Les Rencontres de Physique des Particules (RPP)-2024

Light by Light Scattering at NLO in QCD+QED

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26 Jan 2024

In collaboration with *Ekta Chaubey, Mathijs Fraaije, Valentin Hirschi,* and *Hua-Sheng Shao*

arXiv: 2312.16966 [hep-ph] & arXiv:2312.16956 [hep-ph]

Motivation

- Light by light scattering (LbL) is a fundamental QED process and it is one of the earliest QED predictions : in 1930s by Euler, Heisenberg et al!

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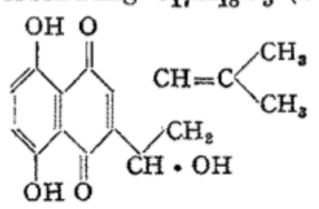
Kurze Originalmitteilungen.

Für die kurzen Originalmitteilungen ist ausschließlich der Verfasser verantwortlich.

Über spiegelbildliche Naturfarbstoffe.

Der in reinem Zustande isolierte rote Farbstoff der Alkannawurzeln (Alkanna tinctoria, Südeuropa), das *Alkannin* besitzt die Formel $C_{16}H_{16}O_5$, Schmp. 149°. Das Alkannin ist, was allen Bearbeitern bisher entgangen ist, optisch aktiv, und zwar linksdrehend $[\alpha]_{Cd}^{20} = -170°$ (Benzol). Ein Farbstoff von gleicher Zusammensetzung ($C_{16}H_{16}O_5$), fast gleichem Schmp. (147°) und gleichem Absorptionsspektrum (Schwerpunkte der Banden bei 570, 560, 543, 522, 488, 454 in Benzin) war bereits aus Shikonwurzeln (Lithospermum erythrorhizon; Japan) isoliert und unter dem Namen *Shikonin* beschrieben worden[1]. Ein Vergleich[2] der beiden Farbstoffe ergab, daß diese nicht identisch sind. Das polarimetrisch ebenfalls noch nicht untersuchte Shikonin erwies sich im Gegensatz zum Alkannin als rechtsdrehend und stellt den optischen Antipoden des linksdrehenden Alkannins dar, eine bei Naturfarbstoffen überraschende Erscheinung.

Durch Behandlung mit methyl-alkoholischer Salzsäure ließen sich Alkannin und Shikonin in ein und dieselbe optisch-inaktive Methoxyverbindung $C_{17}H_{18}O_5$ (Schmp. 105°) überführen[3].



Auf Grund zahlreicher Abbauversuche, über die an anderer Stelle berichtet werden soll, sind die für Alkannin und Shikonin bisher vermuteten Konstitutionsformeln im beistehenden Sinne abzuändern.

Heidelberg, Kaiser Wilhelm-Institut für Medizinische Forschung, Institut für Chemie, den 12. Januar 1935.

HANS BROCKMANN. HUBERT ROTH.

Über die Streuung von Licht an Licht nach der Diracschen Theorie.

HALPERN[4] und DEBYE[5] haben darauf aufmerksam gemacht, daß es nach der DIRACschen Theorie Streuung von sichtbarem Licht an Licht geben muß. Denn es gibt Prozesse, in denen zwei Lichtquanten virtuell ein Paar (Positron und Elektron) erzeugen, das dann sofort wieder zerstrahlt. Diese Prozesse verwandeln also zwei Lichtquanten (ν_1, ν_2) in zwei andere Lichtquanten (ν_1', ν_2'), und sie können auch eintreten, wenn ihre Energie nicht zur Erzeugung eines wirklichen Paares ausreicht.

Für diesen Fall, der durch die Bedingung:

$$\nu_1 \nu_2 (1 - \cos \sphericalangle \nu_1 \nu_2) < \frac{2 (m c^2)^2}{h^2},$$

also in geeignetem Koordinatensystem durch $h\nu < mc^2$ für alle Lichtquanten charakterisiert werden kann,

(m = Masse des Elektrons, e = Ladung des Elektrons, c = Lichtgeschwindigkeit, $h = 2\pi\hbar$ = PLANCKsches Wirkungsquantum)

haben wir den Wirkungsquerschnitt für den Zusammenstoß zweier Lichtquanten bestimmt[1].

Dazu wurde zunächst nach der gewöhnlichen Störungsrechnung der DIRACschen Theorie das Matrixelement 4. Ordnung für diesen Prozeß berechnet und nach Lichtquantenenergien $\frac{h\nu}{mc^2}$ entwickelt.

Das Glied 0. Ordnung in $\frac{h\nu}{mc^2}$ erwies sich als entgegengesetzt gleich dem Glied

$$H_4 = \frac{-1}{12\pi^2}\left(\frac{e^2}{\hbar c}\right)^2 \cdot \frac{1}{\hbar c} \cdot \lim_{r \to 0} \int d\xi \left(\mathfrak{A}(\xi), \frac{r}{r}\right)^4$$

($\mathfrak{A}$ = Potential des Strahlungsfeldes), das nach HEISENBERG[2] zu dem gewöhnlichen Matrixelement 4. Ordnung addiert werden muß, um das wirkliche zu ergeben.

Die Glieder 1., 2. und 3. Ordnung verschwanden, und das Glied 4. Ordnung in $\frac{h\nu}{mc^2}$ ließ sich formal darstellen als Matrixelement einer Funktion vom Strahlungsfeld, so daß für die hier betrachteten Prozesse die gewöhnliche Hamiltonfunktion, die die Energien von Licht *und* Materie enthält[3], ersetzt werden kann durch die folgende, die nur vom Strahlungsfeld abhängt:

$$(1) \quad \begin{cases} \int U\, dV = \int \frac{\mathfrak{B}^2 + \mathfrak{D}^2}{8\pi} dV - \frac{1}{360\pi^2} \frac{\hbar c}{e^2} \frac{1}{E_0^2} \\ \int [(\mathfrak{B}^2 - \mathfrak{D}^2)^2 + 7 (\mathfrak{B}\mathfrak{D})^2]\, dV. \end{cases}$$

Darin bedeutet $\mathfrak{D}$ die elektrische Verschiebung, $\mathfrak{B}$ die magnetische Induktion, $E_0 = \frac{e}{(e^2/mc^2)^2}$ den Betrag der „Feld-

[1] R. MAJIMA u. C. KURODA, Acta phytochim. (Tokyo) 1, 43 (1922).
[2] Der Deutschen Botschaft in Tokyo spreche ich für die freundliche Beschaffung von Shikonwurzeln meinen allerbesten Dank aus.
[3] Dieses Kunstprodukt wurde kürzlich von H. RAUDNITZ u. E. STEIN, Ber. dtsch. chem. Ges. 67, 1955 (1934) für den Naturfarbstoff gehalten.
[4] O. HALPERN, Physic. Rev. 44, 885 (1934) — vgl. auch G. BREIT u. J. WHEELER, Physic. Rev. 46, 1087 (1934).
[5] In einer Diskussion mit Herrn Prof. HEISENBERG.

[1] Die ausführlichen Rechnungen erscheinen später.
[2] W. HEISENBERG, Z. Physik 90, 209 (1934) Formel 61; 92, 692 (1934).
[3] Vgl. HEISENBERG u. PAULI, Z. Physik 56, 1 (1930); 59, 168 (1930).

Consequences of Dirac's Theory of the Positron

W. Heisenberg and H. Euler in Leipzig[1]

22. December 1935

Abstract

According to Dirac's theory of the positron, an electromagnetic field tends to create pairs of particles which leads to a change of Maxwell's equations in the vacuum. These changes are calculated in the special case that no real electrons or positrons are present and the field varies little over a Compton wavelength. The resulting effective Lagrangian of the field reads:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2 (\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{conj.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{conj.}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}$$

$\mathfrak{E}, \mathfrak{B}$ field strengths

$|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{137}\frac{e}{(e^2/mc^2)^2} =$ critical field strengths

The expansion terms in small fields (compared to $\mathfrak{E}$) describe light-light scattering. The simplest term is already known from perturbation theory. For large fields, the equations derived here differ strongly from Maxwell's equations. Our equations will be compared to those proposed by Born.

German title: "Folgerungen aus der Diracschen Theorie des Positrons"
Zeitschr. Phys. *98*, 714 (1936).

