Noisy SUSY

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Rencontres de Physique des Particules
LPTHE Sorbonne Université
26 January 2024

Based on arXiv:1302.2361, arXiv:1405.0820, arXiv:1712.07045, arXiv:1912.12925, arXiv:2107.03194, arXiv:2303.17875 and forthcoming.

Noisy SUSY

Since its invention/discovery, SUSY seems to be considered an *optional* feature of natural phenomena;
is there any way in which it might be understood as an *inevitable* feature of natural phenomena?
Some forty years ago, G. Parisi and N. Sourlas, in "Supersymmetric field theories and stochastic differential equations", *Nucl. Phys.* **B206** (1982) 321
made the case that supersymmetry is an inevitable property of a physical system in equilibrium with a bath of fluctuations.
A key role is played by a quantity introduced, some years previously, by H. Nicolai–within the context of supersymmetric theories–and known, since, as "the Nicolai map".

Outline

The story of a physical system and its fluctuations

The description of the properties of a physical system relies on *two* distinct, but equally important, groups: The dynamical degrees of freedom, that describe the "classical" dynamics and the degrees of freedom, that can resolve the fluctuations, with which they are in equilibrium.
These are (some of) their stories. . .

The hidden properties of the partition function

The canonical partition function of a field theory is given by the expression

$$Z = \int [\mathscr{D}\phi_I]\, e^{-S[\phi_I]}$$

In this expression $\{\phi_I\}$ denote the "dynamical" degrees of freedom, that are assumed in equilibrium with a bath of fluctuations; for a quantum field theory these are quantum fluctuations.

This expression therefore depends on the coupling constant(s), that describe the coupling of the dynamical degrees of freedom with the bath. This coupling leads to the fluctuations affecting the partition function and their effects can be computed, order by order in perturbation theory and, beyond perturbation theory, using lattice simulations. Is it possible to guess what they might look like? Is it possible to provide a "field description" of the bath of fluctuations itself, thereby describing the degrees of freedom, that describe the dynamics and the fluctuations, on equal footing?

The "resolved" partition function

The Euclidian action, $S[\phi]$, must be bounded from below and confine at infinity. Therefore, it can be, schematically, written as

$$S[\phi_I] = \frac{1}{2} U'[\phi_I]^2$$

and an interesting question is, under what circumstances $U'[\phi]$ is a *local* functional of the fields. One way to obtain hints as to its possible form is by noticing that, assuming this form, the contribution of the fluctuations can be *guessed* to be

$$\left|\det U''[\phi_I]\right| ;$$

for, then, we remark that

$$Z = \int [\mathscr{D}\phi] \, e^{-\frac{1}{2} U'[\phi]^2} \left|\det U''[\phi]\right| \stackrel{!}{=} 1$$

since $\left|\det U''[\phi]\right|$ is, indeed, nothing but the Jacobian of the change of variables from the ϕ to "noise fields",η.

Changing variables

We now write the Jacobian in the form

$$\left|\det U''[\phi]\right| \equiv \left|\det \frac{\delta\eta_I}{\delta\phi_J}\right|$$

If we perform the change of variables in the partition function, we find

$$Z = 1 = \int [\mathscr{D}\phi_I] \left|\det \frac{\delta\eta_I}{\delta\phi_J}\right| e^{-S[\phi_I]}$$

and we notice that, absent anomalies, the value of the integral does not change. Therefore the absolute value of the determinant describes *all* of the fluctuations of the action of the scalars, $S[\phi_I]$.

Now we can write

$$\left|\det \frac{\delta\eta_I}{\delta\phi_J}\right| = e^{-\mathrm{i}\theta_{\det}} \det \frac{\delta\eta_I}{\delta\phi_J} =$$
$$e^{-\mathrm{i}\theta_{\det}} \int [\mathscr{D}\psi_I][\mathscr{D}\chi_I]\, e^{\int d^Dx\, \psi_I \frac{\delta\eta_I}{\delta\phi_J}\chi_J}$$

