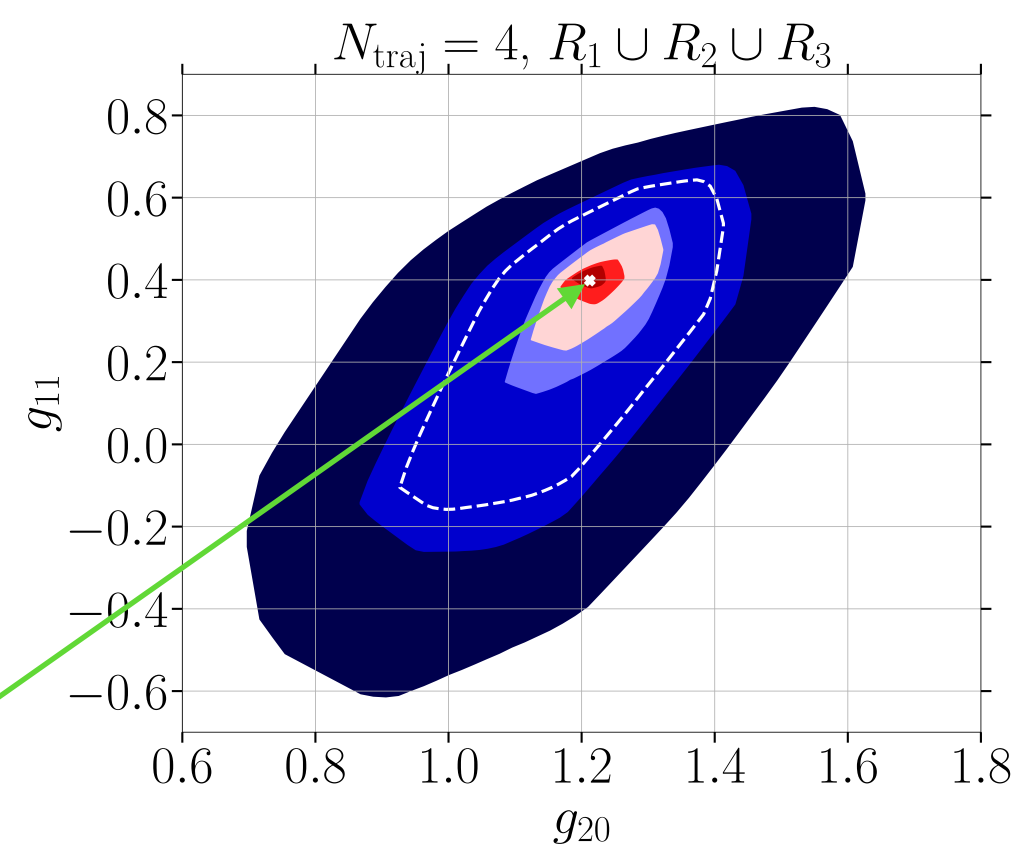
Back to the beginning, dual models

$$A(s,t) = \sum_{n=0}^{N} \frac{\sum_{j=0}^{n} a_{n,j}\, t^j}{s - s_n}$$

$$A(s,t) = \sum_{n=0}^{\infty}$$



- Meromorphic amplitudes: Subclass of non-perturbative S-matrices
- string theory at tree-level, large N QCD
- There was no practical way to construct such amplitudes
- Found a way to do primal bootstrap for meromorphic amplitudes



String theory

[arXiv:2401.08736]
The Regge bootstrap, from linear to non-linear trajectories
C. Eckner, F. Figueroa, P. Tourkine

Open directions

- Remarkable set of results, completely non-obvious from the starting point: Regge trajectories (for scalars, phi^4-like?), graviballs, stringy optimal bounds, etc
- A lot to explore, even at this level of 2-to-2 !
- What is the exact relation between these Regge resonances and unitarity?
- Maximal analyticity? Tension with growing Regge trajectory?
- Construct full scattering amplitudes of pheno use? Pions, kaons, …
- Connect perturbative to non-pert ?
- Add higher-point unitarity/crossing constraints