SCIENCES SORBONNE UNIVERSITÉ

LPNHE PARIS

Development of an unbiased shear estimator measured on galaxies

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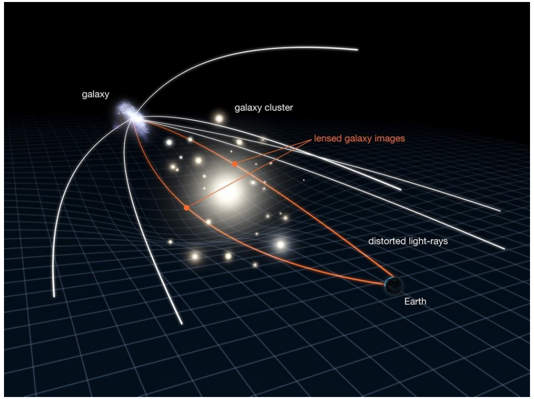
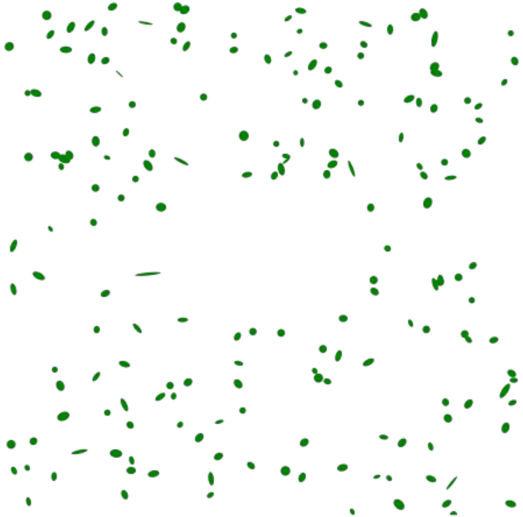
December 2023

Context

Weak lensing (cosmic shear) : statistical detection

Why do we use cosmic shear ?
→ Both matter and expansion sensibility
→ Powerful tool to understand **dark energy**





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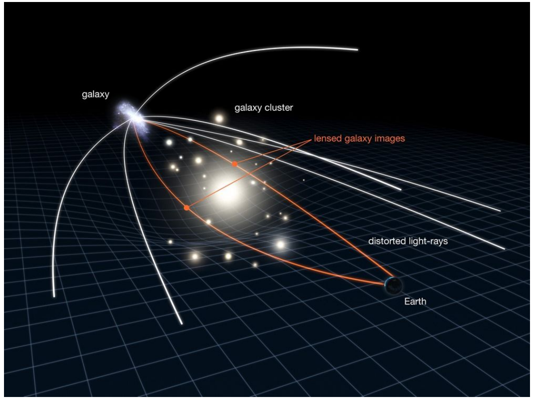

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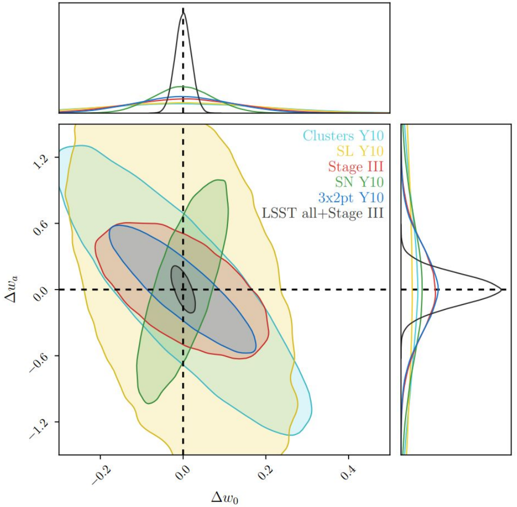
Why do we use cosmic shear ?
→ Both matter and expansion sensibility
→ Powerful tool to understand **dark energy**

Challenge of the next decade : precision cosmology
→ Will be possible with LSST images, but complex measurement : associated biases, one source being the **shear estimator**

My goal : Development of an unbiased cosmic shear estimator measured on galaxy shapes

arxiv:1809.01669



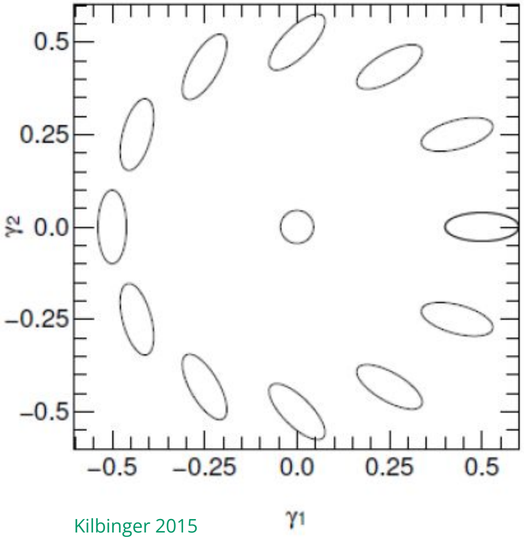
Method : Formalism

Distortion of an image described by matrix : $\mathscr{A}(\boldsymbol{\theta}) = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$

Shear components : $\gamma \equiv \gamma_1 + \mathrm{i}\gamma_2 = |\gamma| \mathrm{e}^{2\mathrm{i}\varphi}$

<u>Weak lensing :</u> $|\gamma| \ll \kappa \ll 1$

$\Longleftrightarrow$ **reduced shear :** $\boxed{g_i = \dfrac{\gamma_i}{1-\kappa}}$



Kilbinger 2015

Method : Seconds moments

$$M = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix}$$

We use the seconds moments to measure the shape of the galaxy

$$M = \int \overbrace{XX^T}^{(X-X_0)(X-X_0)^T} W(X) I(X)\, dX^2$$

Pixel coordinates

Weight function

$$I = I_0 \circledast \psi$$

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Ellipticity second moment estimator
(linear combination of seconds moments)

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} M_{xx} - M_{yy} \\ 2M_{xy} \end{pmatrix}$$

Method : Seconds moments calibration

Ellipticity from seconds moments is not sufficient to measure the shear → **Need to calibrate**

The galaxy profile in unknown **but :** we do know how to insert shear into images !