From the point of view of the (super)partners

In the previous expression we used anticommuting fields, ψ_I and χ_J to introduce the operator $\delta\eta_I/\delta\phi_J$ in the expression of the (Euclidian) action

$$Z = 1 = \int [\mathscr{D}\phi_I][\mathscr{D}\psi_I][\mathscr{D}\chi_I]\, e^{-\mathrm{i}\theta_{\mathrm{det}}} \times e^{-S[\phi_I]+\int\, d^Dx\,\psi_I\frac{\delta\eta_I}{\delta\phi_J}\chi_J}$$

This expression can be understood in two, equivalent, ways:

- The fluctuations of the commuting fields, ϕ_I, are described by the action of the anticommuting fields–along with the phase of the determinant!
- The fluctuations of the anticommuting fields, in interaction with the commuting fields, are described by the phase of the determinant, along with the action of the scalars.

From the point of view of the (super)partners

Said differently:
The anticommuting fields resolve the bath of fluctuations, with which the commuting fields are in equilibrium, as do the commuting fields for the anticommuting fields, **when they are parts of a supermultiplet.**
It is in this way that the no-go theorem pertaining, in particular, to Bell's inequalities can be evaded; this was, in fact, noted by P. G. O. Freund, already, in 1981 in the paper "Fermionic hidden variables and EPR correlations", *Phys. Rev.* **D24** (1981) 1526.
Curiously, this idea wasn't followed up–nor was the relation to the work of Parisi and Sourlas, after it appeared, investigated further...

Worldvolume and target space fermions

The expression for the action of the anticommuting fields that is of particular interest can be written as

$$\frac{\delta \eta_I}{\delta \phi_J} = \sigma^{\mu}_{IJ}\partial_\mu + \frac{\partial^2 W}{\partial \phi_I \partial \phi_J}$$

This raises the question on what properties might it be of interest to impose on the σ^{μ}_{IJ}.

- The σ^{μ}_{IJ} commute. Then the anticommuting fields are not target space fermions, they're worldvolume fermions; this is relevant for particle models.
- The σ^{μ}_{IJ} generate a Clifford algebra,

$$\{\sigma^{\mu}, \sigma^{\nu}\} = 2\delta^{\mu\nu}$$

  Then the anticommuting fields are target space fermions. This is the case relevant for particle physics.

Flavors vs. species

There is a subtle point hidden here, for the indices of the σ^μ_{IJ}, if they generate a Clifford algbra, are, usually, identified as spinor indices; while we would like to identify them with "species" indices (the reason they can't be identified as "flavors" is because these fermions transform in the adjoint and not the fundamental representation of the internal symmetry group). Thanks to Pierre Fayet for stressing this difference.

So there is a hidden "species" matrix at work here, whose consequences remain to be understood.

N.B. We are working in Euclidian signature; this means that the σ^μ realize a Majorana representation iff $D \equiv 2 \bmod 8$.

The Nicolai map

The expression for the action of the anticommuting fields implies that the "noise fields", η_I, are given, in turn, by the expression

$$\eta_I = \sigma^{\mu}_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial\phi_I}$$

This is a relation between the dynamical degrees of freedom, $\{\phi_I\}$, and the fields $\{\eta_I\}$. This is called, now, the "Nicolai map". It is, also, an example of the "trivializing map" by M. Lüscher. It maps a non-trivial theory of scalar fields, $\{\phi_I\}$, to a "trivial" theory of other scalar fields, $\{\eta_I\}$. The latter are supposed to describe the bath–in the absence of the scalars and the Nicolai map is supposed to show how the dynamics is affected upon introducing the scalars. One consequence is the emergence of the fermions; and vice vera.

Illustration: The Nicolai map for the $\mathcal{N} = 2, D = 2$ Wess–Zumino model

Parisi and Sourlas write the following Nicolai map:

$$\eta_1 = \partial_x \phi_2 + \partial_y \phi_1 + \frac{\partial W}{\partial \phi_1}$$
$$\eta_2 = \partial_x \phi_1 - \partial_y \phi_2 + \frac{\partial W}{\partial \phi_2}$$

and take

$$\frac{\partial W}{\partial \phi_1} = g(\phi_1^2 - \phi_2^2)$$
$$\frac{\partial W}{\partial \phi_2} = 2gs\phi_1\phi_2$$

where $s = \pm 1$.

