Kaluza-Klein truncations of gravity to two dimensions

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Based on works with G. Bossard, G. Inverso, A. Kleinschmidt and H. Samtleben

Anly workshop, April 11th 2024

d=2 ?

- d=2 gravity models are usually simpler than in higher-dimensions.

Einstein-Hilbert $$S_{\mathrm{EH}} \sim \int d^2x\,\sqrt{-g}\,R = \text{total derivative}$$

Simplest non-trivial models: dilaton gravity $$S \sim \int d^2x\,\sqrt{-g}\,[\rho\,R - V(\rho)]$$

$V(\rho) \propto \rho$: Jackiw-Teitelboim model

and D=2 is the lowest possible dimension for:

⟶ Riemann curvature

⟶ Black holes

⟶ Boundaries with dynamics

- There are also models which are much harder to study than in higher-dimensions…

For the past years: interested in d=2 (maximal) supergravities.

(in particular, those that arise as Kaluza-Klein truncations of `string theory').

Motivation: holographic duality between string theory on AdS_2 backgrounds and various supersymmetric matrix quantum mechanics.

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and D=2 is the lowest possible dimension for:

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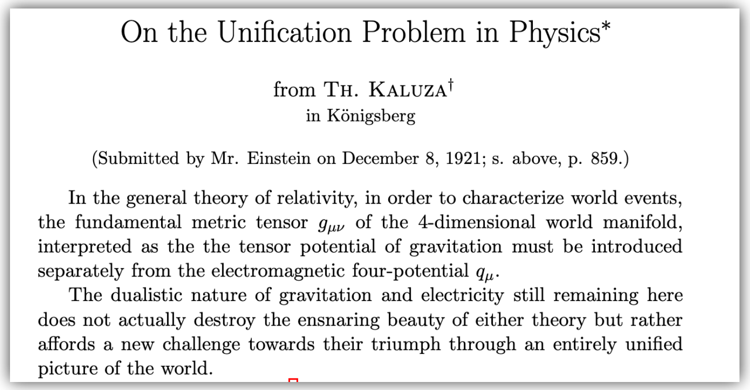
→ Black holes

→ Boundaries with dynamics

Early works:

D=5: **gravity** $\xrightarrow[\text{KK truncation}]{\mathcal{M}_D = \mathcal{M}_d \times S^1}$ **d=4:** **gravity** **+** **Maxwell** **+** **dilaton**



On the Unification Problem in Physics*

from Th. Kaluza†
in Königsberg

(Submitted by Mr. Einstein on December 8, 1921; s. above, p. 859.)

In the general theory of relativity, in order to characterize world events, the fundamental metric tensor $g_{\mu\nu}$ of the 4-dimensional world manifold, interpreted as the the tensor potential of gravitation must be introduced separately from the electromagnetic four-potential q_μ.

The dualistic nature of gravitation and electricity still remaining here does not actually destroy the ensnaring beauty of either theory but rather affords a new challenge towards their triumph through an entirely unified picture of the world.

Kaluza 1921

Klein 1926

What is meant exactly by consistent Kaluza-Klein truncation…?

Toy model

Scalar field Φ on a D-dimensional spacetime $\mathcal{M}_D = \mathcal{M}_d \times S_1$ with coordinates $X^M = (x^\mu \,, y)$

Expand on a Fourier basis: $\Phi(x,y) = \sum_{n\in\mathbb{Z}} \phi_n(x)\, e^{i\frac{n}{R}y}$

circle radius

Dynamics: $\hat{\Box}\,\Phi(x,y) = 0 \quad \longrightarrow \quad \Box\phi_n(x) - \frac{n^2}{R^2}\phi_n(x) = 0$

$\Box + \partial_y^2$

Free massless scalar $\longrightarrow$

$n = 0$: massless mode

+

$n \neq 0$: Infinite tower of massive modes with $m_n = \frac{|n|}{R}$

Consistent Kaluza-Klein truncations: only keep a finite subset of Kaluza-Klein modes that do not source the truncated ones.

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- Always possible on a circle (or torus): truncate all massive modes $\Phi(x,y) = \phi_0(x)$

$\Box\,\phi_n(x) \neq \phi_0(x)$ independent of the dynamics! (ensured by group theory)

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- Generally possible when internal space is a group manifold G:

  Isometry of the internal metric: $\tilde{\mathrm{G}} = \mathrm{G}_L \times \mathrm{G}_R$ — Keep only singlets

DeWitt 1963

Scherk, Schwarz 1979

Toy model

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Consistent Kaluza-Klein truncations: only keep a finite subset of Kaluza-Klein modes that do not source the truncated ones.

Consistent Kaluza-Klein truncations are **few** and **difficult to construct**. Powerful tools:

- Allow to uplift all solutions of the lower-dimensional theory.
- Useful for holography (sphere truncations of gravity).

KK truncation of gravity on T^q

d-dimensional spacetime **q-dimensional internal space**

Consider gravity on a D-dimensional manifold $\mathcal{M}_D$ with coordinates $X^M = (x^\mu, y^m)$

and make the following **gauge choice** for the D-dimensional vielbein:

$$E_M{}^A(x) = \begin{pmatrix} e_\mu{}^\alpha & \rho^{1/q} A_\mu{}^m V_m{}^a \\ 0 & \rho^{1/q} V_m{}^a \end{pmatrix}$$

dilaton **Unimodular matrix** $\in SL(q)$

- Breaks D-dimensional local Lorentz symmetry

$$SO(D-1,1) \longrightarrow SO(d-1,1) \times SO(q)$$

d-dimensional Lorentz **internal symmetry**

- Decomposes into d-dimensional:

vielbein $e_\mu{}^\alpha$ **+** vectors $A_\mu{}^m$ **+** scalars $\rho\,, V_m{}^a$

Fix $\mathcal{M}_D = \mathcal{M}_d \times T^q$ and perform a KK truncation on T^q →

All fields independent of y^m

What are the symmetries inherited by the truncated theory?

KK truncation of gravity on T^q

Consider gravity on a D-dimensional manifold $\mathcal{M}_D$ with coordinates $X^M = (\overset{\text{d-dimensional spacetime}}{x^\mu}, \overset{\text{q-dimensional internal space}}{y^m})$

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- Breaks D-dimensional local Lorentz symmetry

$$SO(D-1,1) \longrightarrow \underset{\substack{\text{d-dimensional} \\ \text{Lorentz}}}{SO(d-1,1)} \times \underset{\substack{\text{internal} \\ \text{symmetry}}}{SO(q)}$$

- Decomposes into d-dimensional:

vielbein $e_\mu{}^\alpha$ + vectors $A_\mu{}^m$ + scalars $\rho\,, V_m{}^a$

D-dimensional diffeomorphisms decompose into:

$$\xi^M(x,y) \xrightarrow{\text{Compatibility with } T^q.}$$

d-dimensional diffeo. : $\quad \xi^\mu(x,y) = \xi(x)$

internal diffeo. : $\quad \xi^m(x,y) = \underset{\substack{U(1)^q \text{ gauge} \\ \text{parameter}}}{L^m(x)} + \underset{\substack{\text{constant} \\ SL(q) \text{ matrix}}}{\Lambda_n{}^m} y^n + \underset{\substack{\text{constant} \\ \text{rescaling}}}{\lambda} y^m$

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Internal diffeomorphims generate:

- Local abelian symmetry $U(1)^q$ $\quad A_\mu^m \longrightarrow A_\mu^m + \partial_\mu L^m(x)$
- Rigid symmetry $\quad R^+ \times SL(q)$ $\quad A_\mu^m \longrightarrow \lambda^{-1/q} \Lambda_n{}^m A_\mu^n \qquad \rho \longrightarrow \lambda\,\rho \qquad V_m{}^a \longrightarrow \Lambda_m{}^n V_n{}^a$

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$$V_m{}^a \longrightarrow V_m{}^b K(x)_b{}^a$$

$SL(q)/SO(q)$ coset space

What about the dynamics?