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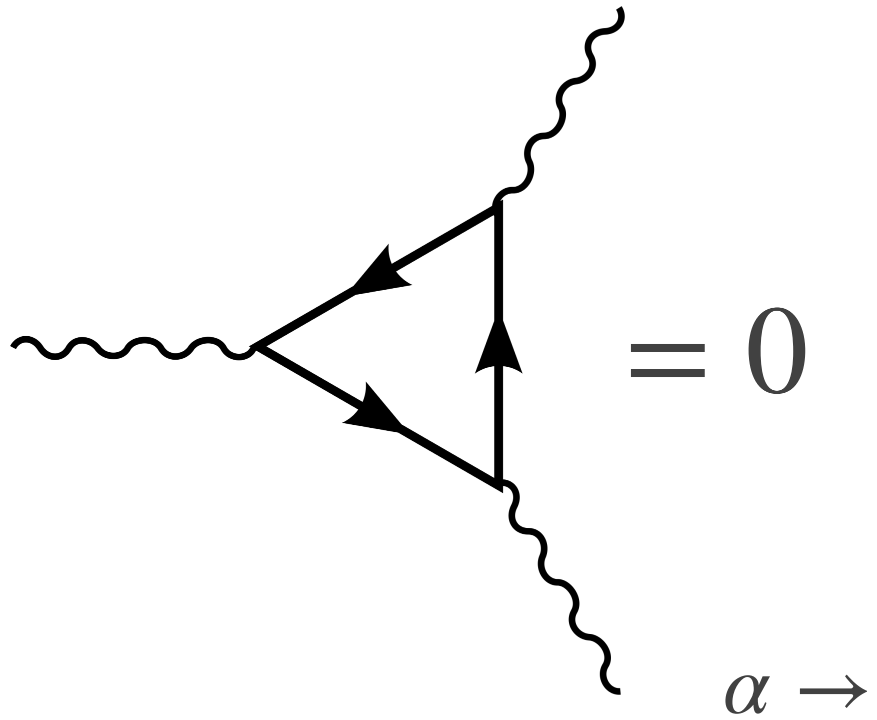
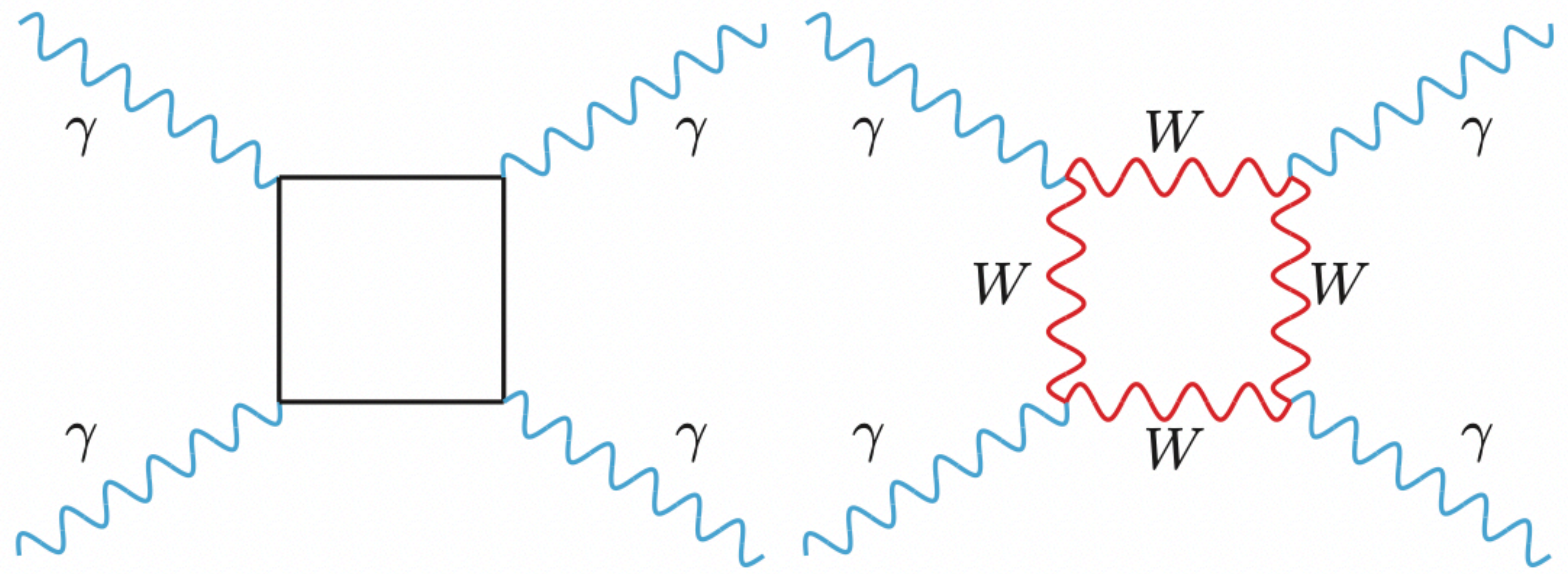
- In Standard Model,
  most viable self interaction is four photon interaction mediated via virtual charged fermion or $W^{\pm}$ boson loops.



$\alpha \rightarrow$ fine structure constant



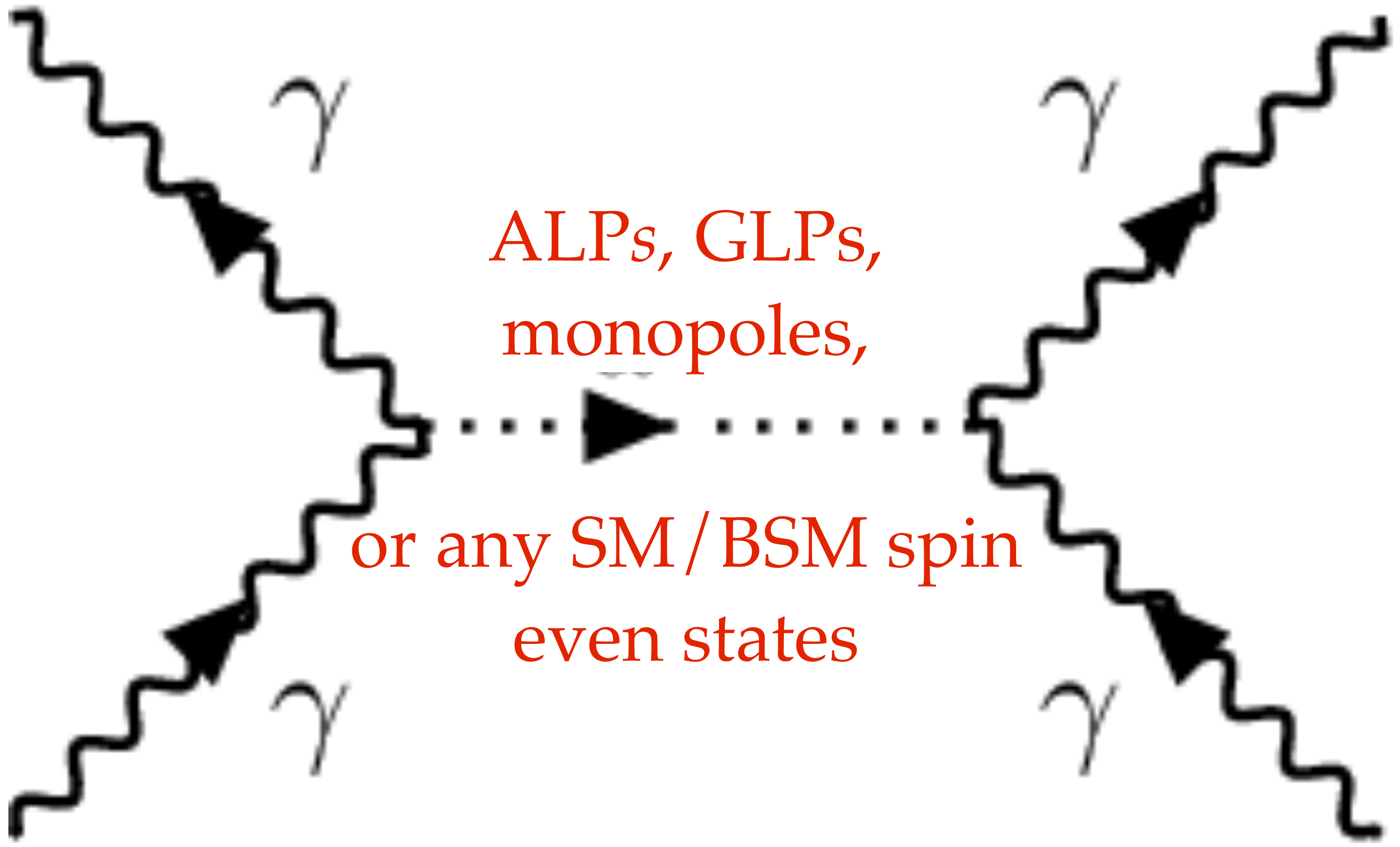
$\sim \mathcal{O}(\alpha^4)$ in QED at lowest order

Experimentally challenging to detect

No direct measurement until 2017!