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Idea : Shear-sensitive algorithm
→ see how it reacts to the introduction of small shear variations

$$M(S) = \int \underbrace{[(YY^T W(Y)) \circledast \psi_-](X)}_{F(X)} I_0(SX)\, dX^2$$

Method used in *Metacalibration*

arXiv:1702.02601

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$$M(S) = \int F(S^{-1}X) I_0(X)\, dX^2$$

$$M(S) = \int F(Sk) I_0(k)\, dk^2$$

$$= \int \frac{F(Sk)}{\tilde{\psi}^*} \tilde{\psi}^* I_0(k)\, dk^2$$

$$= \int G(S, X)[\psi \circledast I_0]\, dX^2$$

No shear applied to the original image !

$$= \int G(S, X) I(X)\, dX^2$$

Method : Seconds moments calibration

Shear dependency : inserting small shear variations, calculating the **numerical derivative**

$$\frac{dM}{d\gamma} = \int \frac{dG(S(\gamma), X)}{d\gamma} I(X)\, dX^2$$

$$g_1 = \pm\, \varepsilon \; ; \; g_2 = 0 \quad \Rightarrow \quad \frac{\partial M}{\partial g_1} = \frac{SM_{1+} - SM_{1-}}{2\epsilon}$$

$$g_1 = 0 \; ; \; g_2 = \pm\, \varepsilon \quad \Rightarrow \quad \frac{\partial M}{\partial g_2} = \frac{SM_{2+} - SM_{2-}}{2\epsilon}$$

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Self calibration factor (linear combination of seconds moments derivatives with respect to the shear)

$$\mathbf{R} = \begin{pmatrix} \frac{\partial e_1}{\partial g_1} & \frac{\partial e_2}{\partial g_1} \\ \frac{\partial e_1}{\partial g_2} & \frac{\partial e_2}{\partial g_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial M_{xx}}{\partial g_1} - \frac{\partial M_{yy}}{\partial g_1} & \frac{2\partial M_{xy}}{\partial g_1} \\ \frac{\partial M_{xx}}{\partial g_2} - \frac{\partial M_{yy}}{\partial g_2} & \frac{2\partial M_{xy}}{\partial g_2} \end{pmatrix}$$

Method : Shear calibration

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} M_{xx} - M_{yy} \\ 2M_{xy} \end{pmatrix}$$

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Shear estimation :

$$\boxed{\langle g \rangle = \langle R \rangle^{-1} \langle e \rangle}$$

(LSST : we aim to have biases less than ‰)

Method : Seconds moments

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- Calculations are based on second moments, rather than maximum likelihood, so we don’t have to make any assumption about the galaxy profile.

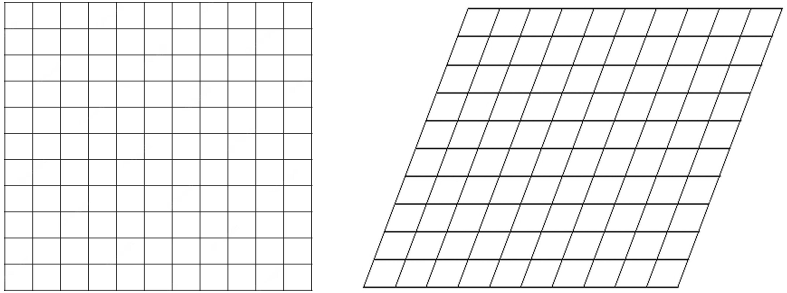
- The *F* function is more extensive than the object image I_0 , and therefore better resolved, it is therefore better to apply shear distortion on it.

Galsim convention

$$S = \frac{1}{\sqrt{1-g^2}} \begin{pmatrix} 1+g_1 & g_2 \\ g_2 & 1-g_1 \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix}$$

Method : Technical aspects

Application of shear variations (±ε) to calculate derivatives : distortion of the coordinate system (with *S* matrix), then interpolation of the image (*F* function) onto the new grid.



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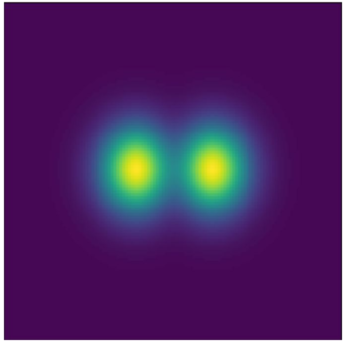

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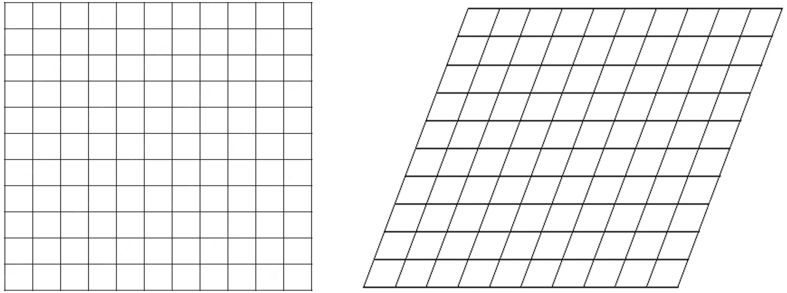