Dynamics of KK reduced gravity

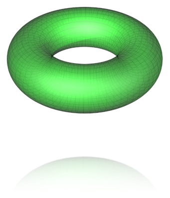
$D = d + q$

Gravity in D=d+q dimensions: $S_D = \frac{1}{\kappa_D^2}\int d^DX\, E\, R^{(D)}$

Substitute KK
truncation Ansatz

$E_M{}^A(x) = \begin{pmatrix} \bullet & \bullet \\ 0 & \bullet \end{pmatrix}$

T^q



$$S_d = \frac{1}{\kappa_d^2}\int d^dx\, e\,\rho\Big[R^{(d)} - \frac{1}{4}\rho^{2/q}(VV^T)_{mn}F^m_{\mu\nu}\,F^{n\,\mu\nu} + \frac{q-1}{q}\rho^{-2}\partial_\mu\,\rho\partial^\mu\rho - \mathrm{tr}(P_\mu P^\mu)\Big]$$

Dynamics in
d dimensions:

d-dimensional gravity + $U(1)^q$ **Maxwell** + **dilaton** + $SL(q)/SO(q)$ **sigma model**

$F^m_{\mu\nu} = 2\,\partial_{[\mu}A^m_{\nu]}$

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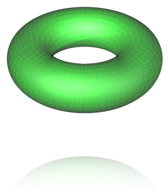

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- Standard kinetic term for the sigma model in terms of the current: $\underbrace{V^{-1}\partial_\mu V}_{\in\, \mathfrak{sl}(q)} = P_\mu + Q_\mu$

$\mathrm{tr}(P_\mu\, P^\mu) = \eta_{\alpha\beta}\, P^\alpha_\mu P^{\alpha\mu}$

$\mathfrak{sl}(q)$
Cartan-Killing
metric

Invariant under rigid SL(q) and local SO(q) transformations with parameters:

$\Lambda \in SL(q)$ $K(x) \in SO(q)$

$V \to \Lambda V - V\,K(x)$

$P_\mu \to K^{-1}P_\mu\, K$

$Q_\mu \to K^{-1}\partial_\mu K + K^{-1}Q_\mu\, K$

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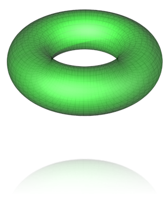

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Rigid symmetry from
KK truncation on T^q : $R^+ \times SL(q)$

- ‘Free’ massless theory: *no non-abelian gauge interactions, no scalar potential.*

⟶ Deformations can be obtained from certain KK truncations on compact manifolds.

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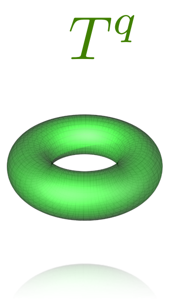

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- *Weyl rescaling:* $e_\mu{}^\alpha \longrightarrow \rho^{\frac{1}{d-2}} e_\mu{}^\alpha$

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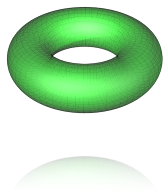

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Rigid symmetry from
KK reduction on T^q : $R^+ \times SL(q)$

D=4:		General relativity	
		$\downarrow$	
d=3:	Rigid symmetry	$SL(2)_E$	*Ehlers 1956*
		$\downarrow$	
d=2:		$\infty - \text{dimensional}$	*Geroch 1987*

Enhancement ?

KK reduction of GR to d=3

$$\mathcal{M}_4 = \mathcal{M}_3 \times S^1$$

Coordinates: $X^M = (x^\mu, y)$

Following the procedure described previously, one finds:

$$S_{\rm EH}[E] \xrightarrow{\text{KK truncation}} S_{d=3} = \int d^3x\, e\, \Big[R^{(3)} - \tfrac{1}{4}\,\varphi^4\, \underbrace{F_{\mu\nu}}_{=\,\partial_{[\mu}A_{\nu]}}\, F^{\mu\nu} - 2\,\varphi^{-2}\partial_\mu\varphi\,\partial^\mu\varphi\Big]$$

vierbein

$$E_M{}^A(x) = \begin{pmatrix} e_\mu{}^\alpha & \varphi\, A_\mu \\ 0 & \varphi \end{pmatrix}$$

d=3 dilaton

d=3 gravity **+** **Maxwell** **+** **dilaton**

In d=3, the vector A_μ can be dualized into a scalar ψ via the duality equation:

$$\boxed{\varphi^4\, F^{\mu\nu} = e^{-1}\epsilon^{\mu\nu\rho}\,\partial_\rho\psi} \longleftarrow$$

Integrability/consistency condition given by the vector field equation: $\partial_\mu\big(e\,\varphi^4\, F^{\mu\nu}\big) = 0$

The d=3 action can then be rewritten without the vector field as:

$$S_{d=3} = \int d^3x\, e\, \Big[R^{(3)} - \tfrac{1}{2}\,\varphi^{-4}\,\partial_\mu\psi\,\partial^\mu\psi - 2\,\varphi^{-2}\partial_\mu\varphi\,\partial^\mu\varphi\Big]$$

Rigid symmetries: scaling $\mathbb{R}^+$: $\begin{array}{l}\varphi \longrightarrow \lambda\varphi \\ \psi \longrightarrow \lambda^2\psi\end{array}$ **+** shift $\psi \longrightarrow \psi + \mathrm{cst}$

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Rigid **symmetry enhanced** to $SL(2)_E$: scaling + shift + hidden sym. generator

Ehlers 1956

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vierbein

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$SL(2)_E/SO(2)$ **sigma model**

$$= \int d^3x\, e\,\left[R^{(3)} - \mathrm{tr}\left[(P_\mu P^\mu)\right]\right]$$

current: $\underset{\in\,\mathfrak{sl}(2)_E}{\underline{V^{-1}\partial_\mu V}} = P_\mu + Q_\mu$ with $V = \begin{pmatrix} \varphi^{-1} & 0 \\ \varphi^{-1}\,\psi & \varphi \end{pmatrix}$

KK reduction: from d=3 to d=2

$\mu = 0\,,1$

$e_\mu{}^\alpha = \text{zweibein}$

Reducing further on S^1 leads to:

$$S_{d=2} = \int d^2x\, e\, \rho\, \left[R^{(2)} - \mathrm{tr}(P_\mu P^\mu)\right]$$

dilaton

d=2 gravity

$SL(2)_E/SO(2)$ **sigma model**

with the current: $$\underbrace{V^{-1}\partial_\mu V}_{\in\, \mathfrak{sl}(2)_E} = P_\mu + \underbrace{Q_\mu}_{\in\, \mathfrak{so}(2)}$$

Specificities of d=2 :

- The vector field is non-dynamical and has been integrated out.
- Constant rescaling of the zweibein is a symmetry.

→ The dilaton cannot be removed.

The action can be simplified by going to the conformal gauge:

$$S_{d=2} = \int d^2x\, \left[\partial_\mu\sigma\, \partial^\mu\rho - \rho\, \mathrm{tr}(P_\mu P^\mu)\right]$$

Conformal gauge

$$e_\mu{}^\alpha = e^\sigma\, \delta_\mu^\alpha$$

σ : conformal factor

Rigid symmetries: $SL(2)_E$ and Weyl rescaling $\sigma \longrightarrow \sigma + \mathrm{cst}$

On-shell symmetries in d=2

Consider the field equation for the coset scalars:

$$\partial_\mu(\rho V P_\mu V^{-1}) = \boxed{\partial^\mu I^\mu_{(1)}} = 0 \xrightarrow{\text{dualisation}} \underset{\in \mathfrak{sl}(2)}{I^\mu_{(1)}} = \rho V P^\mu V^{-1} = \epsilon^{\mu\nu}\partial_\nu \underset{\in \mathfrak{sl}(2)}{Y_1}$$

$$\longrightarrow I^\mu_{(2)} = (\rho\,\tilde\rho\,\delta^\mu_\nu + \epsilon^{\mu\lambda}\eta_{\lambda\nu}\,\rho^2)V P^\nu V^{-1} - \frac{1}{2}\left[Y_1\,, \partial^\mu Y_1\right] = \epsilon^{\mu\nu}\partial_\nu Y_2$$

$\longrightarrow \cdots \longrightarrow$ infinite tower of $\mathfrak{sl}(2)$-valued dual scalar fields Y_n .

Integrability conditions given by the conservation of the currents $I^\mu_{(n)}$.

→ consistency of the tower relies on the field equation.

