Integrability and Mathematical/Theoretical Physics

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IPhT@60
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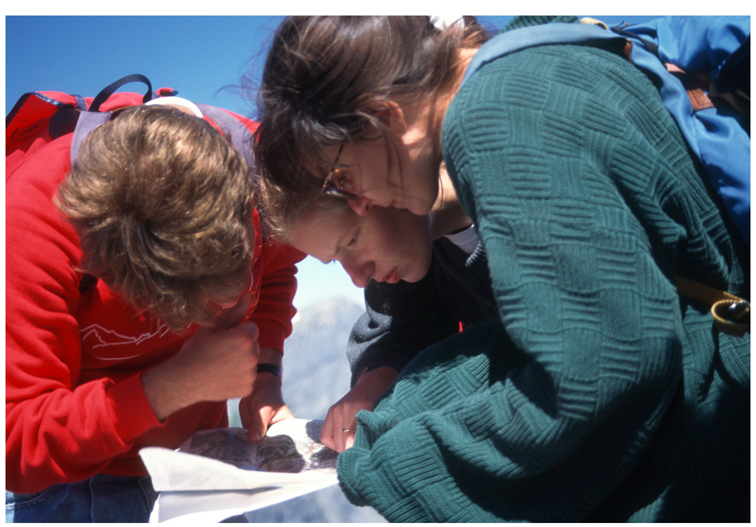
My own story at the IPhT

- Arrived for a DEA (M2) internship in 1993, continued with a PhD with D. Bernard and V. Pasquier (and with J.-B. Zuber as official advisor)
- Joined the SPhT as a permanent researcher in 1998
- I work(ed) on quantum integrable models, 2dCFTs, quantum Hall effect, disordered systems, non-unitary models, quantum dimer models, 2d gravity, AdS/CFT integrability, etc



1994, F. David / IPhT

2018, L. Godart / CEA

Integrability at the SPM/SPhT/IPhT

[using infinite-dimensional symmetries to find exact solutions to problems in physics]

- The towering figure of **integrability** in our institute is **M. Gaudin** (1931-2023) who joined the SPM in 1958 and the SPhT in 1963 [fermion gas with spin with delta interaction; nested Bethe ansatz; norm of the Bethe wave functions - Gaudin determinant; thermodynamic Bethe ansatz for the Heisenberg model, Gaudin model, field theory in curved space]. His book, **La fonction d'onde de Bethe** is a classic reference for the field

- In the 80's and the beginning of the 90's (the orange preprint period) the **matrix models** and the **2dCFTs** and were rapidly developing, in particular at the SPhT (M.L.Mehta, **M. Gaudin**, E. Brézin, F. David, C. Itzykson, J. Zinn-Justin, J.-M. Normand, **J.-B. Zuber**, **M. Bauer**, **P. Di Francesco**, **V. Pasquier, H. Saleur, I. Kostov, B. Eynard, M. Bergère**…). Integrability and quantum groups are closely related to these topics and were very active in the lab. **Integrable field theories** was one of the focal points of the activity of **Al. Zamolodchikov,** who was briefly member of the SPhT.

- Integrability is closely related to the description of **geometrical objects** (polymers, percolation), **SLE, disordered systems, super-spin chains**, **non-unitary systems** (B. Duplantier, **H. Saleur, I. Kostov, D. Bernard, D. Serban, V. Schomerus, J. Jacobsen, S. Ribault, M. Bauer,**…)

Integrability at the SPM/SPhT/IPhT

- Integrability is applied successfully to **stochastic processes** (B. Derrida, **K. Mallik, V. Pasquier**,…), transport in low dimensional systems (**H. Saleur**) and out of equilibrium dynamics (**V. Pasquier**, **G. Misguich**, J.-M. Luck,…)

- Other important related fields are **combinatorics, random maps, quantum gravity** (**P. Di Francesco**, E. Guitter, **J. Bouttier, S. Ramassamy,** F. David, B. Duplantier), **topological recursion** (**B. Eynard**), **representation theory** (**P. Di Francesco**, **R. Kedem**)