How target space fermions emerge

If $g = 0$ we find that

$$\frac{\delta \eta_I}{\delta \phi_J} = \sigma^x_{IJ} \partial_x + \sigma^z_{IJ} \partial_y$$

whence they notice the emergence of target space fermions!

Doubling

If $D \not\equiv 2 \bmod 8$, (e.g. $D = 3$ or $D = 4$ spacetime dimensions) then the σ^μ_{IJ} have imaginary entries, therefore, in the map,

$$\eta_I = \sigma^\mu_{IJ}\partial_\mu\phi_J + \frac{\partial W}{\partial \phi_I}$$

the RHS is complex, so the LHS must be, too. Therefore, we must introduce the complex conjugate:

$$\eta_I^\dagger = \sigma^\mu_{JI}\partial_\mu\phi_J^\dagger + \left(\frac{\partial W}{\partial \phi_I}\right)^\dagger$$

(since the σ^μ are Hermitian) and the partition function for the noise fields is, now,

$$Z = 1 = \int [\mathscr{D}\eta_I][\mathscr{D}\eta_I^\dagger]\, e^{-\int\, d^D x\, \frac{\eta_I(x)\eta_J(x)^\dagger\delta^{IJ}}{\sigma^2}}$$

Testing the idea in practice

The question is, whether the fluctuations of the scalars can reproduce the absolute value of the determinant, i.e. the Jacobian between the scalars and the noise fields. This can be answered by computing the correlation functions of the noise fields, $\eta_I[\phi]$, (resp. $\eta_I[\phi, \phi^\dagger]$, $\eta_I[\phi, \phi^\dagger]^\dagger$), sampled with the action of the scalars, i.e. with the measure

$$[\mathscr{D}\phi_I]\, e^{-S[\phi_I]}$$

(resp. for the generalization, when $D \not\equiv 2 \bmod 8$) and checking that

$$\langle \eta_I(\phi(x)) \rangle \stackrel{?}{=} 0$$
$$\left\langle \left(\eta_I(x) - \langle \eta_I(x) \rangle\right) \left(\eta_J(x') - \left\langle \eta_J(x') \right\rangle\right) \right\rangle \stackrel{?}{=} \mathrm{const}\delta_{IJ}\delta(x - x')$$

and the higher order correlators of the η_I should be given by Wick's theorem.

Any significant deviation is the signal for the appearance of anomalies.

Tests

- For probability distributions, these identities *do* have anomalies, that can be understood (cf. arXiv:1302.2361[hep-th]). In addition, while the Jacobian does do the job expected of it, it can't be generated by the fluctuations, since the identities aren't satisfied.
- For a non-relativistic particle, these identities do *not* show anomalies–the identities are satisfied to numerical precision and up to lattice artifacts (cf. arXiv:1405.0820[hep-th]).
- For two dimensional scalar field theories, these identities do not show anomalies, either (cf. arXiv:1712.07045[hep-th]).
  Work for the cases $D = 3$ and $D = 4$ is ongoing–the simulations take considerably more time...

How about gauge theories?

Gauge theories, with compact gauge group, can be described by scalar fields, taking values on the group manifold. The "natural" noise distribution isn't a Gaussian, with ultra–local 2–point function, but uniform over the group manifold.
This has been studied on the lattice, through the so-called "trivializing maps", introduced by Lüscher. These are, indeed, the avatars of the Nicolai map for the group manifolds.
However their construction is, still, work in progress.

Abelian gauge theories

For abelian gauge fields it's possible to take a shortcut (in Lorenz–Feynman gauge):

$$\eta_I = \sigma^\mu_{IJ} \partial_\mu \phi_J$$
$$\xi_I = \sigma^\mu_{IJ} \nabla_\mu \varphi_J + \frac{\partial W}{\partial \varphi_I}$$
$$\xi_I^\dagger = \sigma^\mu_{JI} [\nabla_\mu \varphi_J]^\dagger + \left(\frac{\partial W}{\partial \varphi_I} \right)^\dagger$$
$$\nabla_\mu \equiv \partial_\mu - \mathrm{i} q A_\mu \equiv \partial_\mu - \mathrm{i} q \phi_\mu$$

where

$$\phi_\mu \equiv \phi_I \equiv A_\mu$$

and q is the charge of the matter fields under the gauge field. Here φ_I are the scalar superpartners of the fermions of the hypermultiplet(s).