Infinite number of duality relations encoded in:

- a generating function known as the *linear system* *Belinsky, Zakarov 1978* *Breitenlohner, Maison 1986*

or equivalently

- a *twisted self-duality equation* *Julia, Nicolai 1996* *Paulot 2004*

On-shell symmetries in d=2

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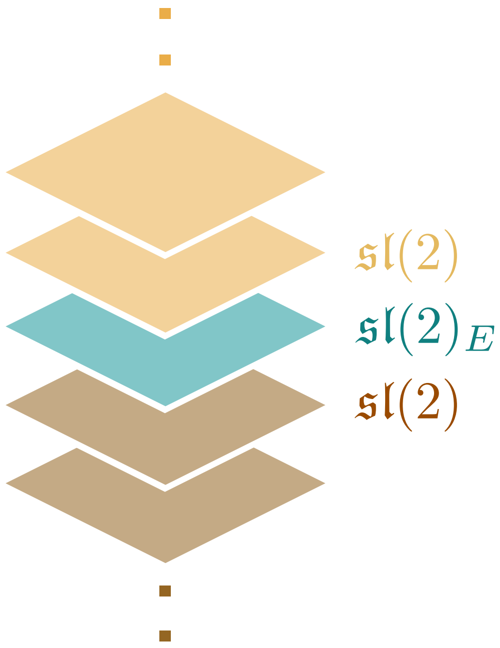
$\rightarrow \cdots \rightarrow$ infinite tower of $\mathfrak{sl}(2)$-valued dual scalar fields Y_n .

Extra rigid symmetries of the field equations:

- $Y_n \longrightarrow Y_n + \underset{\mathfrak{sl}(2)}{\underline{C_n}}$ infinite number of shifts
- *Non-linearly realised ('hidden') symmetries...*

$\widetilde{SL(2)}$
loop group



Classical integrability

On-shell symmetries in d=2

Consider the field equation for the coset scalars:

$$\partial_\mu(\rho V P_\mu V^{-1}) = \partial^\mu I^\mu_{(1)} = 0 \xrightarrow{\text{dualisation}} \underset{\in\,\mathfrak{sl}(2)}{I^\mu_{(1)}} = \rho V P^\mu V^{-1} = \epsilon^{\mu\nu}\partial_\nu \underset{\in\,\mathfrak{sl}(2)}{Y_1}$$

$$\longrightarrow \quad I^\mu_{(2)} = (\rho\,\tilde\rho\,\delta^\mu_\nu + \epsilon^{\mu\lambda}\eta_{\lambda\nu}\,\rho^2)V P^\nu V^{-1} - \frac{1}{2}\left[Y_1\,,\partial^\mu Y_1\right] = \epsilon^{\mu\nu}\partial_\nu Y_2$$

→ ⋯ → infinite tower of $\mathfrak{sl}(2)$-valued dual scalar fields Y_n .

Integrability conditions given by the conservation of the currents $I^\mu_{(n)}$.

→ consistency of the tower relies on the field equation.

Full on-shell symmetry of the theory: $\widetilde{SL(2)}$ loop group and $\sigma \longrightarrow \sigma + c$ Weyl rescaling

central extension

Affine group

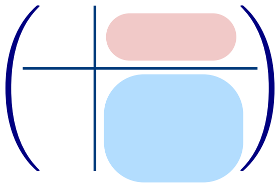
- Symmetry of space of stationary axisymmetric solutions of GR.

Geroch 1987

Always two paths to d=2

D=2+q: Gravity

$\searrow T^{q-1}$



d=3: $R^+ \times SL(q-1)$ symmetry

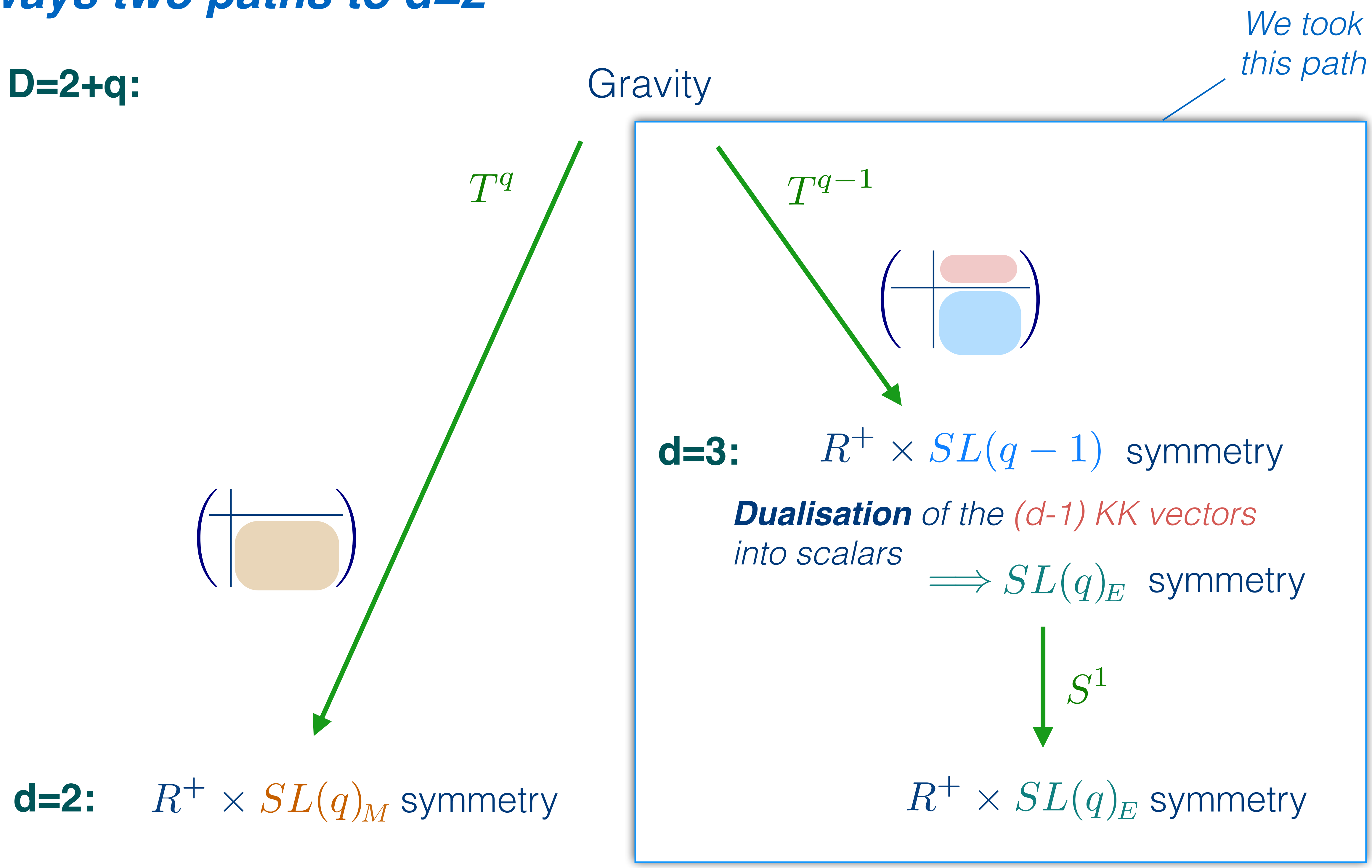
Dualisation *of the (d-1) KK vectors into scalars*