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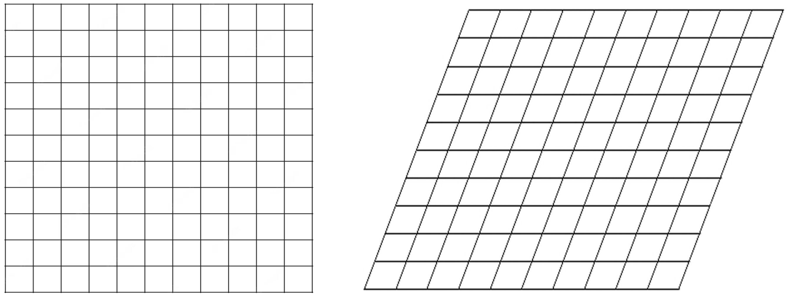

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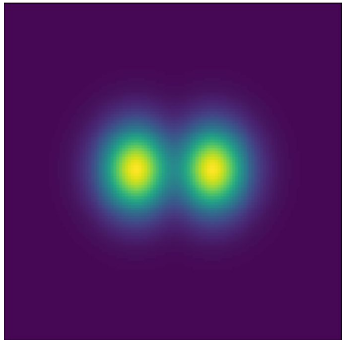

$g_1 = \pm\,\varepsilon\ ;\ g_2 = 0$

grid distortion
+
interpolation

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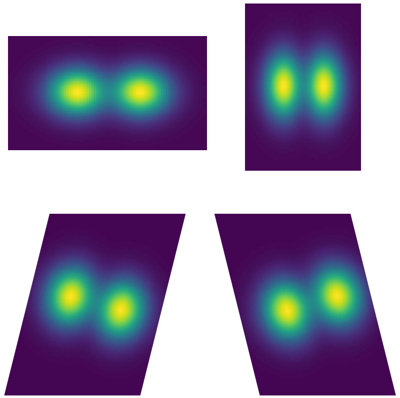

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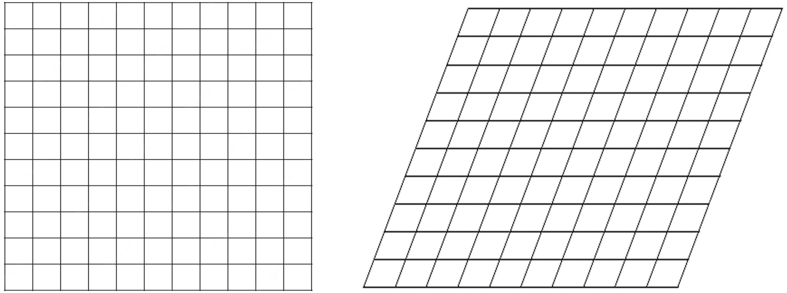

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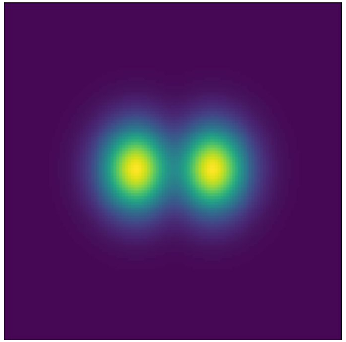

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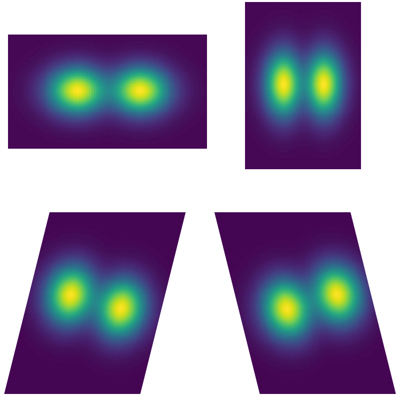

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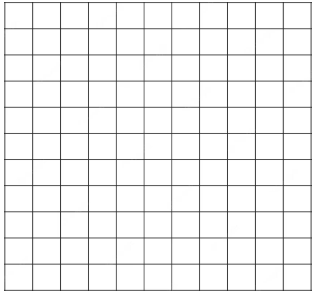

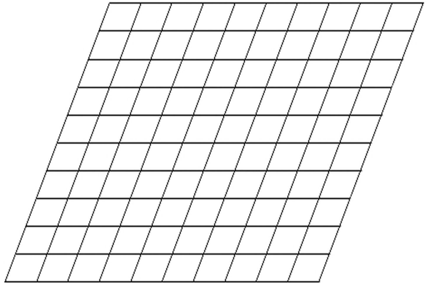

Sampling :

$$I = I_c \otimes \Pi \quad \Leftrightarrow \quad M = \boxed{M_c} + \boxed{M_{pix}}$$

$$M = \boxed{\int XX^T I(X)\, d^2X} + \boxed{\frac{JJ^T}{12}}$$

with :

- **J** : the Jacobian involved in the affine transformation of coordinates (pixel ↔ physical)

- **s** : the image pixel scale (arcsec/pixel)

$$J = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

Method : Technical aspects - shear & sampling cross-effect

As we measure the *SM* matrices on a distorted pixels grid, **we should subtract a *distorted* pixels second moments matrix *Mpix*** to recover the real object second moments.