- In the mid-90's integrability started to be applied to **high energy QCD** (**G. Korchemsky**; then at LPT Orsay, joined the IPhT in 2009) and since the beginning of 00's to **gauge/string dualities** (**D. Serban, G. Korchemsky, I. Kostov**), and computation of **amplitudes** (D. Kosower, **G. Korchemsky**, P. Vanhove, M. von Hippel)

- Integrability is also important in string theory, also for the **AdS/CFT correspondence** (I. Bena, M. Guica, **V. Schomerus**), **TTbar deformations** (M. Guica)

- And last but not least, it has close connections with **bootstrap in CFTs** (**S. Ribault, H. Saleur, J. Jacobsen**, M. Guica, E. Perlmutter, D. Mazač,…)

- **Correlation functions of gauge invariant operators in planar N=4 SYM theory from integrability**

  past collaborators: **I. Kostov, D. Volin, O. Foda, Y. Jiang, A. Petrovskii, F. Loebbert, S. Komatsu D.-L. Vu, V. Petkova,**

  present collaborators: **I. Kostov, G. Lefundes, F. Levkovich-Maslyuk, A. Klemenchuk, B. Basso**

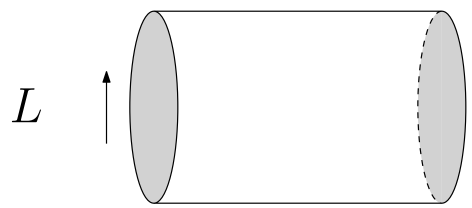
- **Integrable long range models with extended symmetry**

  past collaborators: **D. Bernard, V. Pasquier, F. Lesage, R. Santachiara, B. Estienne, J. Lamers**

  present collaborators: **J. Lamers, G. Ferrando, F. Levkovich-Maslyuk, A. Toufik, A. Ben Moussa,**

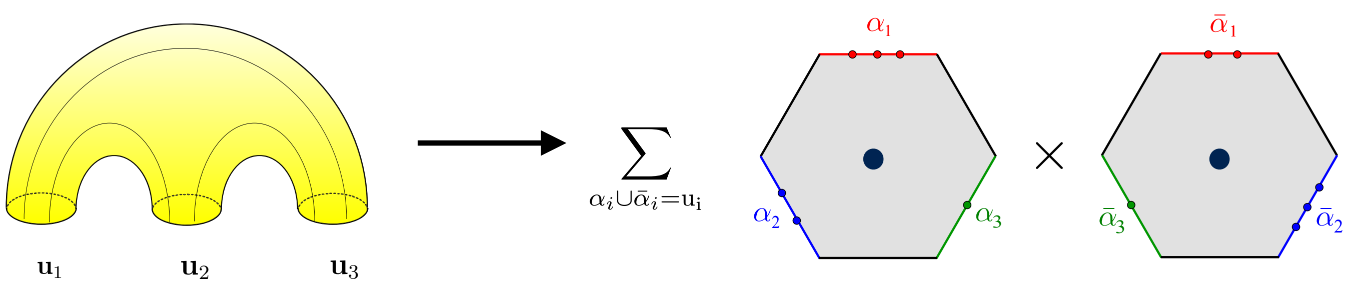
Correlation functions in planar N=4 SYM from integrability

- The planar N=4 SYM theory is the ideal playground for testing ideas for dualities between gauge and string theories ($\rightarrow$ finding the appropriate degrees of freedom)
- The problem of finding the spectrum of conformal dimensions of N=4 SYM was reduced to a set of difference equations involving 8 complex functions (**quantum spectral curve**)
- This was achieved by mapping the problem to a **long range spin chain**/an **integrable 2d field theory** and **reformulating the Thermodynamic Bethe Ansatz (TBA)**. Considering the spectral problem amounts to putting the problem on an infinite cylinder with circumference L, with prescribed insertions at infinity

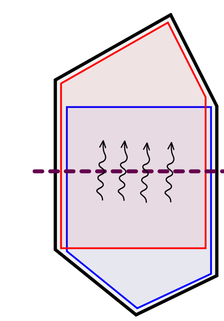


Correlation functions in planar N=4 SYM from integrability

- for more complicated objects (correlation functions, amplitudes, Wilson loops, form factors) “tailoring” procedures were devised