$\Longrightarrow SL(q)_E$ symmetry

$\downarrow S^1$

d=2: $R^+ \times SL(q)_E$ symmetry

Always two paths to d=2



We took this path

D=2+q: Gravity

T^q

T^{q-1}

d=3: $R^+ \times SL(q-1)$ symmetry

Dualisation *of the (d-1) KK vectors into scalars*

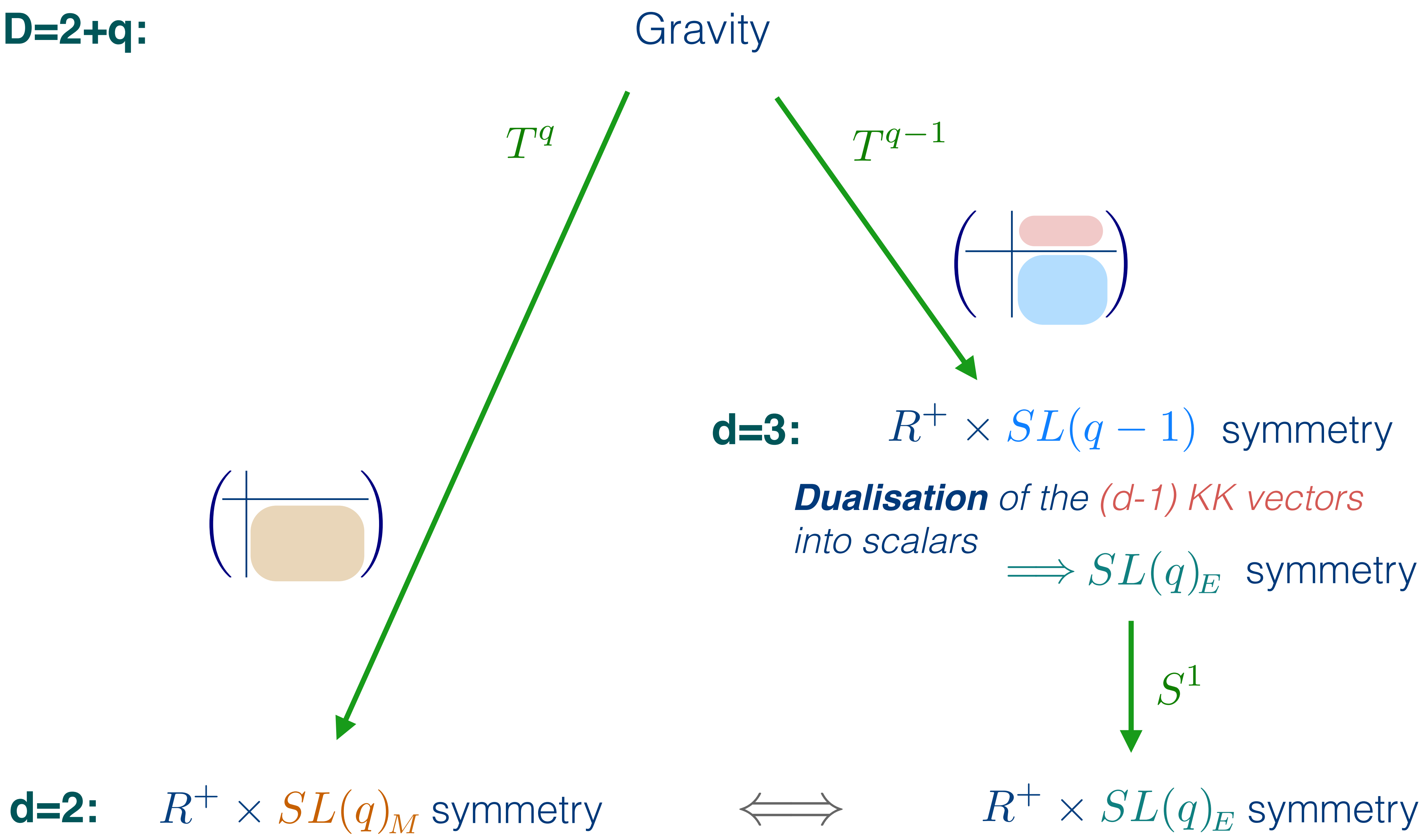
$\Longrightarrow SL(q)_E$ symmetry

S^1

$R^+ \times SL(q)_E$ symmetry

d=2: $R^+ \times SL(q)_M$ symmetry

Always two paths to d=2



D=2+q: Gravity

T^q

T^{q-1}

d=3: $R^+ \times SL(q-1)$ symmetry

Dualisation *of the (d-1) KK vectors into scalars*

$\Longrightarrow SL(q)_E$ symmetry

S^1

d=2: $R^+ \times SL(q)_M$ symmetry $\Longleftrightarrow$ $R^+ \times SL(q)_E$ symmetry

Realised on scalars dual to each other

Two (on-shell) equivalent versions of the D=2 theory

Realising the two *SL(q)* simultaneously requires an infinite number of dual scalars

" $SL(q)_M \times SL(q)_E = \widetilde{SL(q)}$ " loop group

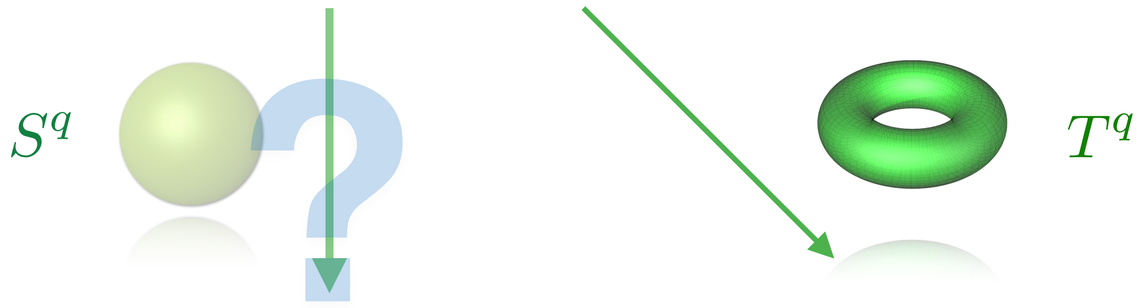
Consistent sphere truncations

Necessary condition:

$$SO(q+1) \subset \mathcal{G}$$

→ Requires symmetry enhancement of $\mathcal{G}$.

Cvetic, Gibbons, Lü, Pope 2003



D: Gravity theory

d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity

Consistent sphere truncations

Necessary condition:

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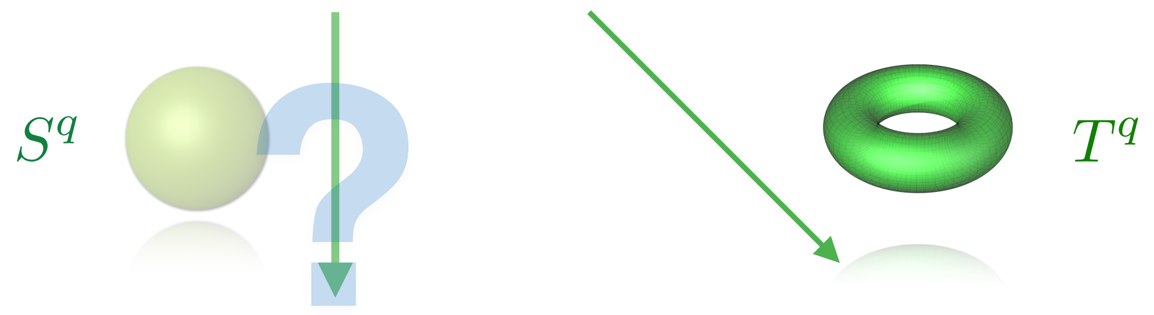


D: Gravity theory

S^q

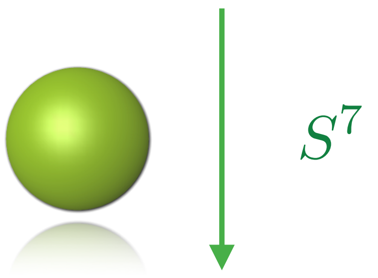
T^q

d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

<u>Only rare examples</u>: must start from matter-coupled gravity

D=10 Gravity + 3-form



S^7

d=5 Gravity + SO(8) YM + scalars

De Wit, Nicolai 1980's

Maximal SUSY context:

$AdS_4 \times S^7$ ⟷ M2-branes

Prototypical examples of AdS/CFT

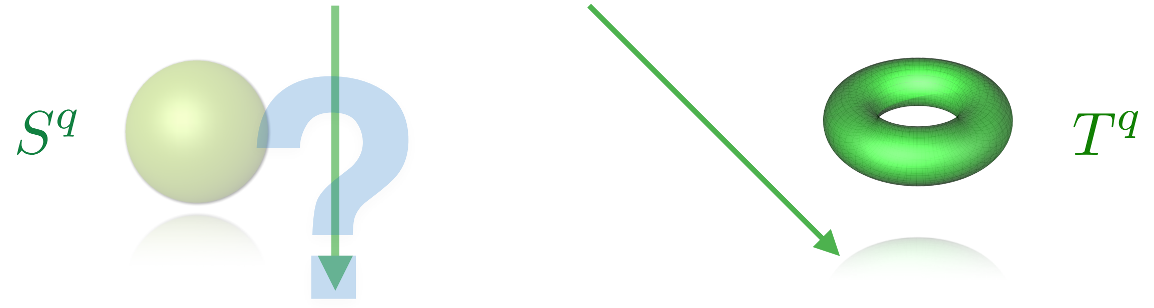
Consistent sphere truncations

Necessary condition:

$SO(q+1) \subset \mathcal{G}$

→ Requires symmetry enhancement of $\mathcal{G}$.