<u>New second moments formalism :</u>

$$M(s,\epsilon) \propto \underbrace{\boxed{\gamma + \alpha\epsilon + \alpha'\epsilon^2}}_{\text{theoretical second moment}} + \underbrace{\boxed{\beta s^2 + \beta' s^4}}_{\text{sampling correction}} + \boxed{\delta s^2 \epsilon} + \boxed{\delta' s^4 \epsilon}$$

sampling x shear correction

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Bias introduced by a cross-effect between shear and sampling

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theoretical second moment

sampling correction

sampling x shear correction

We subtract only the last term to correct the second moment, since the sampling correction will cancel (they only influence M_{xx} and M_{yy}, which are subtracted in ***e*** and ***R***)

Bias introduced by a cross-effect between shear and sampling

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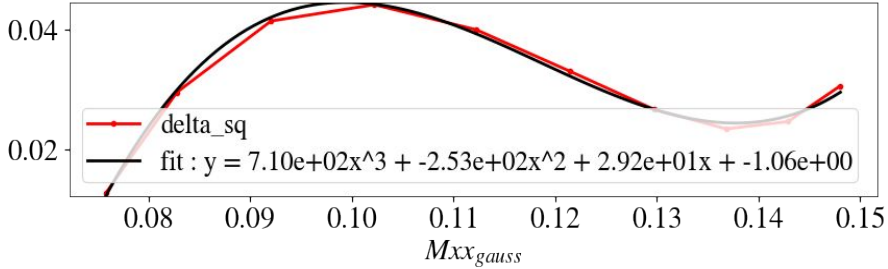
No big impact of the galaxy profile on the parameters estimation, but highly related to the size of the galaxy.

→ Relation between δ' and galaxy second moment (Gaussian profile)



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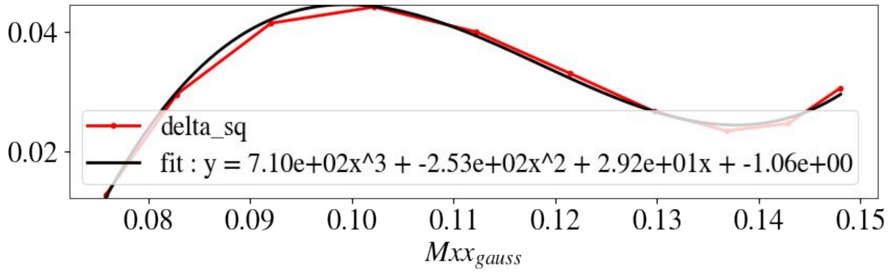
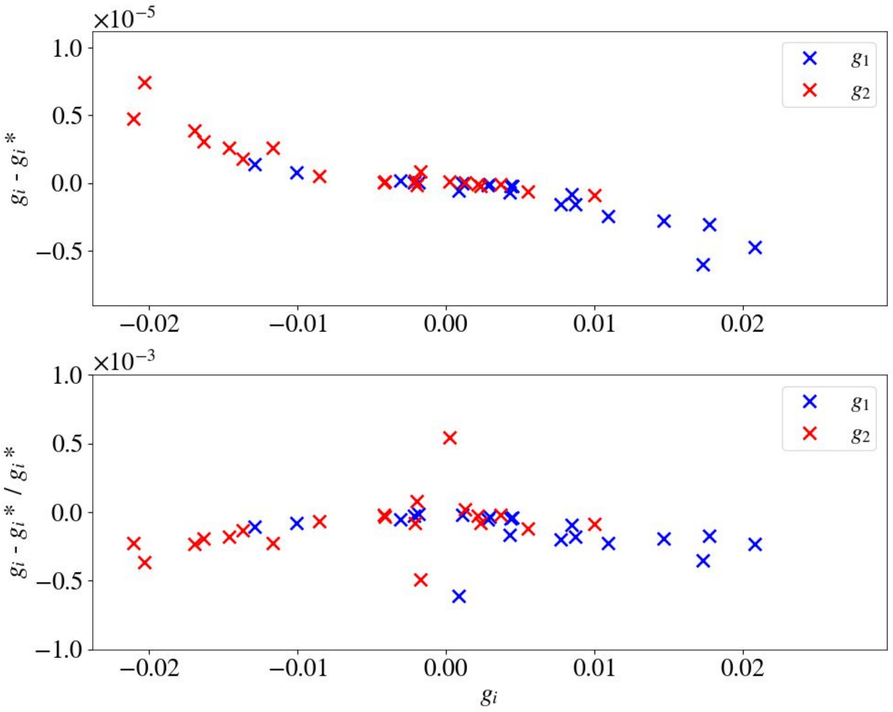


The result of the fit works for any galaxy profile (including realistic profiles from the COSMOS catalog), **but not when the PSF profile is different**

→ **BUT** this is not a problem : the PSF profile is sufficiently well known

Results : Elliptical gaussians galaxies

Comparison : Mean over elliptical galaxies ($\langle g \rangle = \langle R \rangle^{-1} \langle e \rangle$)



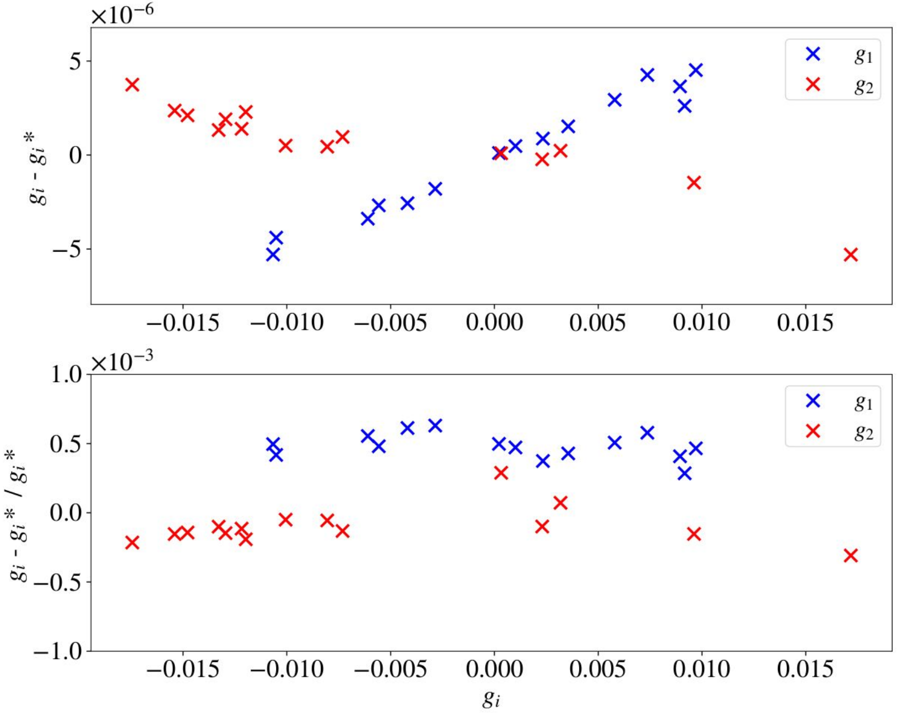
Upper panel : absolute difference between applied shear and estimated shear