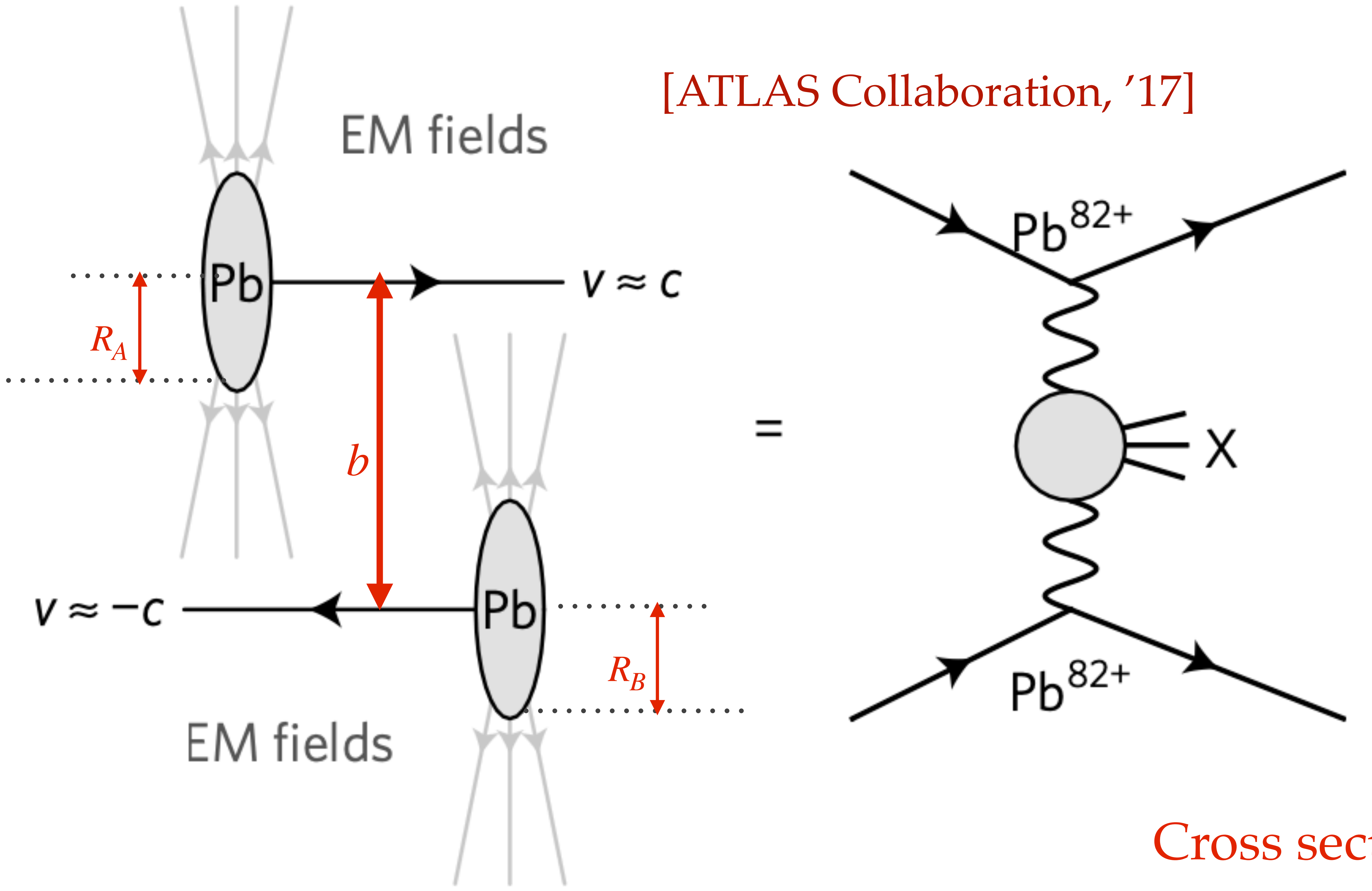
Motivation

- Interest to this process revived recently, because it is sensitive to searches for new physics..
- It has been also acknowledged that LbL can be used to probe the quartic anomalous gauge couplings, large extra dimensions, supersymmetric particles etc.
- It is also background for looking for new particles in the SM, such as ditauonium



Measurement of LbL

- First evidence for direct detection reported by ATLAS in 2017 [Nature Physics 13 (2017) 852] followed by CMS in 2018 [*Phys.Lett.B* 797 (2019) 134826]
- Observed in **Ultra-peripheral heavy ion collisions (UPCs)** such as Lead (Pb)



[ATLAS Collaboration, '17]

$b_{min} > R_A + R_B$

EM field associated with highly relativistic charged particles can be treated as a beam of coherent photons with small virtuality (Equivalent photon approximation)

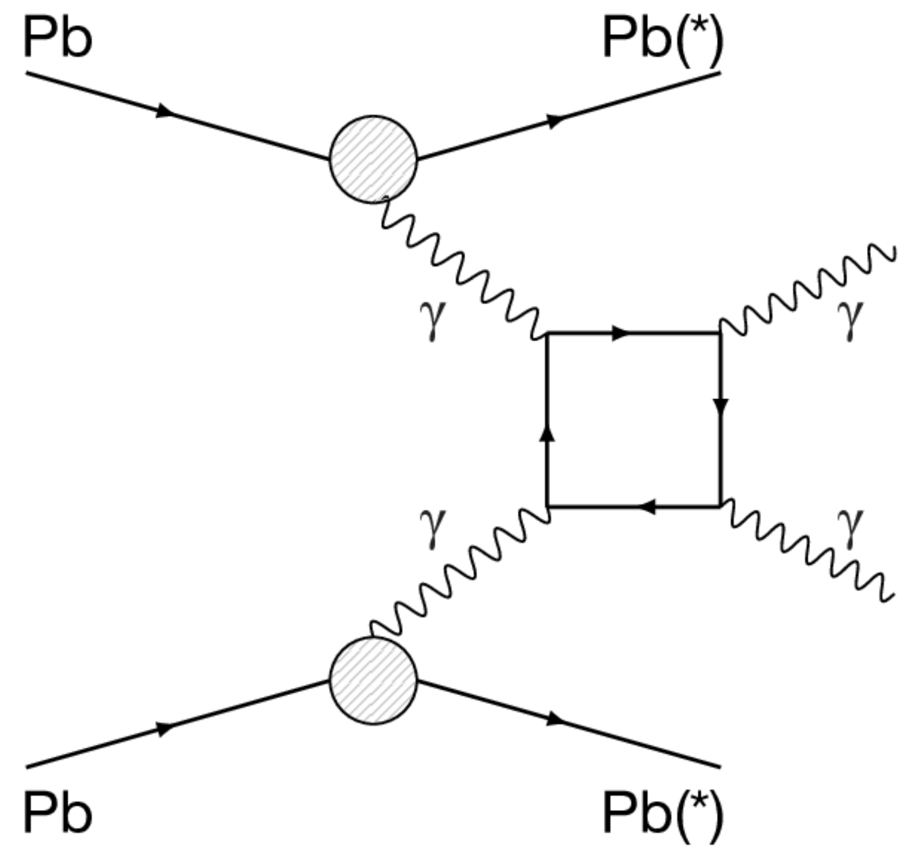
Large photon flux $\sim Z^2, Z = 82$ for Pb

Cross section (PbPb) scales like $Z^4 \sim 5 \cdot 10^7$ larger than pp or $e^\pm$

Data-theory comparison

- Integrated fiducial cross section

[H.S Shao, D. d'Enterria '22] [L. A. Harland-Lang, V. A. Khoze, and M. G. Ryskin '19]



Process, system	ATLAS data [15]	gamma-UPC σ			SUPERCHIC σ
		EDFF	ChFF	average	
$\gamma\gamma \to \gamma\gamma$, Pb-Pb at 5.02 TeV	120 ± 22 nb	63 nb	76 nb	70 ± 7 nb	78 ± 8 nb

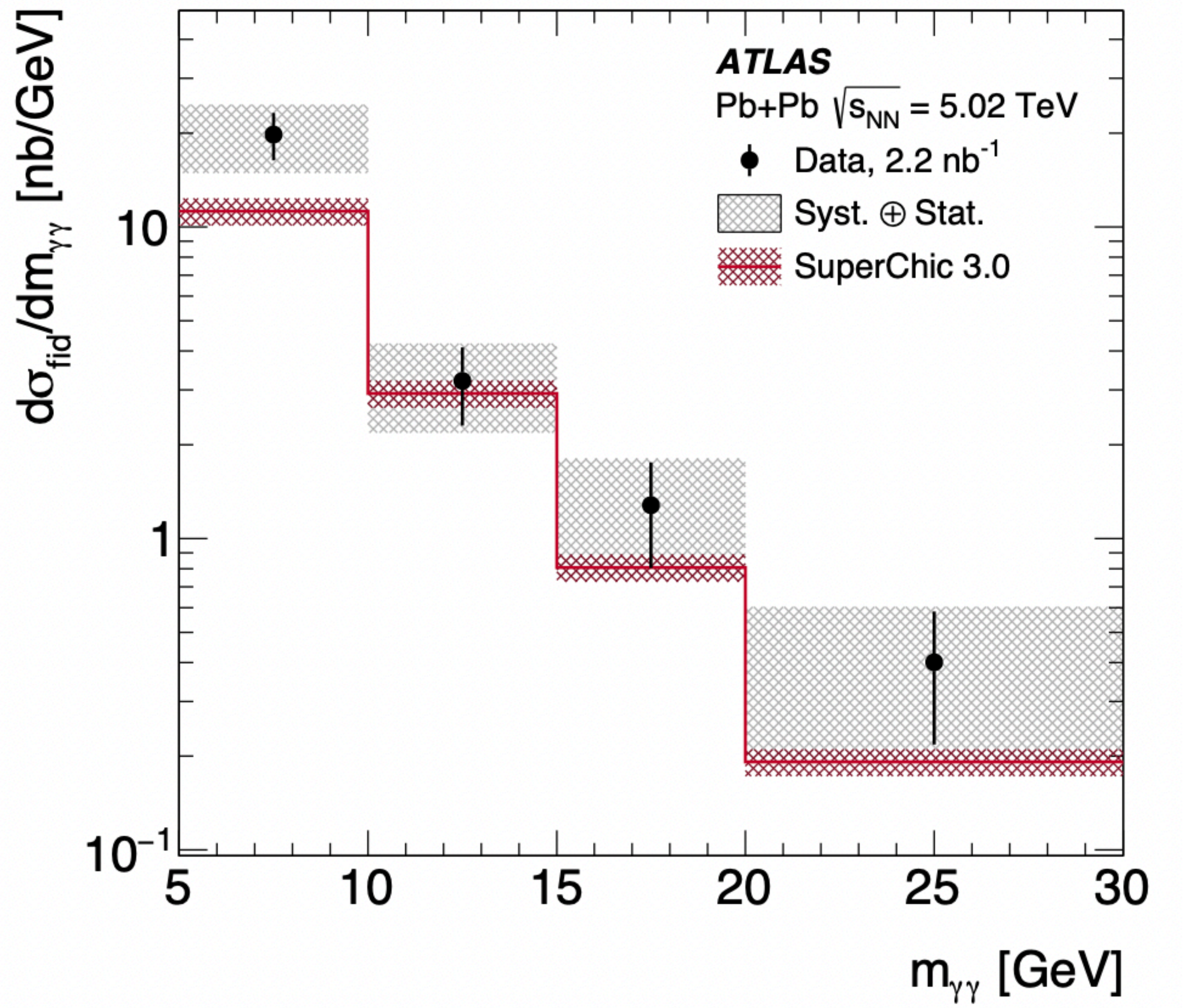
[ATLAS : JHEP 03 (2021) 243]
[CMS : Phys. Lett. B **797** (2019) 134826]