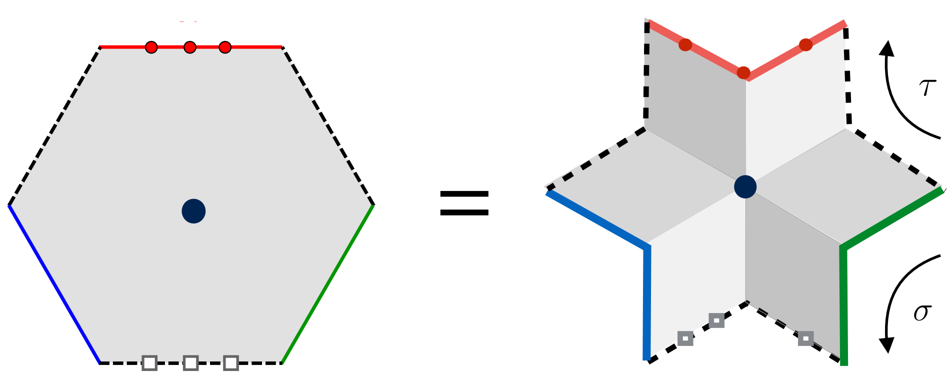


3-pt function



6-gluon amplitude

- the elementary building blocks are non-local form factors with curvature excess



- gluing involves summing over an infinite number of virtual particles

Correlation functions in planar N=4 SYM from integrability

- gluing involves summing over an infinite number of virtual particles: this was successfully done for the octagon → result in terms of Fredholm determinant (~fermions at finite temperature)

  [Kostov, Petkova, D.S. 19; Belitsky, Korchemsky, 19-22]

- but the resummation procedure is heavy and not efficient for e.g. the three point function; conjectures by **[Basso, Georgoudis, Klemenchuk-Sueiro, 22]**

- idea: use/devise a formalism which is friendlier to the TBA and the spectral curve , namely the **separation of variables;** preliminary results by **[Bercini, Homrich, Vieira, 22]**

- the difficulty consists in working with long-range deformations of spin chains/higher rank and supersymmetric

- work in progress with **I. Kostov, G. Lefundes, F. Levkovich-Maslyuk**

Long-range interacting models with extended symmetry

- We are studying a family of models based on **[Bernard, Gaudin, Haldane, Pasquier, 93]**

- They include the spin-Calogero-Sutherland model and the Haldane-Shastry model and their **q-deformations** (difference/anisotropic);

- One of the features of these models is the existence of an extended symmetry: **Yangian** in the undeformed case and **quantum affine** symmetry for the deformations [the integrals of motion are generated by the quantum determinant of the monodromy matrix]

- One purpose is to understand separation of variables in long range spin chains solvable by Bethe Ansatz. We find **extra integrals of motion** inside the spin CS models [generated by the trace of the (twisted) monodromy matrix] which can be **diagonalised by Bethe Ansatz** **[Ferrando, Lamers, Levkovich-Maslyuk, D.S., 23]**

- On the side of the deformed models, of particular interest are the cases where q is a root of unity, when extra (super) symmetry may occur. We have looked in detail first at the case q=i **[Ben Moussa, Lamers, D.S., Toufik, to appear].**

Long-range interacting models with extended symmetry

[Bernard, Gaudin, Haldane, Pasquier, 93]

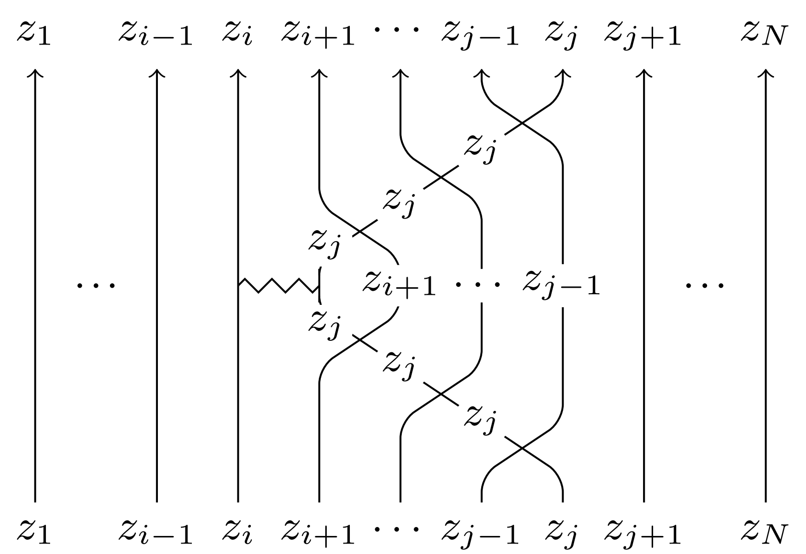
[Uglov 95; Lamers 18; Lamers, Pasquier, D.S., 22]