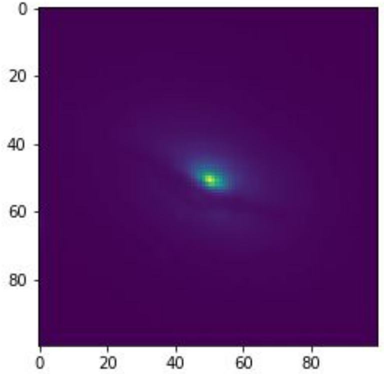
Lower panel : relative difference

Estimation performed over 20 random shear values and averaging over 20 pairs of random (and opposite) intrinsic ellipticities.

Results : Realistic galaxy profile (COSMOS catalog)

Comparison : Mean over random rotations applied to the galaxy





$$M = \int \overbrace{XX^T}^{(X-X_0)(X-X_0)^T} W(X)I(X)\,dX^2$$

Results : Noisy simulations

Analytical calculation of position variance :

$$\boxed{\sigma(x_0)^2 = K^2\sigma^2_{noise}\frac{M_W}{2}\sum_i W_i^2}$$

with :

$$K = \frac{1}{F}[\mathbb{1} - M_W^{-1} M_P]^{-1}$$

second moment of I*W

second moment of W

$$F = \sum W(x_i - x_0) I_i$$

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M_W: second moment of W

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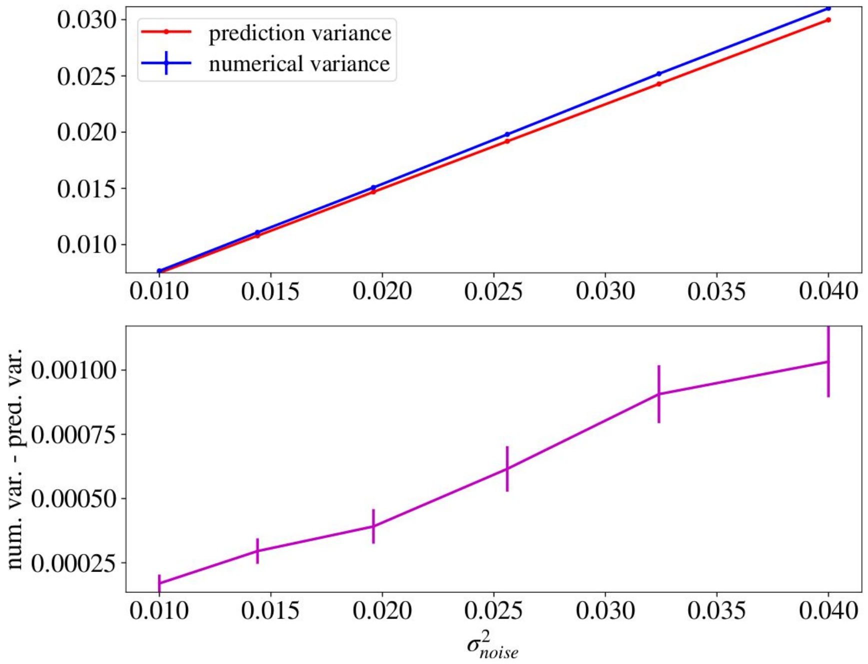

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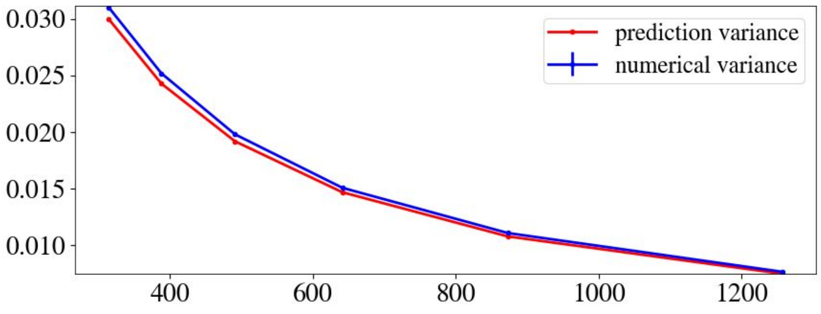

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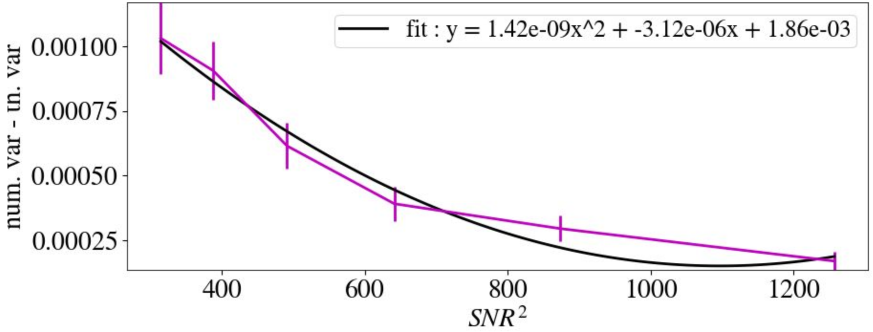