Theory predictions are based on LO cross section

- Differential cross section

Shape well reproduced except for the lowest bin

Data is 2σ larger than theory predictions

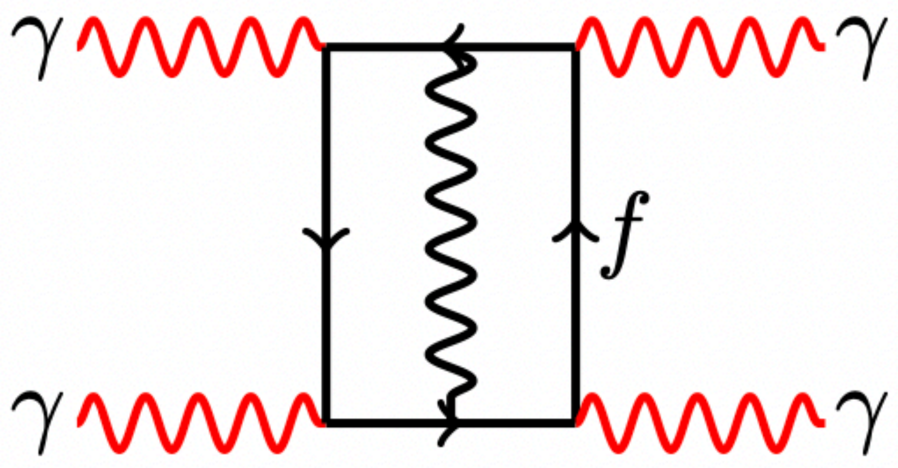


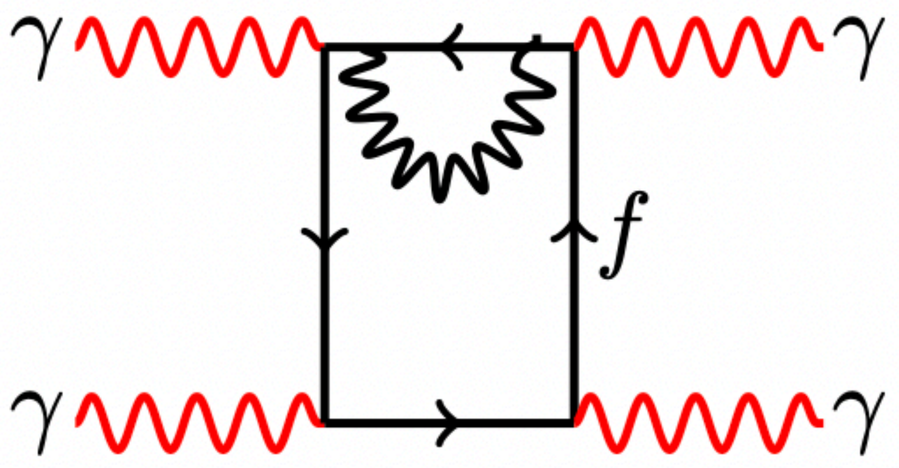
Status - LbL

- Earlier works at two-loop
  - Low energy approximation: two loop corrections to Euler-Heisenberg Lagrangian using string-inspired approach [L.C. Martin,C. Schubert, V.M.V Sandoval '03]
  - Massless limit of two-loop amplitudes with internal fermions in QCD and QED [Z. Bern, A. Freitas, L.J. Dixon, A. Ghinculov, & H.L. Wong '01], [T. Binoth, E.W.N. Glover, P. Marquard, & J. J. van der Bij '02]
- Aim : QCD & QED corrections at NLO with *massive* fermion loops
  - Analytic two-loop helicity amplitudes with general massive internal fermions and get the fully differential cross section
  - Fully differential cross section using two radically different and independent method : Analytic and Numerical Local unitarity method

LbL at NLO

$$\gamma(p_1,\lambda_1)+\gamma(p_2,\lambda_2)+\gamma(p_3,\lambda_3)+\gamma(p_4,\lambda_4)\to 0,$$





- Lorentz decomposition :

$$\mathcal{M}_{\vec{\lambda}} = \left(\prod_{i=1}^{4}\varepsilon_{\lambda_i,\mu_i}(p_i)\right)\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4),$$

$$\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4} = A_1 g^{\mu_1\mu_2}g^{\mu_3\mu_4} + A_2 g^{\mu_1\mu_3}g^{\mu_2\mu_4} + A_3 g^{\mu_1\mu_4}g^{\mu_2\mu_3} + \sum_{j_1,j_2=1}^{3}\Big(B^1_{j_1j_2}g^{\mu_1\mu_2}p^{\mu_3}_{j_1}p^{\mu_4}_{j_2} + B^2_{j_1j_2}g^{\mu_1\mu_3}p^{\mu_2}_{j_1}p^{\mu_4}_{j_2}$$
$$+B^3_{j_1j_2}g^{\mu_1\mu_4}p^{\mu_2}_{j_1}p^{\mu_3}_{j_2} + B^4_{j_1j_2}g^{\mu_2\mu_3}p^{\mu_1}_{j_1}p^{\mu_4}_{j_2} + B^5_{j_1j_2}g^{\mu_2\mu_4}p^{\mu_1}_{j_1}p^{\mu_3}_{j_2} + B^6_{j_1j_2}g^{\mu_3\mu_4}p^{\mu_1}_{j_1}p^{\mu_2}_{j_2}\Big)$$
$$+\sum_{j_1,j_2,j_3,j_4=1}^{3} C_{j_1j_2j_3j_4}p^{\mu_1}_{j_1}p^{\mu_2}_{j_2}p^{\mu_3}_{j_3}p^{\mu_4}_{j_4},$$

$A_i, B^i_{jk}, C_{ijkl} : (s,t,u;m^2)$
$\sim r(s,t,u;m^2)\ I(s,t,u;m^2)$

138 different coefficients

↓ Tranversality ($\varepsilon_{\lambda_i}\cdot p_i = 0$)
Bose sym.
Gauge sym.

5 independent ones

- 5 helicity amplitudes

$$\begin{pmatrix}\mathcal{M}_{++++}\\ \mathcal{M}_{-+++}\\ \mathcal{M}_{--++}\\ \mathcal{M}_{+-+-}\\ \mathcal{M}_{+--+}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & 2 & 2 & 2 & -1\\ 1 & 0 & 0 & 0 & 1\\ 1 & 2 & -2 & -2 & -1\\ 1 & -2 & -2 & 2 & -1\\ 1 & -2 & 2 & -2 & -1\end{pmatrix}\begin{pmatrix}A_S(s,t,u)\\ u\Delta\hat{B}^1_{11}(s,t,u)\\ s\Delta\hat{B}^1_{11}(t,u,s)\\ t\Delta\hat{B}^1_{11}(u,s,t)\\ su\Delta\hat{C}_{2111}(s,t,u)\end{pmatrix},$$

Computation

Generate Feynman diagrams : Qgraf/FeynArts → Color/Dirac/Lorentz algebraic manipulation and get the amplitude : FORM/Mathematica → IBP Reduction: Reduce 10000+ integrals into 30 + crossing Master integrals (MIs) FiniteFlow/Kira

$$\text{Helicity amplitudes} = \sum_j r_j^{(2)}\left(s,t,u;m^2,\epsilon\right) I^{(2)}_{j;\,a_1,\cdots,a_9}(s,t,u;m^2,\epsilon)$$

$$I^{(2)}_{a_1,\cdots,a_9}(s,t,u;m^2,\epsilon) = \int \mathrm{d}^d\ell_1 \mathrm{d}^d\ell_2 \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7} D_8^{a_8} D_9^{a_9}}$$

$$D_i = q^2 - m_f^2 + i\epsilon$$
$$d = 4 - 2\epsilon$$

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$$D_i = q^2 - m_f^2 + i\epsilon$$
$$d = 4 - 2\epsilon$$

Standard techniques to reduce the integrals → linearly independent *Master integrals* : basic idea is Feynman integrals obey linear relations such as Integration by parts identities.