D: Gravity theory

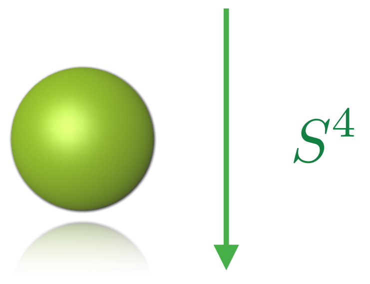


d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity

D=11 Gravity + 3-form



d=7 Gravity + SO(5) YM + scalars+…

Nastase, Vaman, van Nieuwenhuizen 1999

Maximal SUSY context:

$AdS_4 \times S^7$ ⟷ M2-branes

$AdS_7 \times S^4$ ⟷ M5-branes

Prototypical examples of AdS/CFT

Consistent sphere truncations

Necessary condition:

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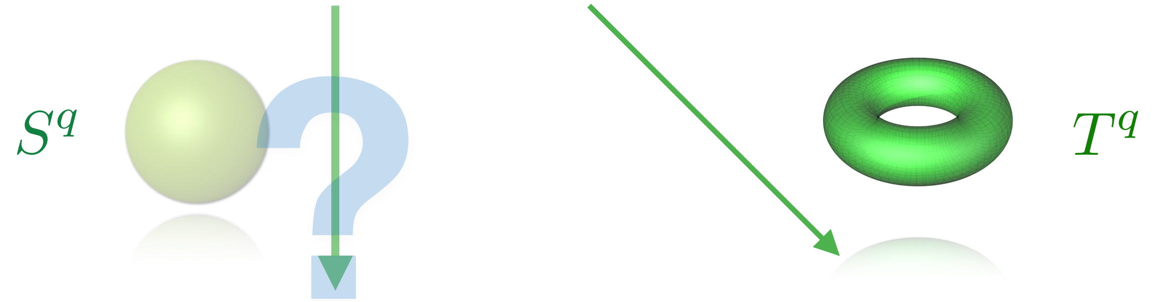
D: Gravity theory

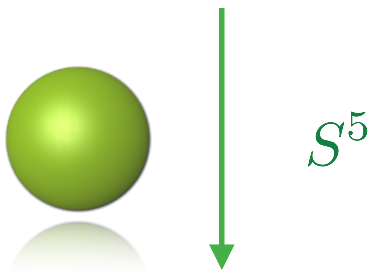


d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity

D=11 Gravity + 5-form



d=4 Gravity + SO(6) YM + scalars

Baguet, Hohm, Samtleben 2015

Maximal SUSY context:

$AdS_4 \times S^7$ ⟷ M2-branes

$AdS_7 \times S^4$ ⟷ M5-branes

$AdS_5 \times S^5$ ⟷ D3-branes

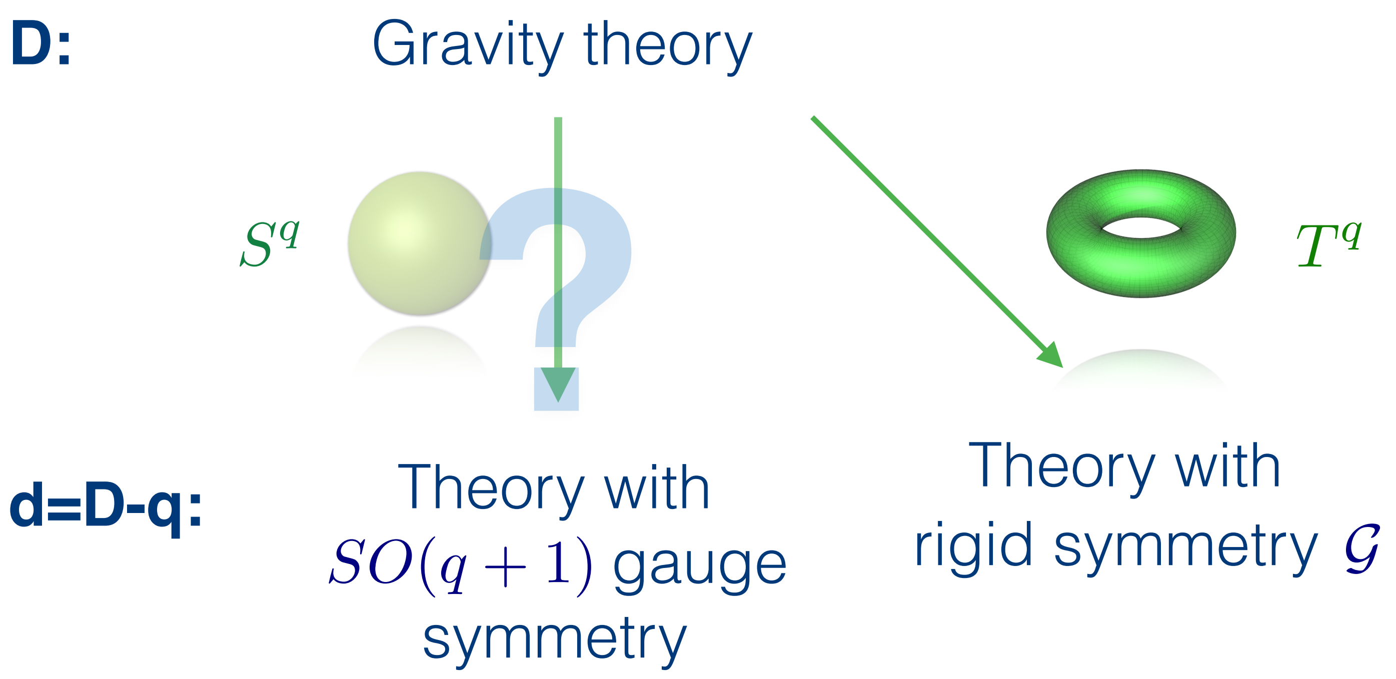
Prototypical examples of AdS/CFT

Consistent sphere truncations

Necessary condition:

$SO(q+1) \subset \mathcal{G}$

→ Requires symmetry enhancement of $\mathcal{G}$.



D: Gravity theory

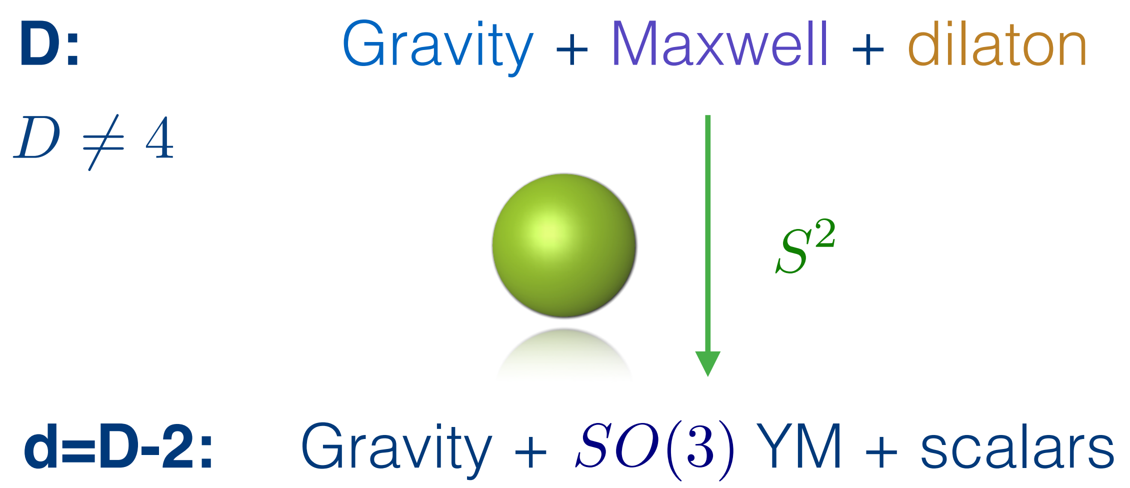
S^q

T^q

d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity



D: Gravity + Maxwell + dilaton

$D \neq 4$

S^2

d=D-2: Gravity + $SO(3)$ YM + scalars

+ 2 other examples

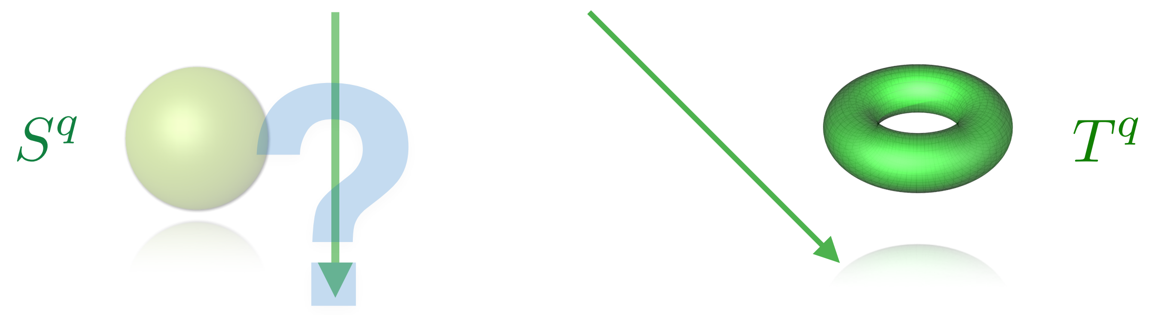
Cvetic, Lü, Pope 2000

Consistent sphere truncations

Necessary condition:

$SO(q+1) \subset \mathcal{G}$

→ Requires symmetry enhancement of $\mathcal{G}$.