Results : Noisy simulations

Analytical calculation of second moment noise bias :

$$m_I(I+n) = m_I(I) + \sum_k \frac{dm_I}{dI_k} n_k + \frac{1}{2} \sum_{kl} \frac{d^2 m_I}{dI_k dI_l} n_k n_l$$

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6 correction terms, depending on :

- Position variance
- W size
- I*W size
- I*W flux
- 4th moments of W and I*W

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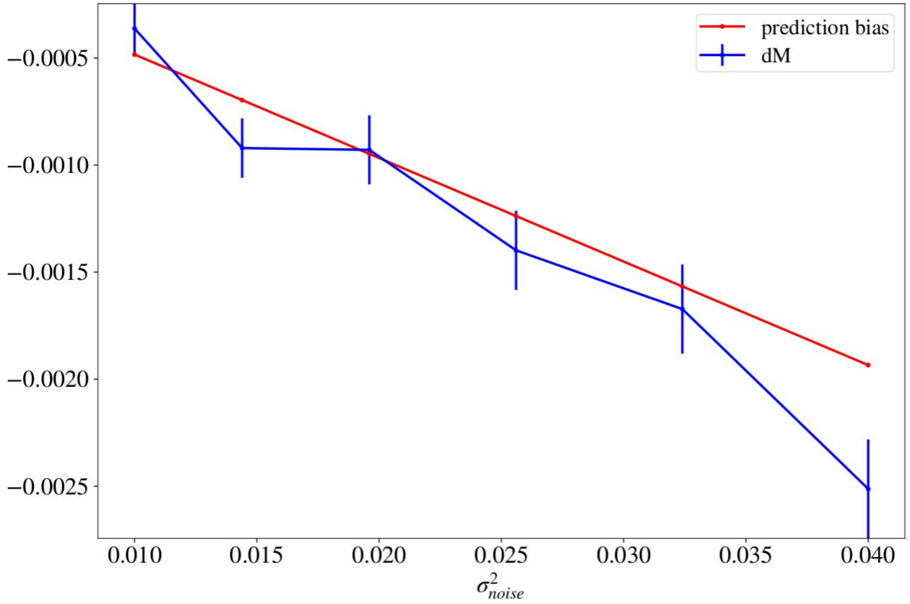

Thank you

Backup

Method : Theoretical prediction

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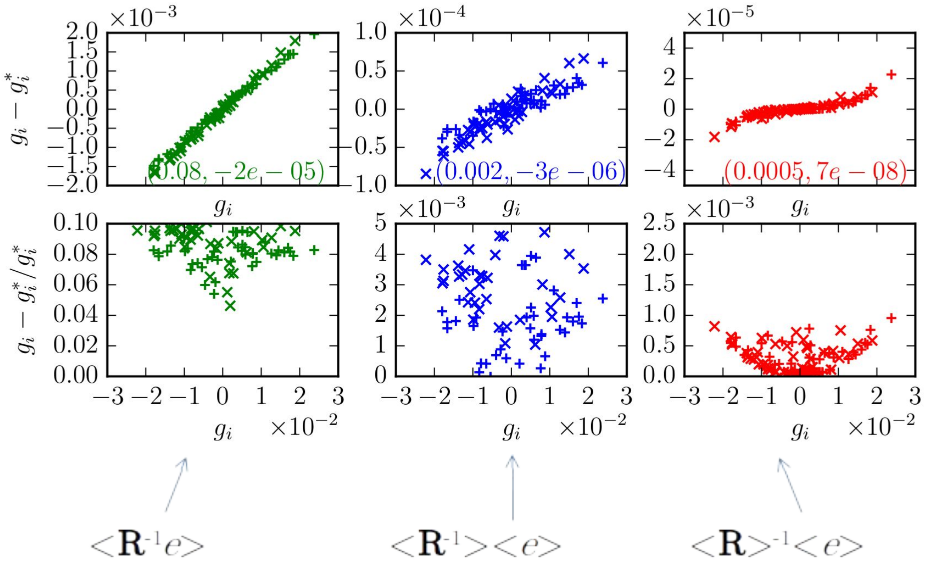


Upper panel : absolute difference between applied shear and estimated shear

Lower panel : relative difference

Credit : Maude Le Jeune

e : ellipticity second moment estimator

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} M_{xx} - M_{yy} \\ 2M_{xy} \end{pmatrix}$$

R *(self calibration factor)* : linear combination of seconds moments derivatives with respect to the shear

$$\mathbf{R} = \begin{pmatrix} \frac{\partial e_1}{\partial g_1} & \frac{\partial e_2}{\partial g_1} \\ \frac{\partial e_1}{\partial g_2} & \frac{\partial e_2}{\partial g_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial M_{xx}}{\partial g_1} - \frac{\partial M_{yy}}{\partial g_1} & \frac{2\partial M_{xy}}{\partial g_1} \\ \frac{\partial M_{xx}}{\partial g_2} - \frac{\partial M_{yy}}{\partial g_2} & \frac{2\partial M_{xy}}{\partial g_2} \end{pmatrix}$$