Automatised

$$I_j^{(2)} = \sum_k C_{jk}\, f_k^{(2)}, \quad f_k^{(2)} \to \text{Master integrals at } 2-\text{loop}$$

30 + crossing master integrals

LiteRed (FiniteFlow) [Lee '13] [Peraro '19]

KIRA [Klappert, Lange, Maierhöfer, Usovitsch, `20]

Computation

Generate Feynman diagrams :
Qgraf/FeynArts

→

Color/Dirac/Lorentz algebraic manipulation and get the amplitude :
FORM/Mathematica

→

IBP Reduction: Reduce 10000+ integrals into 30 + crossing Master integrals (MIs)
LiteRed-FiniteFlow/Kira

↓

Solve MIs in terms of known special functions (Multiple polylogarithms)
In-house routine

Computation

Generate Feynman diagrams : Qgraf/FeynArts → Color/Dirac/Lorentz algebraic manipulation and get the amplitude : FORM/Mathematica → IBP Reduction: Reduce 10000+ integrals into 30 + crossing Master integrals (MIs) LiteRed-FiniteFlow/Kira → Solve MIs in terms of known special functions (Multiple polylogarithms) In-house routine

One of the most effective method to solve the MIs : use differential equations [Kotikov '91] [Gehrmann, Remiddi '00]

$$d\vec{f}^{(2)} = \epsilon \; dA^{(2)} \vec{f}^{(2)}$$

Canonical basis [Caron-huot, Henn, '14]

differential equation matrix $dA^{(2)}$ depends only on kinematics, ϵ is fully decoupled

Computation

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Further series expansion over dimensional regulator ϵ : $\vec{f}^{(2)} = \sum_w \epsilon^w \vec{f}^{(2,w)}$

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For the canonical differential equations, the solution can be written in terms of *Chen's iterated integrals*, mostly they can be expressed in terms of *Multiple polylogarithms* : well known functions and can be numerically evaluated

$$G(a_1, \dots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)\,,$$

$$G(a_1; z) = \int_0^z \frac{dt}{t - a_1}\,, \qquad a_1 \neq 0\,.$$

Computation

Generate Feynman diagrams : Qgraf/FeynArts

Color/Dirac/Lorentz algebraic manipulation and get the amplitude : FORM/Mathematica

IBP Reduction: Reduce 10000+ integrals into 30 + crossing Master integrals (MIs) LiteRed-FiniteFlow/Kira

Solve MIs in terms of known special functions (Multiple polylogarithms) In-house routine

Simplification Finiteflow/Mathematica

$$\text{Helicity amplitudes} = \sum_j \sum_w \epsilon^w \, \tilde{r}_j^{(2,w)}\left(s,t,u;m^2\right) \, f_j^{(2,w)}\left(\frac{s}{m^2},\frac{t}{m^2},\frac{u}{m^2}\right)$$

300+ master integrals at different weights, $f_j^{(2,w)}$

We use properties of iterated integrals to find relations among them : shuffle properties

Naively,

$$\begin{aligned} X_1(t)X_1(t) &= \int_{0\le t_1\le t} dt_1 H(t_1) \int_{0\le t_2\le t} dt_2 H(t_2) \\ &= \iint_{0\le t_1\le t_2\le t} dt_1 dt_2 H(t_1)H(t_2) + \iint_{0\le t_2\le t_1\le t} dt_1 dt_2 H(t_1)H(t_2) \\ &= X_{(12)}(t) + X_{(21)}(t), \end{aligned}$$

Computation

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↓

Solve MIs in terms of known special functions (Multiple polylogarithms) In-house routine

↓

Simplification Finiteflow/Mathematica

$$\text{Helicity amplitudes} = \sum_j \sum_w \epsilon^w \, \tilde{r}_j^{(2,w)}\left(s,t,u;m^2\right)\, f_j^{(2,w)}\left(\frac{s}{m^2},\frac{t}{m^2},\frac{u}{m^2}\right)$$

We can also find out relations between the rational coefficients.

With a set of rational functions, we can solve a linear fit problem such that the most complex ones can be written in terms of rather simple ones.

Computation

Generate Feynman diagrams : Qgraf/FeynArts

→ Color/Dirac/Lorentz algebraic manipulation and get the amplitude : FORM/Mathematica

→ IBP Reduction: Reduce 10000+ integrals into 30 + crossing Master integrals (MIs) FiniteFlow/Kira

↓ MIs into canonical basis and then set up differential equations to solve them in terms of known functions (Multiple polylogarithms) In-house routine

↓ Simplification Finiteflow/Mathematica

Number of scalar integrals to Master integrals (MIs)	10k+	30 + crossing	IBP/Lorentz invariance/ Symmetries
Master integrals in UT basis at different weights	300+	84	Using different properties of integrals
Rational coefficients	200+	31+ crossing	Symmetries and linear relations
Total size	300 Mb	Few pages	arXiv:2312.16966

Amplitude - compact structure

- Most compact structure for M_{++++} : fully symmetric in (s,t,u)

$$i\mathcal{M}^{(1,0,f)}_{++++} = 4N_{c,f}Q_f^4\alpha^2\frac{\alpha_s}{\pi}C_{F,f}\Bigg\{3-4\sum_{\substack{(i,j,k)=(s,t,u),\\(t,u,s),(u,s,t)}}\Bigg[\left(\frac{2}{x_i}+\frac{2}{x_k}-\frac{1}{x_j}\right)f_5^{(1,2)}(x_i,x_j,x_k)-\frac{1}{x_i^2}f_{24}^{(2,4)}(x_i,x_j,x_k)+\frac{2}{\sqrt{x_ix_j(x_ix_j+4x_k)}}\left(\frac{x_ix_j+4x_k}{x_k}f_{19}^{(2,2)}(x_i,x_j,x_k)+f_{22}^{(2,3)}(x_i,x_j,x_k)\right)$$

$$+\frac{1}{2x_ix_j}\left(6f_3^{(2,3)}(x_i,x_j,x_k)-f_{14}^{(2,3)}(x_k,x_i,x_j)-4f_{16}^{(2,3)}(x_k,x_i,x_j)-8f_{20}^{(2,3)}(x_i,x_j,x_k)+2f_{15}^{(2,2)}(x_k,x_i,x_j)f_3^{(1,1)}(x_k,x_i,x_j)\right)\Bigg]+\sum_{\substack{(i,j,k)=(s,t,u),(s,u,t),(t,s,u),\\(t,u,s),(u,s,t),(u,t,s)}}\left[\frac{8\,f_{19}^{(2,3)}(x_i,x_j,x_k)}{\sqrt{x_ix_j(x_ix_j+4x_k)}}-\frac{2(x_i-2)f_{27}^{(2,4)}(x_i,x_j,x_k)}{\sqrt{x_i(x_i-4)}\sqrt{x_ix_j(x_ix_j+4x_k)}}\right]\Bigg\}$$

- M_{-+++} is slightly bigger, fully symmetric in (s,t,u)

$$i\mathcal{M}^{(1,0,f)}_{-+++} = N_{c,f}Q_f^4\alpha^2\frac{\alpha_s}{\pi}C_{F,f}\Bigg\{\sum_{\substack{(i,j,k)=(s,t,u),\\(t,u,s),(u,s,t)}}\Bigg[\frac{1}{\sqrt{x_j(x_j-4)}}\Big(2r_1^{(1)}(x_j,x_k,x_i)f_3^{(1,1)}(x_i,x_j,x_k)+r_2^{(2)}(x_i,x_j,x_k)\left(f_3^{(2,3)}(x_i,x_j,x_k)+\frac{1}{3}f_3^{(1,1)}(x_k,x_i,x_j)f_{15}^{(2,2)}(x_k,x_i,x_j)\right)+r_5^{(2)}(x_i,x_j,x_k)f_{14}^{(2,3)}(x_i,x_j,x_k)+r_6^{(2)}(x_i,x_j,x_k)f_5^{(1,2)}(x_i,x_j,x_k)$$

$$+r_1^{(2)}(x_i,x_j,x_k)\left(f_3^{(1,2)}(x_i,x_j,x_k)-\frac{1}{6}f_{15}^{(2,2)}(x_i,x_j,x_k)\right)\Big)-\frac{2x_ix_jf_6^{(1,2)}(x_i,x_j,x_k)}{\sqrt{x_ix_j(x_ix_j+4x_k)}}+\frac{r_3^{(2)}(x_i,x_j,x_k)}{\sqrt{x_i(x_i-4)}}f_5^{(2,3)}(x_i,x_j,x_k)+\frac{r_4^{(2)}(x_i,x_j,x_k)}{\sqrt{x_i(x_i-4)}}f_{13}^{(2,4)}(x_i,x_j,x_k)+r_7^{(2)}(x_i,x_j,x_k)f_5^{(1,3)}(x_i,x_j,x_k)-\left(\frac{6}{x_i}+\frac{4}{x_j}+\frac{6}{x_k}\right)f_{17}^{(2,4)}(x_i,x_j,x_k)$$