D: Gravity theory

S^q

T^q

d=D-q:

Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity

D: Gravity + Maxwell + dilaton

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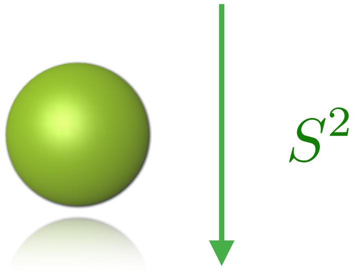
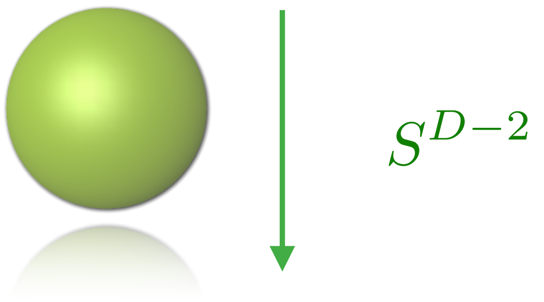


S^2

d=D-2: Gravity + $SO(3)$ YM + scalars

D: Gravity + Maxwell + dilaton



S^{D-2}

d=2: Gravity + $SO(D-1)$ YM + scalars

Ciceri, Samtleben 2023

Consistent sphere truncations

Necessary condition:

$SO(q+1) \subset \mathcal{G}$

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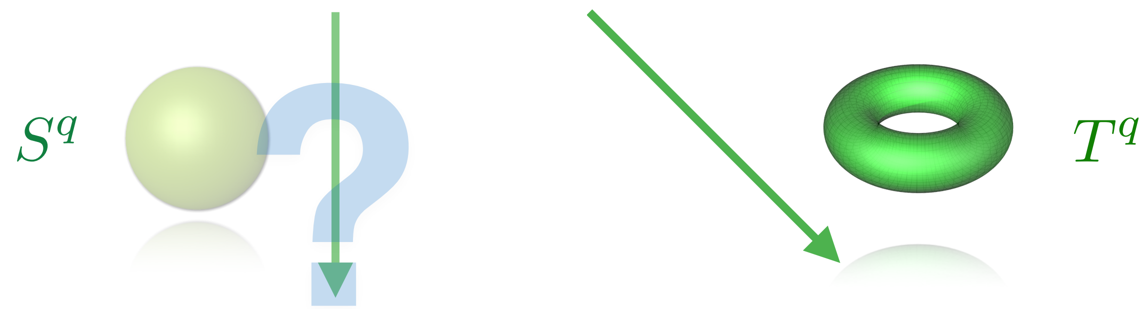
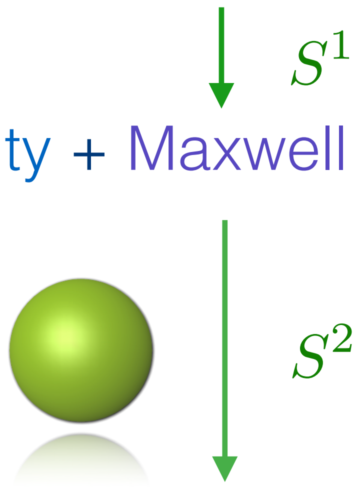


D: Gravity theory

d=D-q: Theory with $SO(q+1)$ gauge symmetry | Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity

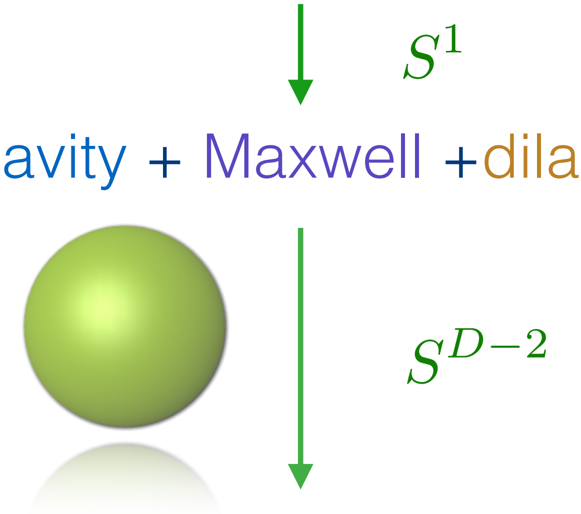


D+1: Gravity

D: Gravity + Maxwell + dilaton

$D \neq 4$

d=D-2: Gravity + $SO(3)$ YM + scalars



D+1: Gravity

D: Gravity + Maxwell +dilaton

d=2: Gravity + $SO(D-1)$ YM + scalars

Ciceri, Samtleben 2023

Consistent sphere truncations

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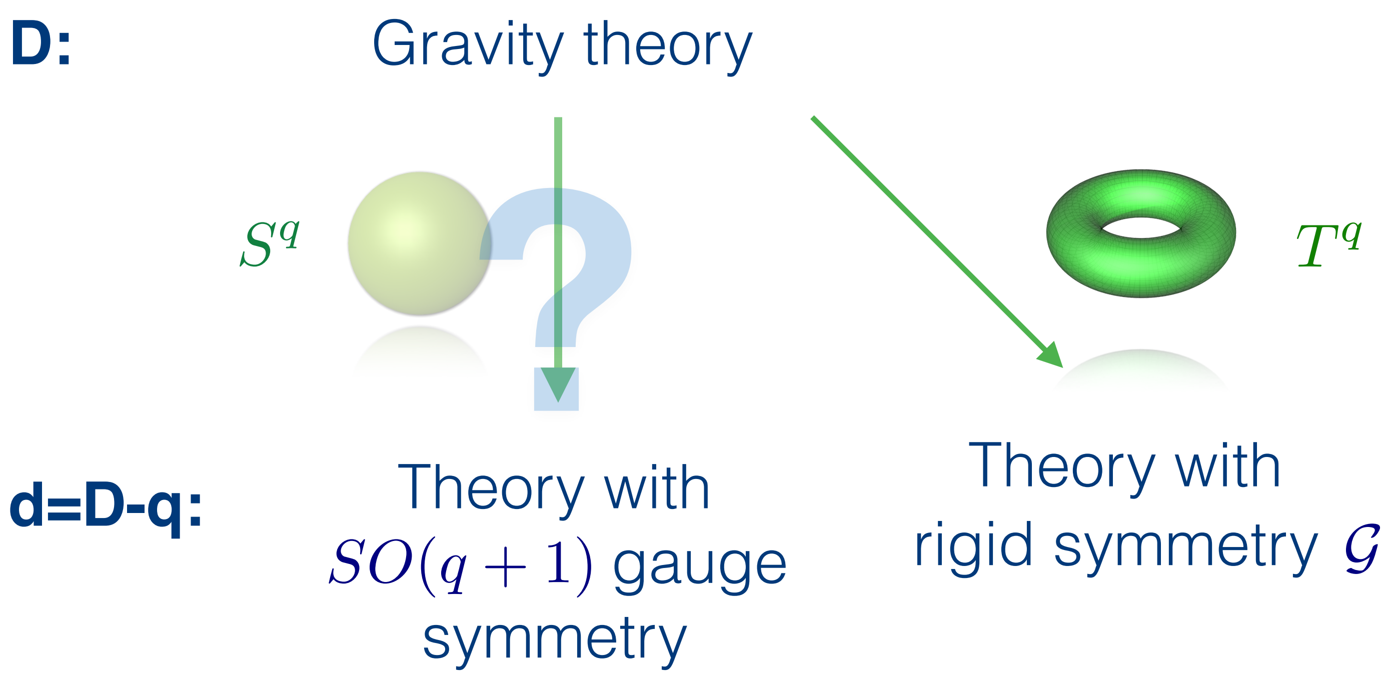