Shear estimation :

$$\boxed{\langle g \rangle = \langle R \rangle^{-1} \langle e \rangle}$$

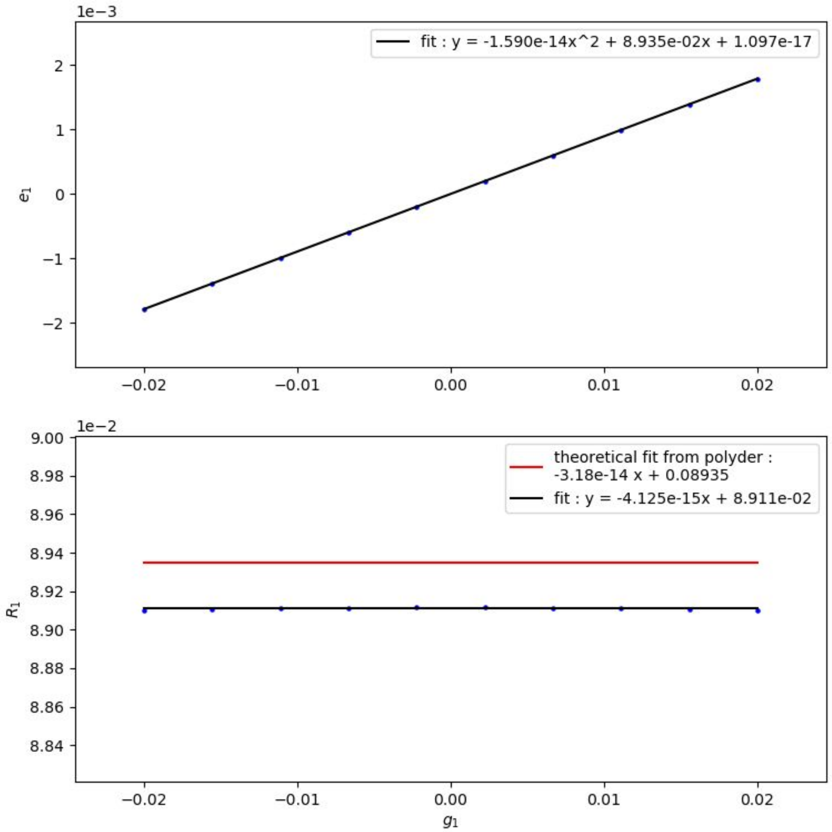
(LSST : we aim to have biases less than ‰)

g1 bias estimation

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} M_{xx} - M_{yy} \\ 2M_{xy} \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \boxed{\frac{\partial e_1}{\partial g_1}} & \frac{\partial e_2}{\partial g_1} \\ \frac{\partial e_1}{\partial g_2} & \frac{\partial e_2}{\partial g_2} \end{pmatrix} = \begin{pmatrix} \boxed{\frac{\partial M_{xx}}{\partial g_1} - \frac{\partial M_{yy}}{\partial g_1}} & \frac{2\partial M_{xy}}{\partial g_1} \\ \frac{\partial M_{xx}}{\partial g_2} - \frac{\partial M_{yy}}{\partial g_2} & \frac{2\partial M_{xy}}{\partial g_2} \end{pmatrix}$$

Observed gap between the derivative of e1 calculated with *polyder* and the value of R1 → explains the bias on the estimate of g1.



Results : Simulations

Parameters :

N = 100 (image size)

Pixel scale = 0.2 arcsec/pixel (similar to LSST)

Ngal = 10 (mean over Ngal galaxies)

Ellip. Intrinsèques = [-0.3 ; 0.3] (zero in average)

Cosmic shear = [-0.02 ; 0.02] (5 or 10 values)

Simulations performed using the *Galsim* package

		PSF (FWHM = 0.8" ; σ = 0.34")		
	Profils	Gaussien	Kolmogorov	Moffat (β = 3.5)
Galaxies	Gaussien	σ = [0.15 – 0.4]	FWHM = [0.36 - 1.0]	FWHM = [0.35 - 1.0]
	Sersic n = 0.5	R_H = [0.18 – 0.5]	R_H = [0.18 – 0.48]	R_H = [0.18 – 0.5]
	Sersic n = 1.5	R_H = [0.14 – 0.54]	R_H = [0.14 – 0.52]	R_H = [0.14 – 0.52]

Results : Trace ratio

Trace Ratio : $$TR = \frac{Tr(M_{image})}{Tr(M_{PSF})}$$

DES Y1 : From TR = 1.5 (and below), shear estimation becomes poor.

To test different galaxy and PSF profiles (Gaussian, Moffat, Sersic...), we look at certain key TR values (between 1.2 and 2.5), to test the limits of the estimator.

- We set the FWHM of the PSF to a given value (0.8''), then vary the FWHM of the galaxy to achieve the desired TR values.

- The FWHM of the weight W is set between 20 and 30% higher than the FWHM of the PSF (if FWHM(PSF) = 0.8'', then FWHM(W) = 1'').

Shear application to seconds moments : *Metacalibration*

Image after shear application : $I(s) = P \otimes [\mathbf{s}(P^{-1} \otimes I)]$

with s the shear operator and P the atmospheric seeing + PSF + pixel response function.

To remove noise amplified by deconvolution, creation of a **dilated PSF Γ** : $\Gamma(x) = P((1 + 2|\gamma|)x)$

New sheared image : $I(s) = \Gamma \otimes [\mathbf{s} * (P^{-1} \otimes I)]$

shear distorsion

→ This procedure introduces correlated anisotropic noise, which can lead to a systematic multiplicative bias.