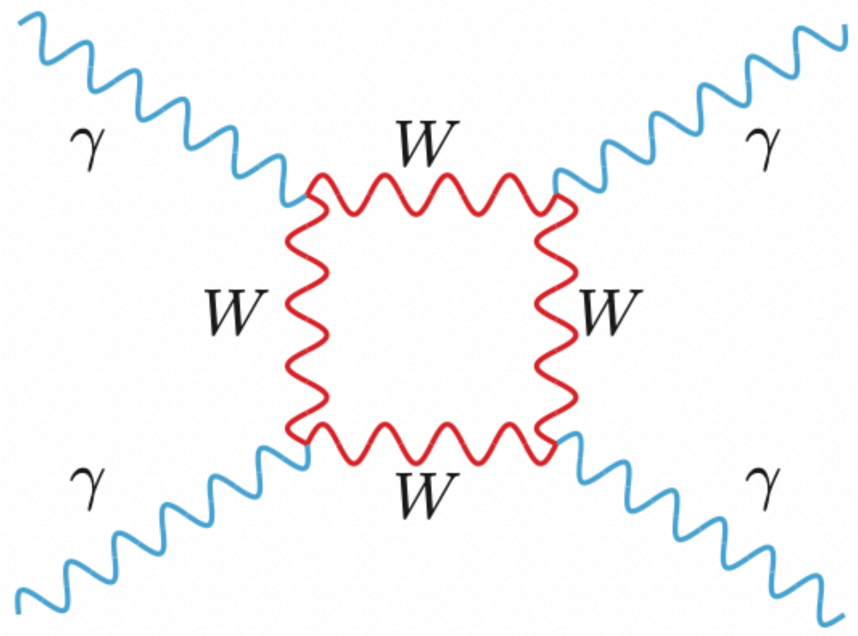
$$+\left(\frac{4}{x_i}+\frac{4}{x_j}+\frac{8}{x_k}\right)\left(f_{21}^{(2,4)}(x_i,x_j,x_k)-\frac{1}{2}f_{24}^{(2,4)}(x_k,x_i,x_j)\right)+r_8^{(2)}(x_i,x_j,x_k)f_{20}^{(2,3)}(x_i,x_j,x_k)+r_{10}^{(2)}(x_i,x_j,x_k)\frac{f_{19}^{(2,2)}(x_i,x_j,x_k)}{\sqrt{x_ix_j(x_ix_j+4x_k)}}-\left(32+\frac{16x_ix_j}{x_k}\right)\frac{f_{22}^{(2,3)}(x_i,x_j,x_k)}{\sqrt{x_ix_j(x_ix_j+4x_k)}}\Bigg]$$

$$-\sum_{\substack{(i,j,k)=(s,t,u),(s,u,t)\\(t,u,s),(t,s,u),(u,s,t),(u,t,s)}}\Bigg[\left(\frac{4}{x_i}+\frac{6}{x_j}+\frac{6}{x_k}\right)f_{26}^{(2,4)}(x_i,x_j,x_k)+\left(\frac{2}{x_j}-\frac{2}{x_k}\right)f_{29}^{(2,4)}(x_i,x_j,x_k)-\frac{r_9^{(2)}(x_i,x_j,x_k)f_{18}^{(2,3)}(x_i,x_j,x_k)}{\sqrt{x_i\left(x_i(x_j-1)^2-4x_j^2\right)}}-\left(32+\frac{16x_ix_j}{x_k}\right)\frac{f_{19}^{(2,3)}(x_i,x_j,x_k)}{\sqrt{x_ix_j(x_ix_j+4x_k)}}$$

$$-r_{11}^{(2)}(x_i,x_j,x_k)\frac{f_{25}^{(2,4)}(x_i,x_j,x_k)}{\sqrt{x_ix_j(x_ix_j+4x_k)}}+\left(2+\frac{x_ix_j}{x_k}\right)\frac{4(x_i-2)}{\sqrt{x_i(x_i-4)}}\frac{f_{27}^{(2,4)}(x_i,x_j,x_k)}{\sqrt{x_ix_j(x_ix_j+4x_k)}}-\left(\frac{4(x_i-2)}{x_k}-\frac{2x_k}{x_j}\right)\frac{f_{28}^{(2,4)}(x_i,x_j,x_k)}{\sqrt{x_i(x_i-4)}}\Bigg]\Bigg\}$$

Cross section

- In addition to two loop QCD and QED corrections coming from massive fermion loop, we also include one-loop corrections from W-boson loop.



- Cross section for $AB \xrightarrow{\gamma\gamma} A\gamma\gamma B$ with heavy ions A and B :

$$\sigma^{\mathrm{NLO}'_{\mathrm{QCD+QED}}} = \int dx_1 dx_2 \mathcal{L}^{(AB)}(x_1, x_2) \frac{1}{2s} \int d\Phi_2 \overline{\sum}_{\mathrm{helicity}} \left| \mathcal{M}^{(0,0)}_{\vec{\lambda}} + \mathcal{M}^{(1,1)}_{\vec{\lambda}} \right|^2,$$

$$\mathcal{M}^{(0,0)}_{\vec{\lambda}} = \sum_{l=f,W} \mathcal{M}^{(0,0,l)}_{\vec{\lambda}}$$

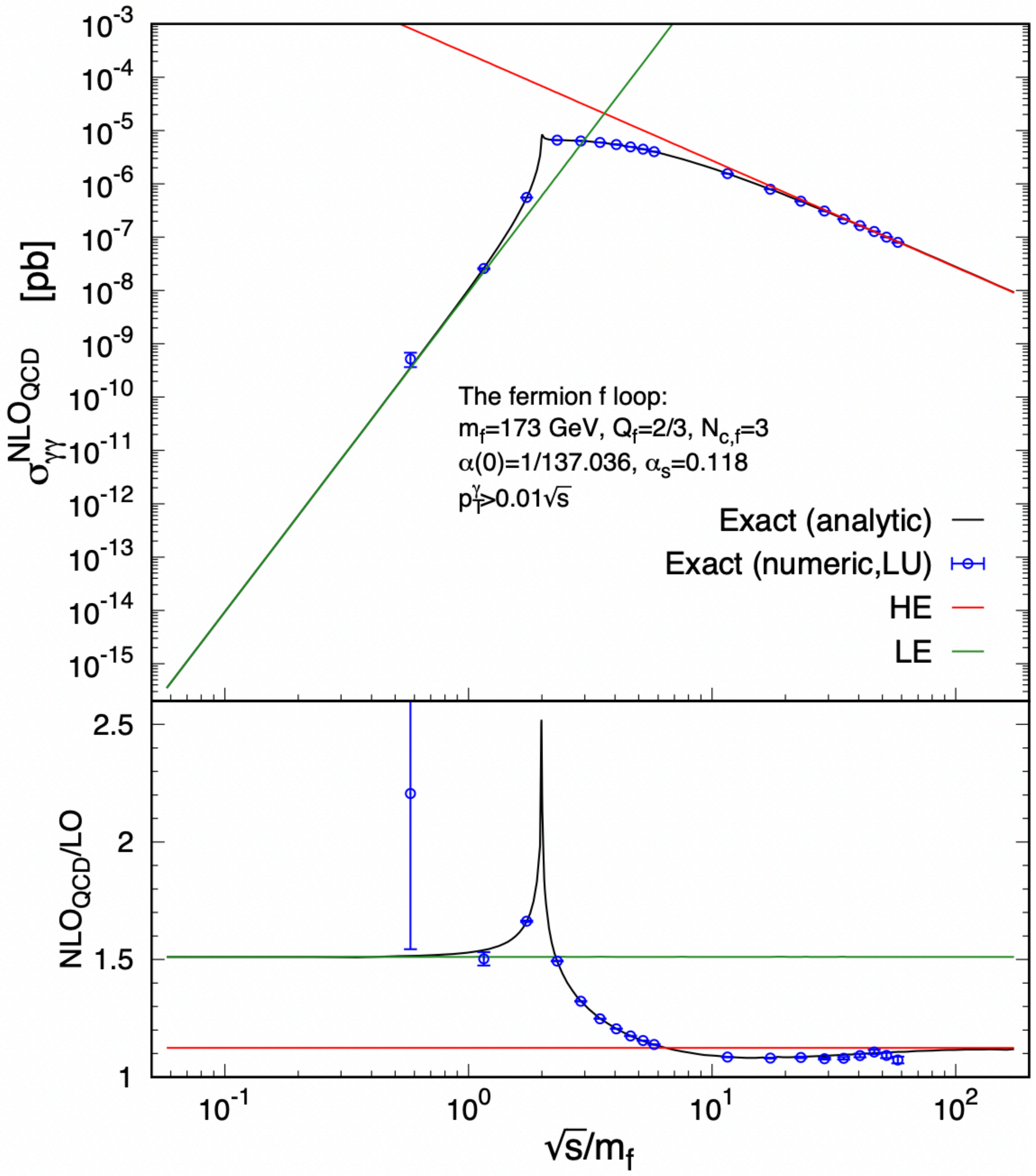
$$\mathcal{M}^{(1,1)}_{\vec{\lambda}} = \sum_f \mathcal{M}^{(1,0,f)}_{\vec{\lambda}} + \mathcal{M}^{(0,1,f)}_{\vec{\lambda}}$$

$M^{(i,j,f)}_{\vec{\lambda}} \rightarrow$ two loop amplitude for fermion f in QCD (i) and / or QED(j).