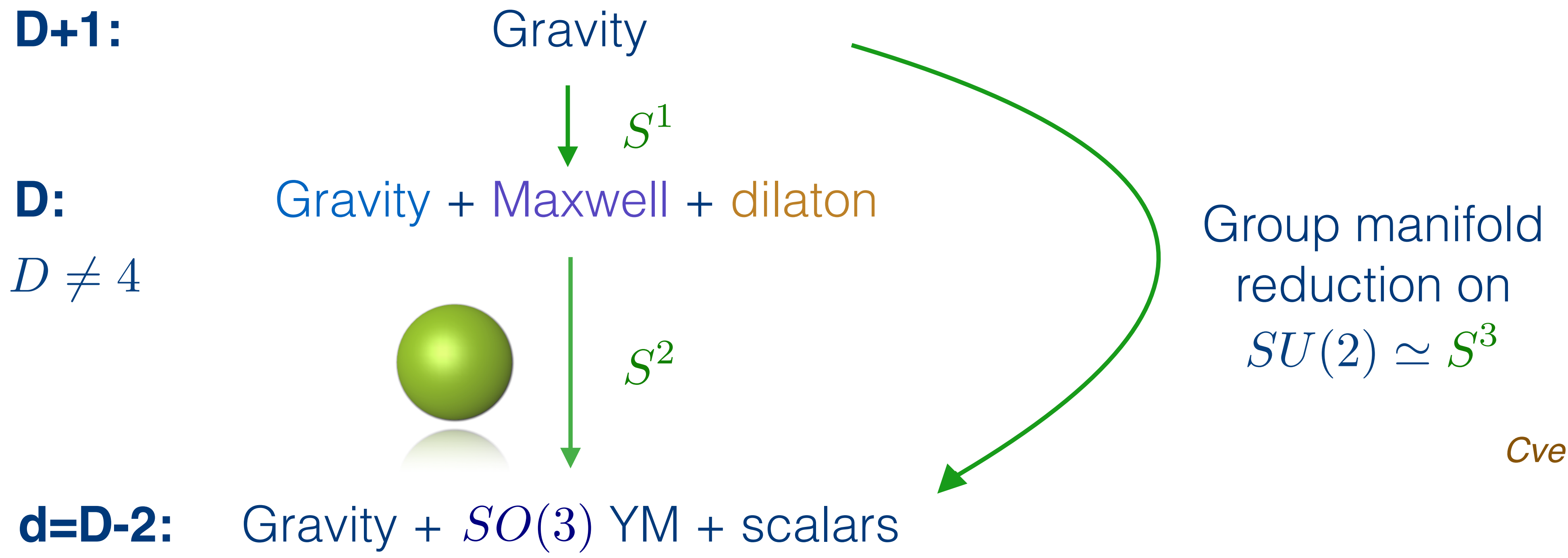
D: Gravity theory

S^q T^q

d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity



D+1: Gravity

S^1

D: Gravity + Maxwell + dilaton

$D \neq 4$

S^2

d=D-2: Gravity + $SO(3)$ YM + scalars

Group manifold reduction on $SU(2) \simeq S^3$

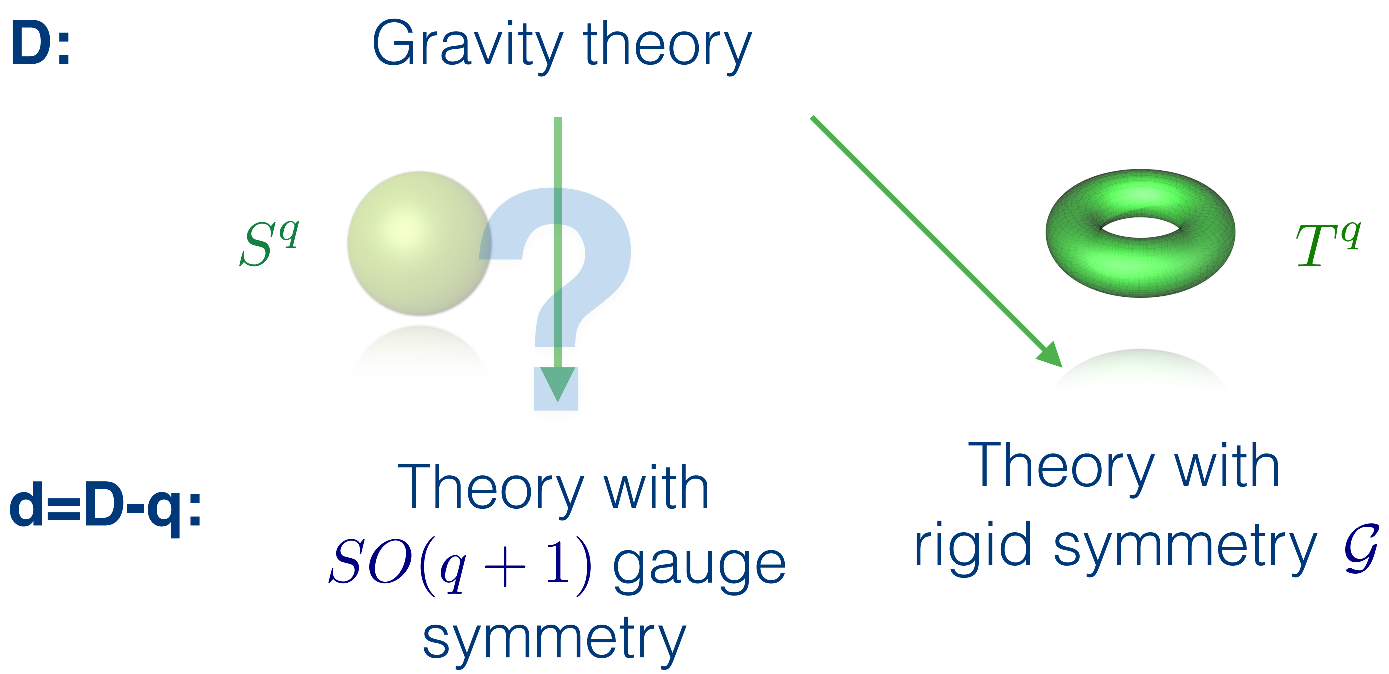
Cvetic, Gibbons, Lü, Pope 2003

Consistent sphere truncations

Necessary condition:

$SO(q+1) \subset \mathcal{G}$

→ Requires symmetry enhancement of $\mathcal{G}$.