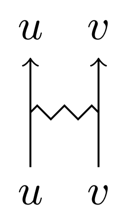
We are studying a Heisenberg anisotropic (XXZ-type) spin chain with **multi-spin interaction**

- N spins 1/2 on a circle with positions given by the N-th root of unity $z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

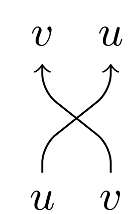
- **anisotropy** controlled by a parameter $\Delta = \dfrac{\mathrm{q} + \mathrm{q}^{-1}}{2}$

$$\widetilde{\mathrm{H}}^{\mathrm{L}} = \sum_{i<j} V(z_i, z_j)\, S^{\mathrm{L}}_{[i,j]} \qquad\qquad V(z_i, z_j) = \frac{z_i\, z_j}{(\mathsf{q}\, z_i - \mathsf{q}^{-1} z_j)(\mathsf{q}^{-1} z_i - \mathsf{q}\, z_j)}$$

$S^{\mathrm{L}}_{[i,j]} :=$ 



$$:= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{q}^{-1} & -1 & 0 \\ 0 & -1 & \mathrm{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = e_i$$



$:= \check{R}(u/v)$

$$\check{\mathrm{R}}_{k,k+1}(u) = 1 - f(u)\, e_k\ , \qquad f(u) = \frac{u-1}{\mathrm{q}\, u - \mathrm{q}^{-1}}$$

based on **Temperley-Lieb** algebra but **long range**

Long-range interacting models with extended symmetry

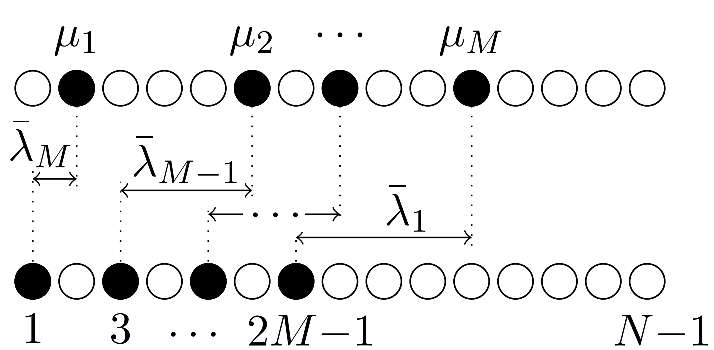
- At q=1 this becomes the **Haldane-Shastry model**

$$H_{\rm HS} = \sum_{i\neq j} V(z_i, z_j)\; P_{ij}$$

$$V(z_i, z_j) = \frac{z_i z_j}{(z_i - z_j)^2} = -\frac{4}{\sin^2 \pi(i-j)/N}$$

$$P_{jk} = \frac{1}{2}\left(\sigma_j^a \sigma_k^a + 1\right)$$ **spin permutation**

- The model is **Yangian symmetric** (huge degeneracy) and the spectrum is encoded by **motifs**:



M magnon motif consisting of M integers

$$\mu_{m+1} > \mu_m + 1$$

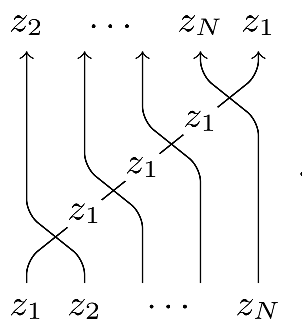
statistical interaction; ideal magnons

$$E(\mu) - E_0 = \sum_{m=1}^{M} \varepsilon(\mu_m) = \sum_{m=1}^{M} \mu_m (N - \mu_m)$$

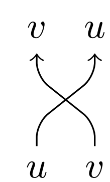
- Yangian and **spinon** description of $su(2)_{k=1}$ CFT: **[Bernard, Pasquier, D.S. 94]**