$L^{(AB)} \rightarrow$ photon-photon flux for the heavy ions A and B. Obtained from **gamma-UPC**

[H.S Shao, D. d'Enterria '22]

Cross section : Analytic and numeric methods



Black curve → results from analytic calculation

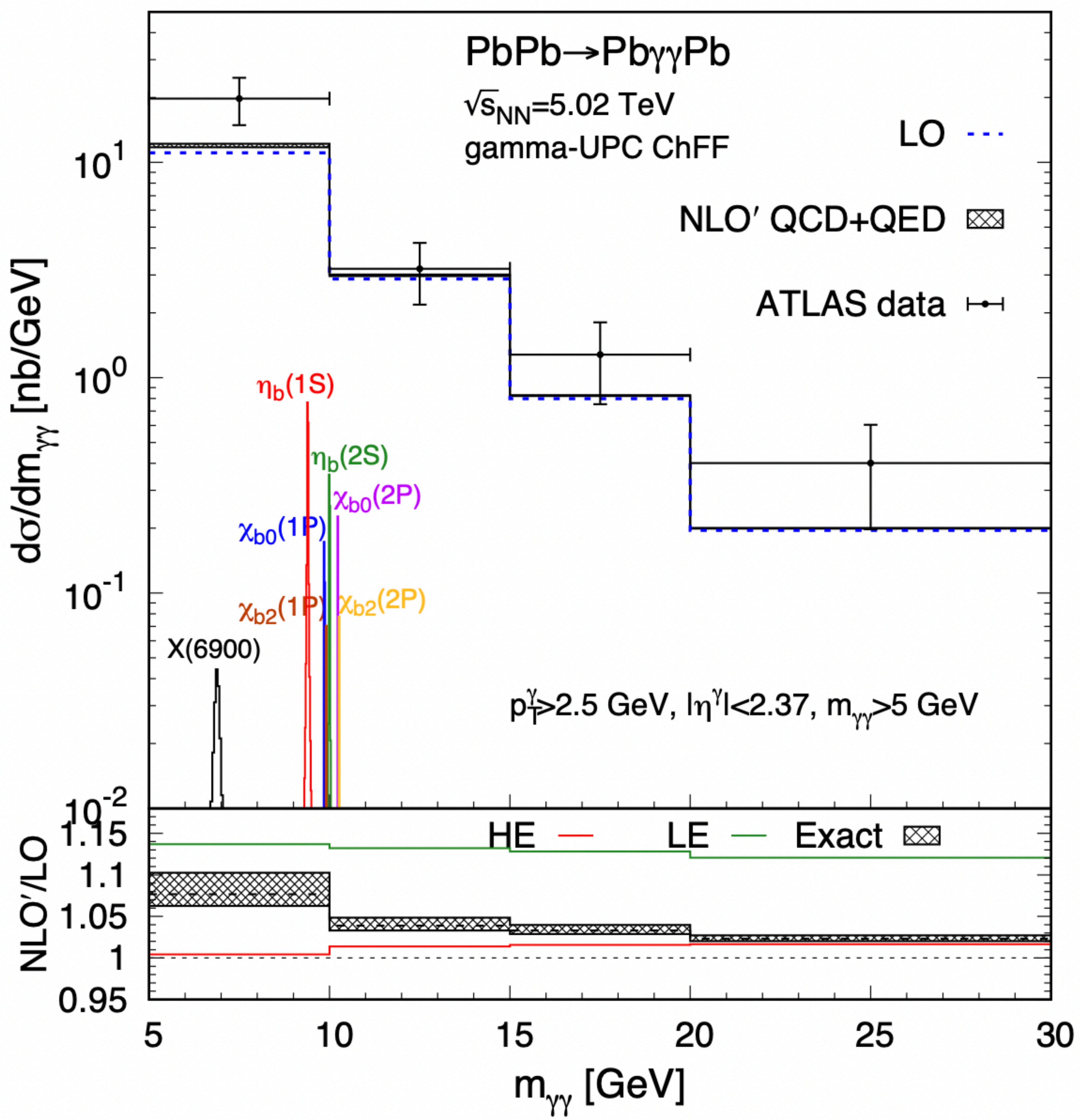
Blue dots → results from numerical Local unitarity method

Green line → Low energy approximation

Red line → High energy approximation

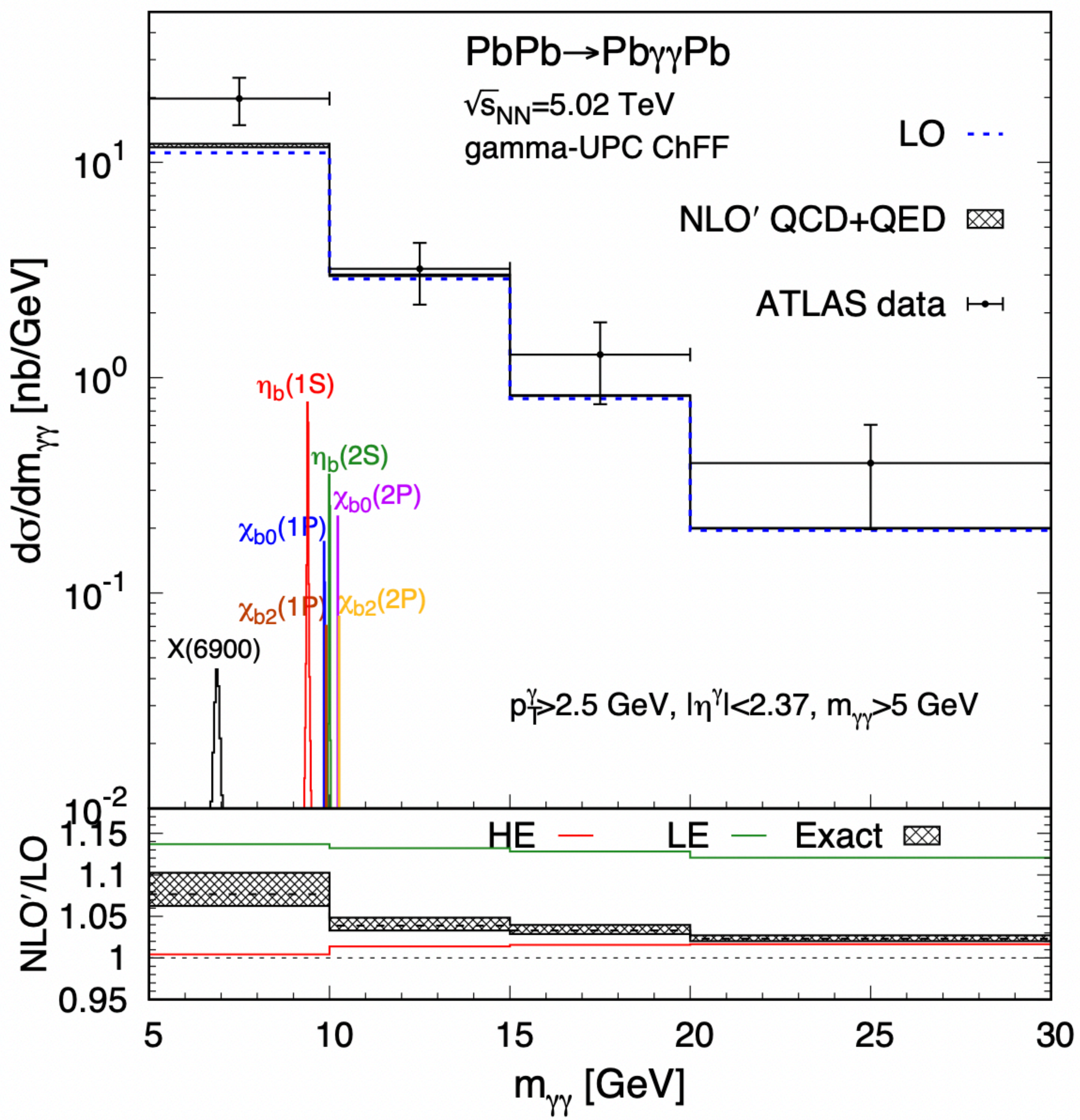
- Two radically different and independent approach match well within the errors, except at the asymptotic limits $\sqrt{s} \ll m_f$ & $\sqrt{s} \gg m_f$, where numerical instability prohibit a fair comparison.
- The full mass dependent result match with known results from Low energy and high energy limits.
- At $\sqrt{s} \to 2m_f$, the two loop amplitudes suffer from Coulomb singularity
  - but is integrable, hence harmless while convoluting with photon flux
  - proper treatment required Coulomb resummation.
- K-factor for the full mass dependent computation exhibit non-trivial behaviour.
- K-factor for HE limit → 1.124 and LE limit → 1.512 .