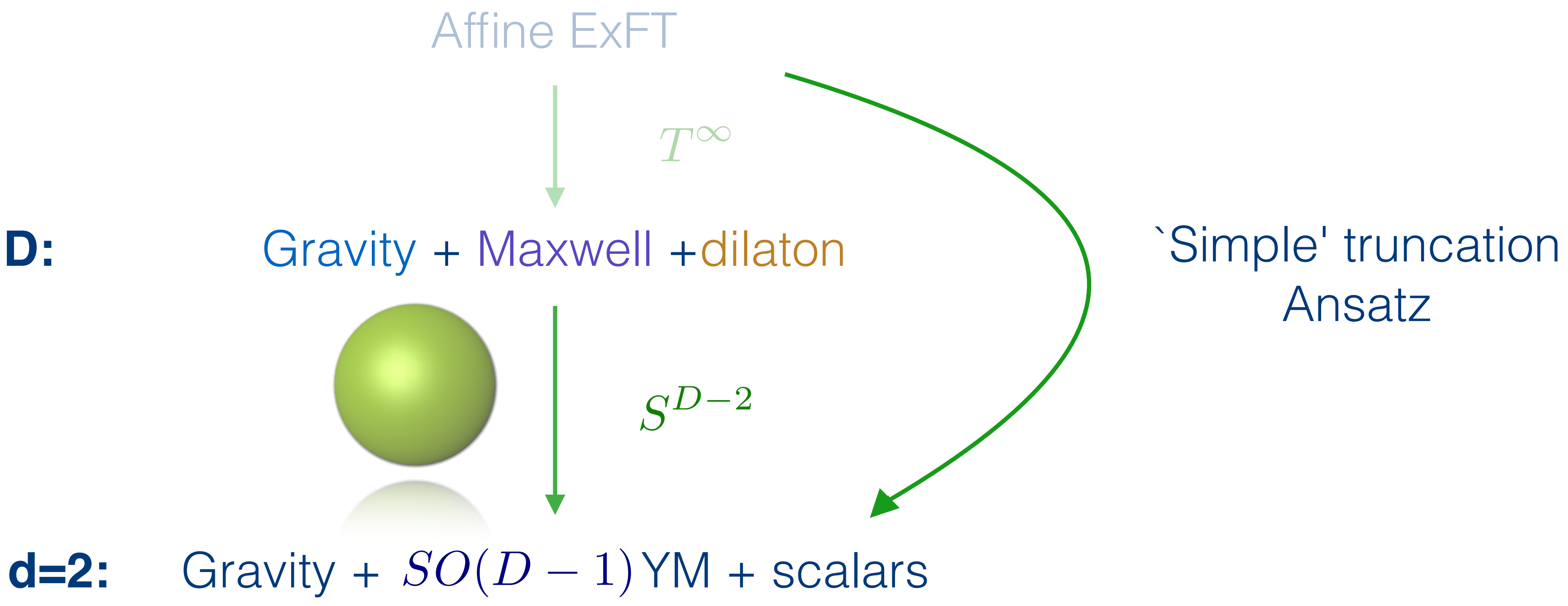
D: Gravity theory

S^q ? T^q

d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

Only rare examples: must start from matter-coupled gravity



Affine ExFT

T^∞

D: Gravity + Maxwell +dilaton

S^{D-2}

`Simple' truncation Ansatz

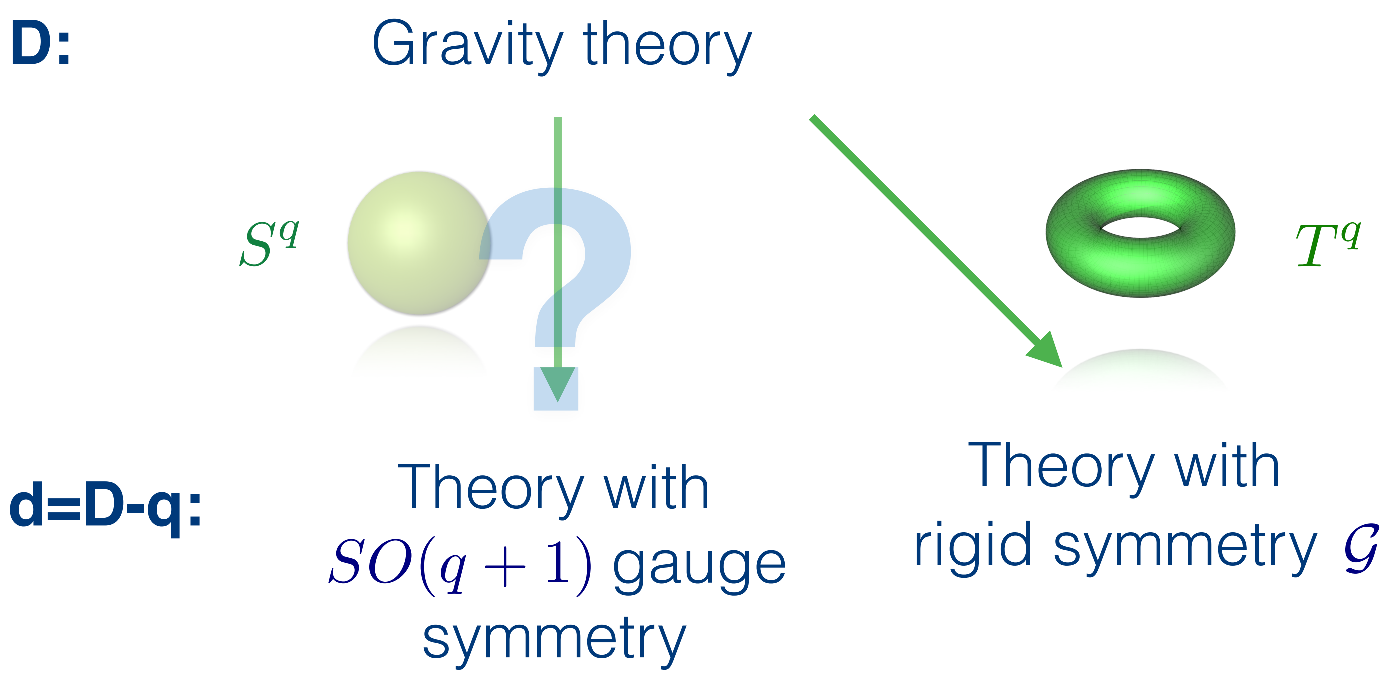
d=2: Gravity + $SO(D-1)$ YM + scalars

Consistent sphere truncations

Necessary condition:

$SO(q+1) \subset \mathcal{G}$

→ Requires symmetry enhancement of $\mathcal{G}$.



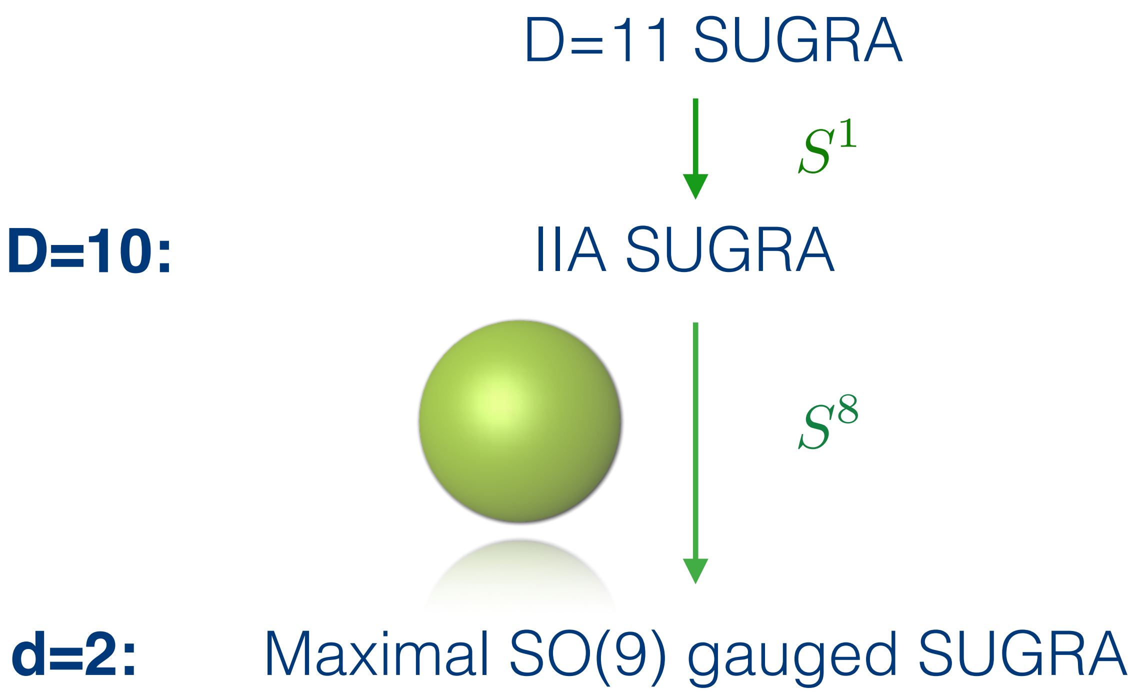
D: Gravity theory

S^q ? T^q

d=D-q: Theory with $SO(q+1)$ gauge symmetry

Theory with rigid symmetry $\mathcal{G}$

<u>Only rare examples</u>: must start from matter-coupled gravity



D=11 SUGRA

$\downarrow S^1$

D=10: IIA SUGRA

$\downarrow S^8$

d=2: Maximal SO(9) gauged SUGRA

Maximal SUSY context:

‘ $AdS_2 \times S^8$ ’ ⟷ D0-branes

Holographic description of BFSS matrix quantum mechanics

Bossard, Ciceri, Inverso, Kleinschmidt, Samtleben 2021

Bossard, Ciceri, Inverso, Kleinschmidt 2023

Thank you for your attention

and

congratulations Anamaria.