Estimation method :

$$\boxed{\langle \gamma \rangle \approx \langle \boldsymbol{R}_\gamma \rangle^{-1} \langle \boldsymbol{e} \rangle \approx \langle \boldsymbol{R}_\gamma \rangle^{-1} \langle \boldsymbol{R}_\gamma \gamma \rangle \,.}$$

$$\left\{ \begin{aligned} \langle \boldsymbol{e} \rangle &\approx \int d\boldsymbol{e} \frac{\partial P(\boldsymbol{e})\boldsymbol{e}}{\partial \gamma}\bigg|_{\gamma=0} \gamma \, d\boldsymbol{e} = \langle \boldsymbol{R}_\gamma \gamma \rangle \\ \langle \boldsymbol{R}_\gamma \rangle &= \int \frac{\partial P(\boldsymbol{e})\boldsymbol{e}}{\partial \gamma}\bigg|_{\gamma=0} d\boldsymbol{e} \approx \int d\boldsymbol{e} \left(\frac{P^+ \boldsymbol{e}_i^+ - P^- \boldsymbol{e}_i^-}{\Delta \gamma_j} \right) d\boldsymbol{e} \\ &= \frac{\langle e_i^+ \rangle - \langle e_i^- \rangle}{\Delta \gamma_j}, \qquad (10) \end{aligned} \right.$$

Shear application to seconds moments

Theoretical second moments (after shear application) :

$$\boxed{M(S) = SMS^T = A^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$S = \underbrace{\frac{1}{\sqrt{1-g^2}}}_{= A} \begin{pmatrix} 1+g_1 & g_2 \\ g_2 & 1-g_1 \end{pmatrix}$$

If g1 ≠ 0 and g2 = 0 :

$$a = (1+g_1)^2 M_{xx}$$
$$b = (1-g_1^2) * M_{xy}$$
$$c = (1-g_1^2) * M_{yx}$$
$$d = (1-g_1)^2 * M_{yy}$$

If g2 ≠ 0 and g1 = 0 :

$$a = M_{xx} + g_2 * (M_{xy} + Myx + g_2 * M_{yy})$$
$$b = M_{xy} + g_2 * (M_{xx} + M_{yy} + g_2 * M_{yx})$$
$$c = M_{yx} + g_2 * (M_{xx} + M_{yy} + g_2 * M_{xy})$$
$$d = M_{yy} + g_2 * (M_{xy} + Myx + g_2 * M_{xx})$$

Pixel second moment calculation

Xc : pixel center
M : Jacobian

$$M_{pix} = \int_{pixel} (\vec{X} - \vec{X}_c)(\vec{X} - \vec{X}_c)^T d^2\vec{X} / \int_{pixel} d^2\vec{X}$$

$$(\vec{X} - \vec{X}_c) = M(\vec{i} - \vec{i}_c)$$
$$d^2\vec{X} = |det(M)|\, d^2\vec{i}$$

change of variable

$$\begin{aligned} M_{pix} &= \int_{pixel} (\vec{X} - \vec{X}_c)(\vec{X} - \vec{X}_c)^T d^2\vec{X} \,/\, \int_{pixel} d^2\vec{X} \\ &= |det(M)| \int_{pixel} M(\vec{i} - \vec{i}_c)(\vec{i} - \vec{i}_c)^T M^T\, d^2\vec{i} \,/\, |det(M)| \int_{pixel} d^2\vec{i} \\ &= M\left[\int_{pixel} (\vec{i} - \vec{i}_c)(\vec{i} - \vec{i}_c)^T\, d^2\vec{i}\right] M^T \\ &= MM^T/12 \end{aligned}$$

$$\int_0^1 (x - 1/2)^2 dx = 1/12$$

Position's variance calculation

$$f(x_0, I) = \sum (x_i - x_0) W(x_i - x_0) I_i$$

$$\frac{\partial x_0}{\partial I_i} = -\frac{\frac{\partial f}{\partial I_i}}{\frac{\partial f}{\partial x_0}} \quad \left\{ \begin{aligned} \frac{\partial f}{\partial I_i} &= (x_i - x_0) W_i \\ \frac{\partial f}{\partial x_0} &= -\sum WI + M_W^{-1} \sum (x_i - x_0)(x_i - x_0)^T WI \\ &= -F\mathbb{1} + M_W^{-1} M_P^* F \end{aligned} \right.$$

$$\Rightarrow \quad \frac{\partial x_0}{\partial I_i} = \frac{1}{F}[\mathbb{1} - M_W^{-1} M_P]^{-1} (x_i - x_0) W_i$$

When noise is added (ε) :

$$\begin{aligned} \delta x_0 &= \frac{\partial x_0}{\partial I_i} \epsilon_i \\ &= K(x_i - x_0) W_i \epsilon_i \end{aligned}$$

$$\boxed{\begin{aligned} \sigma(x_0)^2 &= \sum (\frac{\partial x_0}{\partial \epsilon_i})^2 \sigma_{(}\epsilon_i)^2 \\ &= \sum (\frac{\partial x_0}{\partial I_i})^2 \sigma_{noise}^2 \\ &= K^2 \sigma_{noise}^2 \sum (x_i - x_0)(x_i - x_0)^T W_i^2 \\ &= K^2 \sigma_{noise}^2 \frac{M_W}{2} \sum W_i^2 \end{aligned}}$$

Noise bias analytical calculation

$$m_I(I+n) = m_I(I) + \underbrace{\sum_k \left(\frac{dm_I}{dx_0} + \frac{dm_I}{dI_k}\right) n_k}_{\mathbf{0}} + \frac{1}{2}\sum_k \left(\underset{\substack{\downarrow \\ \textbf{4 terms}}}{\frac{d^2 m_I}{dx_0^2}} + \underset{\substack{\downarrow \\ \textbf{2 terms}}}{2\frac{d^2 m_I}{dx_0 dI_k}} + \underbrace{\frac{d^2 m_I}{dI_k^2}}_{\mathbf{0}}\right) n_k^2$$

$$t1 = 2FV_x$$
$$t2 = -2(V_x M_W^{-1} m_i)$$
$$t3 = t2$$
$$t4 = -m_i Tr(V_x M_W^{-1}) + M_{4i}(M_W^{-1} V_x M_W^{-1})$$
$$t5 = -4KM_{W^2}$$
$$t6 = M_{4W^2}(KM_W^{-1})$$

Sersic profile

$$I(r) \sim e^{-b(r/r_e)^{1/n}}$$

re : half light radius

n : Sersic index (controls profile curvature)