Data-Theory comparison



- ATLAS measured value = 120 ± 22 nb.
- Cross section at LO = 76 nb.
- NLO′ QCD+QED increases by 6.5 $\%^{+2.1\%}_{-1.2\%}$ wrt LO.
- Best prediction → $81.2^{+1.6}_{-0.9}$ nb : 1.8σ below ATLAS measurements
- HE approximation increment from LO by 0.7% : underestimate the corrections specifically for smaller $m_{\gamma\gamma}$
- LE approximation increment from LO by 13% : significantly overestimates them specifically for large values of $m_{\gamma\gamma}$

Data-Theory comparison



- Tension between data-theory is highest in first di-photon invariant mass bin, $m_{\gamma\gamma} \in [5,10]$ GeV.
- This motivated to study the impact from resonances like C-even bottomonia states and fully-charmed tetra quark states X(6900): obtained from HELAC-Onia event generator. [H.S Shao, '13 , '16]
- We find the contributions of these resonances to LbL cross section is negligible.
- Our result reduces, but not eliminate the data-theory tension.
- The NLO corrections are largest in first bin of 10%, and reduces to 2% in the highest mass bin.

Summary and Outlook

- LbL with full mass dependence in the internal fermion loops at NLO in QCD & QED.
- Efforts on simplifying the amplitude to more compact and concise form so that the helicity amplitudes can be expressed in few pages in the paper.
- Cross section has been computed using radically different methods and are well within errors
- The corrections at NLO is around 6.5%
- Does not eliminate, but reduces the theory-data tension compared to LO cross section

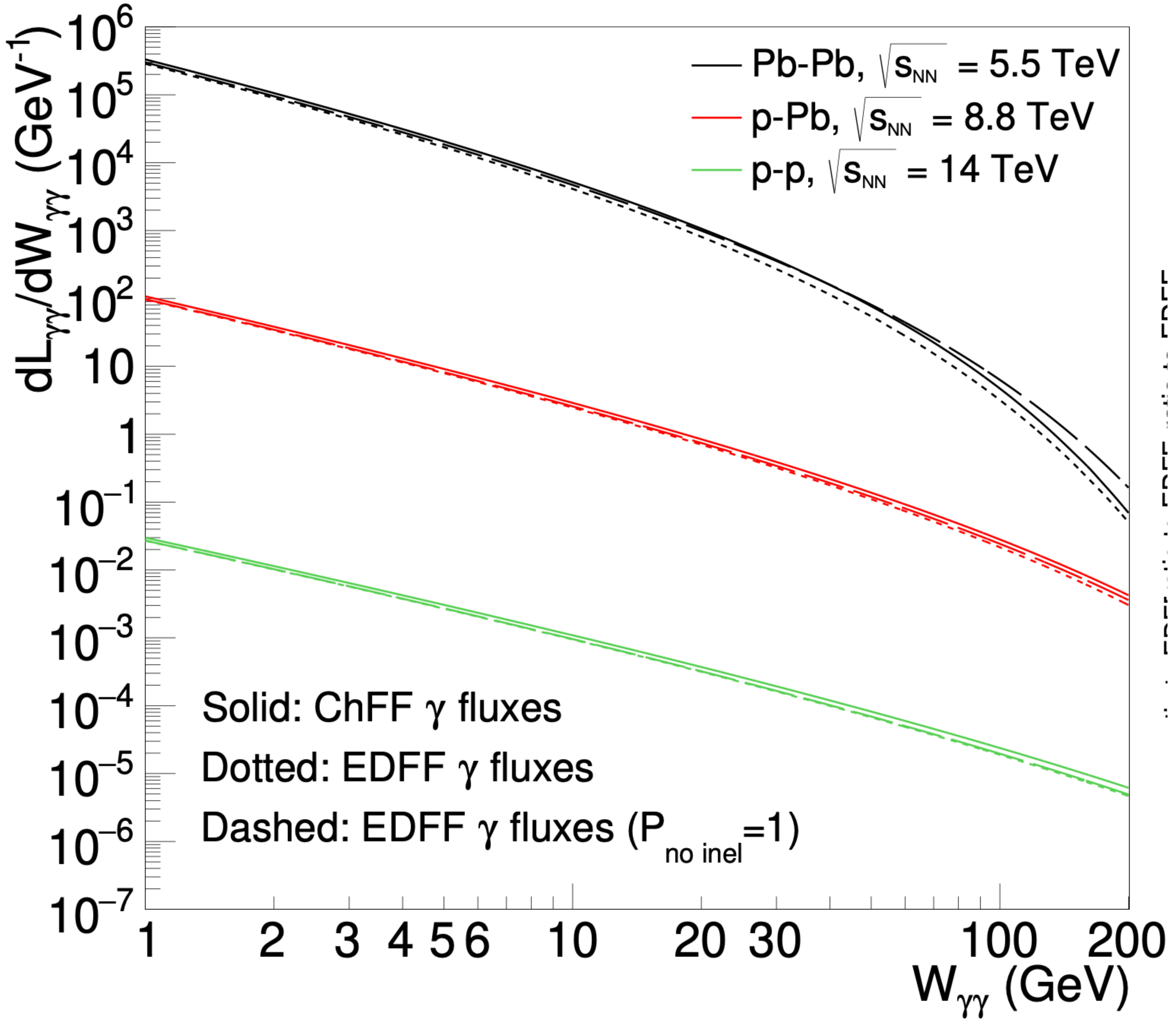
Summary and Outlook

- There are possible extensions to this work
  - LbL is an ideal process to look at complicated higher order level
    - LbL at three loop with massless internal lines : same number of mass scales as NLO.
    - EW corrections at NLO - more complex due to additional internal mass scale. Integrals are unknown
    - Coulomb resummation at the energy scale $\sqrt{s} = 2m_f$

Thank you for the attention!

Backup Slides

Photon luminosity



Photon luminosity

- **Cross section:**

$$\sigma(\mathrm{A\,B} \xrightarrow{\gamma\gamma} \mathrm{A}\ X\ \mathrm{B}) = \int \frac{dE_{\gamma_1}}{E_{\gamma_1}} \frac{dE_{\gamma_2}}{E_{\gamma_2}} \frac{\mathrm{d}^2 N^{(\mathrm{AB})}_{\gamma_1/\mathrm{Z}_1,\gamma_2/\mathrm{Z}_2}}{\mathrm{d}E_{\gamma_1}\mathrm{d}E_{\gamma_2}} \sigma_{\gamma\gamma\to X}(W_{\gamma\gamma})$$

- **Effective two-photon luminosity:**

$$\frac{\mathrm{d}^2 N^{(\mathrm{AB})}_{\gamma_1/\mathrm{Z}_1,\gamma_2/\mathrm{Z}_2}}{\mathrm{d}E_{\gamma_1}\mathrm{d}E_{\gamma_2}} = \int \mathrm{d}^2\boldsymbol{b}_1 \mathrm{d}^2\boldsymbol{b}_2 \boxed{P_{\text{no inel}}\left(|\boldsymbol{b}_1 - \boldsymbol{b}_2|\right)} N_{\gamma_1/\mathrm{Z}_1}(E_{\gamma_1}, \boldsymbol{b}_1) N_{\gamma_2/\mathrm{Z}_2}(E_{\gamma_2}, \boldsymbol{b}_2)$$
$$\times\theta(b_1 - \epsilon R_\mathrm{A})\theta(b_2 - \epsilon R_\mathrm{B})$$

- **No hadronic/inelastic interaction probability density:**

$$P_{\text{no inel}}(b) = \begin{cases} e^{-\sigma^{\mathrm{NN}}_{\mathrm{inel}}\cdot T_{\mathrm{AB}}(b)}, & \text{nucleus-nucleus} \\ e^{-\sigma^{\mathrm{NN}}_{\mathrm{inel}}\cdot T_{\mathrm{A}}(b)}, & \text{proton-nucleus} \\ |1 - \Gamma(s_{\mathrm{NN}}, b)|^2, \quad \text{with } \Gamma(s_{\mathrm{NN}}, b) \propto e^{-b^2/(2b_0)} & \text{p-p} \end{cases}$$

Photon number density calculated using EdFF (Electron dipole form factor or ChFF (charge form factor)

Iterated integrals

2.2. **Iterated integrals.** Let k be the real or complex numbers, and let M be a smooth manifold over k. Let $\gamma : [0,1] \to M$ be a piecewise smooth path on M, and let $\omega_1, \ldots, \omega_n$ be smooth k-valued 1-forms on M. Let us write

$$\gamma^*(\omega_i) = f_i(t)dt \ ,$$

for the pull-back of the forms ω_i to the interval $[0,1]$. Recall that the ordinary line integral is given by

$$\int_\gamma \omega_1 = \int_{[0,1]} \gamma^*(\omega_1) = \int_0^1 f_1(t_1)dt_1 \ ,$$

and does not depend on the choice of parameterization of γ.

Definition 2.1. The iterated integral of $\omega_1, \ldots, \omega_n$ along γ is defined by

$$\int_\gamma \omega_1 \ldots \omega_n = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} f_1(t_1)dt_1 \ldots f_n(t_n)dt_n \ .$$

There is the shuffle product formula

$$\int_\gamma \omega_1 \ldots \omega_r \int_\gamma \omega_{r+1} \ldots \omega_{r+s} = \sum_{\sigma \in \Sigma(r,s)} \int_\gamma \omega_{\sigma(1)} \ldots \omega_{\sigma(r+s)} \ ,$$