The partition function for (S)QED

$$Z = \int \underbrace{[\mathscr{D}\eta_I][\mathscr{D}\xi_I][\mathscr{D}\xi_I^\dagger]}_{[\mathscr{D}h_I]} e^{-\int d^Dx \left\{\frac{1}{2}\eta_I\eta_J\delta^{IJ}+\xi_I\xi_J^\dagger\delta^{IJ}\right\}} = 1 =$$

$$\int \underbrace{[\mathscr{D}\phi_I][\mathscr{D}\varphi_I][\mathscr{D}\varphi_I^\dagger]}_{[\mathscr{D}\Phi_I]} \left|\det \frac{\delta h_I}{\delta \Phi_J}\right| e^{-S[\phi_I,\varphi_I,\varphi_I^\dagger]}$$

The fermions are "hidden" in the determinant and "emerge" upon introducing it in the exponent.
For $D = 4$, we must double the degrees of freedom correspondingly and find, by another argument, the same number of scalars–eight real scalars, or four complex doublets–as in the "conventional" formulation of the minimal supersymmetric extension of the Standard Model.

Conclusions and Outlook

Any field theory (and that includes particle models, in the path integral formalism), whose fields take all possible real values, can be understood as providing a mapping between white noise fields and commuting fields; the anticommuting fields "emerge" from the Jacobian. The relation between the commuting and anticommuting fields is that they are superpartners. This is extended supersymmetry.
The superpartners may be thought of as "BSM" particles; but, in fact, they are part of the SM, since they resolve the quantum fluctuations of the fields of the SM!
That's the essence of the proposal of Parisi and Sourlas; and the way to understand it, in practice, is by computing the identities that should be satisfied by the Nicolai map.

Conclusions and Outlook

Beware of "false prophets", however!
The position, $x^\mu(\tau)$ and the spin, $\psi^\mu(\tau)$, of the spinning relativistic particle are related by target space SUSY; however they *don't* resolve the fluctuations of the other!
The fluctuations of the position are *different* anticommuting fields, say $\chi^\mu(\tau)$, and the fluctuations of the spin define a *different* commuting fields, $\phi^\mu(\tau)$.
Of course they are all related–and how is an interesting exercise to solve. . .

Conclusions and Outlook

Consequences for the Standard Model:

- One scalar field is a semi-classical property, relevant within perturbation theory; in a relativistic field theory, it's not possible to describe, fully, the fluctuations of just one scalar field; there are, inevitably, more-in $D = 4$ the least number is 8, which leads to, at least, two "Higgs-like" scalars (the other scalars becoming, for example, the longitudinal polarization states of gauge bosons).
- "Species" (non-)universality can be straightforwardly accommodated, since the fermion determinant doesn't, inevitably, "factorize" over the flavors. How it does is of interest to spell out. How this is related to "flavor" (non-)universality and can lead, in particular, to constraints on the number of flavors and their mixing properties can now be addressed more explicitly.
- Chiral fermions can be described using the domain–wall construction, that can lead to "partial" SUSY breaking

Conclusions and Outlook

Another issue of practical significance is that, insofar as the absolute value of the determinant–that describes the contribution of the fermions–is generated by the fluctuations, this means that it is possible, in principle, to express fermionic correlators in terms of the correlators of their bosonic superpartners, sampled using the bosonic action, which is much easier to do, than the fermionic action. This remains to be spelled out for practical applications.

Conclusions and Outlook

There's a "natural" way to understand the relevance of SUSY for any field theory and the SM, in particular. There's, still, considerable work to be done to understand how this approach can be realized for non-abelian gauge theories and how this can lead to search strategies in real experiments. However SUSY isn't an optional property of Nature (or of the SM) but an inevitable part of it.
It's necessary to learn how to see it. How it can be realized can be quite unexpected (recall that the quarks cancel the gauge anomalies of the leptons and vice versa!)

Conclusions and Outlook

All theories are supersymmetric.
Some theories are born supersymmetric;
some become supersymmetric;
some have supersymmetry thrust upon them...