Long-range interacting models with extended symmetry

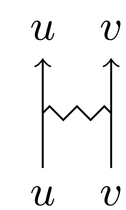
-at $q \neq 1$ the model is **not translationally invariant** (but there is a **q-translation operator**, G)

$G =$ 

$$G^N = 1$$

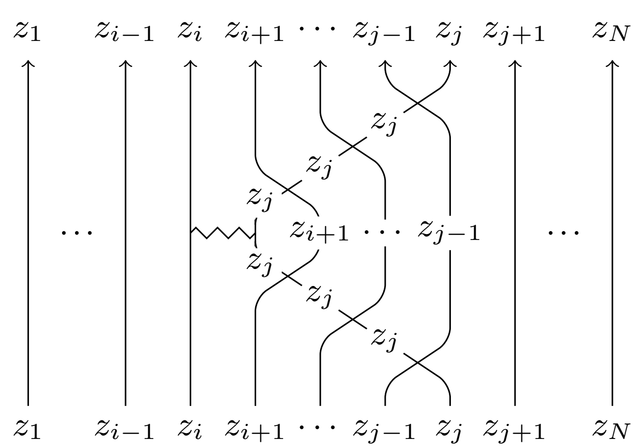


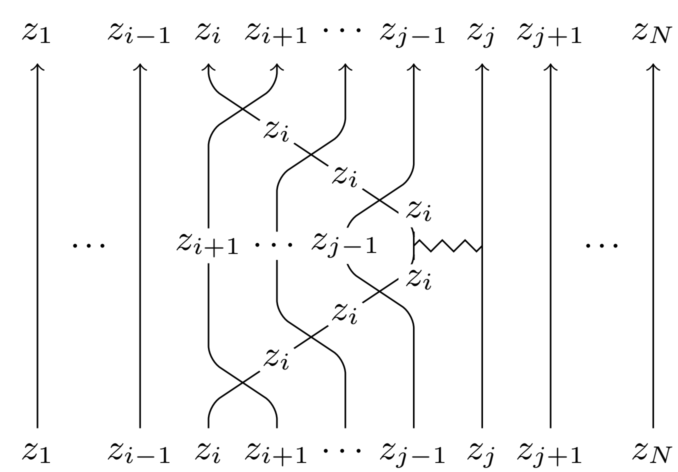
$:= \check{R}(u/v)$



$= e_i$

- there exists **another Hamiltonian with the opposite "chirality"**

$S^{\mathrm{L}}_{[i,j]} :=$ 

$S^{\mathrm{R}}_{[i,j]} :=$ 

.

$$\widetilde{\mathrm{H}}^{\mathrm{L}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\; S^{\mathrm{L}}_{[i,j]}$$

$$\widetilde{\mathrm{H}}^{\mathrm{R}} = \frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\; S^{\mathrm{R}}_{[i,j]}$$

$$[\widetilde{\mathrm{H}}^{\mathrm{L}}, \widetilde{\mathrm{H}}^{\mathrm{R}}] = 0$$

$$[G, \widetilde{\mathrm{H}}^{\mathrm{L/R}}] = 0$$

Fermionic model with extended symmetry

- the model is formulated in terms of the generators of the **Temperley-Lieb algebra** :

$$
\begin{aligned}
e_j^2 &= (\mathrm{q} + \mathrm{q}^{-1})\, e_j \\
e_j e_{j\pm 1} e_j &= e_j \,, \\
e_j e_k &= e_k e_j \qquad (\text{for } j \neq k,\ k \pm 1)\,,
\end{aligned}
$$

$$
\check{\mathrm{R}}_{k,k+1}(u) = 1 - f(u)\, e_k \;, \qquad f(u) = \frac{u-1}{\mathrm{q}\,u - \mathrm{q}^{-1}}
$$

- **at q=i** we have $e_j^2 = 0$ and $f(u^{-1}) = -f(u)$ ⟶ great simplification

- non-unitary **fermions** $\{f_j^+, f_k\} = (-1)^j\, \delta_{jk}$ $\quad e_k = (f_k^+ + f_{k+1}^+)(f_k + f_{k+1})$

- the interaction can be expressed in terms of **nested commutators of TL generators,** which are **quadratic in fermions**

$$
e_{[l,m+1]} := [e_l, [e_{l+1}, \dots [e_{m-1}, e_m] \dots]] = [[\dots[e_l, e_{l+1}], \dots e_{m-1}], e_m]
$$

↑ Jacobi identity and TL algebra

Fermionic model with extended symmetry

- the total Hamiltonian is **zero for odd number of sites** $N = 2L + 1:$

$$\mathrm{H} = \sum_{1 \le p \le q < N} \left(h^{\mathrm{L}}_{p,q} + h^{\mathrm{R}}_{p,q}\right) \; e_{[p,q+1]}$$

$$\mathrm{H} = \frac{1}{2}(\widetilde{\mathrm{H}}^{\mathrm{L}} + \widetilde{\mathrm{H}}^{\mathrm{R}})$$

$$h^{\mathrm{L}}_{p,q} = -h^{\mathrm{R}}_{p,q} \;, \qquad 1 \le p \le q < N$$

explicit but tedious expressions/proof

$$h^{\mathrm{L}}_{p,q} = \sum_{k=1}^{N-q} \left(t_{q-p,0}(k) - (-1)^p t_{q-p,p}(k)\right)$$

with

$$t_{p,q}(n) = \prod_{i=0}^{p-1} \tan \frac{\pi(n+i)}{N} \prod_{j=p}^{p+q-1} \tan^2 \frac{\pi(n+j)}{N}$$

$$\varepsilon^{\mathrm{L}}(n) = -\varepsilon^{\mathrm{R}}(n) = \begin{cases} -n \;, & n = 2k \\ N - n \;, & n = 2k+1 \end{cases}$$

- free fermionic long-range Hamiltonian, not translationally invariant

Fermionic model with extended symmetry

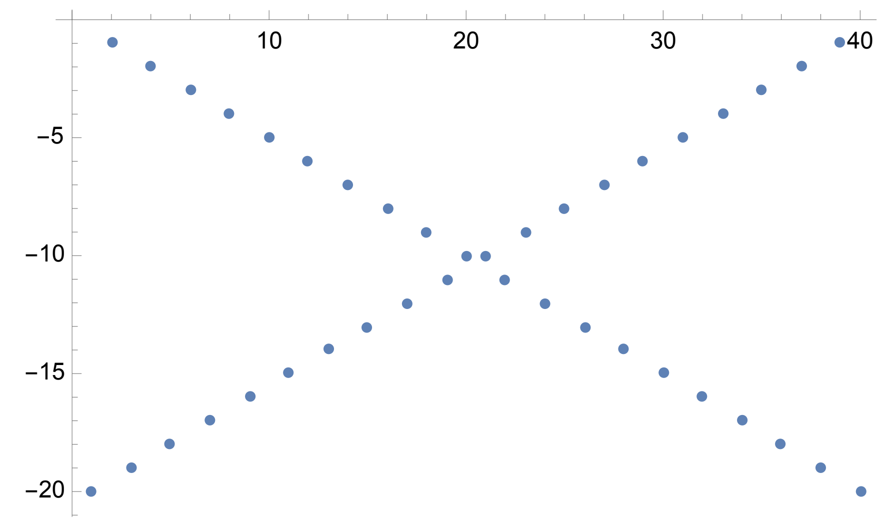
- Another conserved Hamiltonian for the odd number of sites, non-chiral:

$$\widetilde{\mathrm{H}} = \sum_{k\leq l<m\leq n} h_{k,l;m,n} \, \{e_{[k,l+1]}, e_{[m,n+1]}\}$$

anti-commutators of nested commutators of TL generators, **quartic in fermions**

- result for the **one-magnon dispersion relation**:

$$\tilde{\varepsilon}(n) = \lim_{q\to i} \frac{\varepsilon(n)}{\mathrm{q}+\mathrm{q}^{-1}} = (-1)^{L-1} \begin{cases} \frac{n}{2}\,, & n = 2k \\ \frac{N-n}{2}\,, & n = 2k+1 \end{cases}$$



- For even number of sites N=2L all the energies are zero but there are Jordan blocks of size up to L+1

Conclusion

- Long range models show rich mathematical structure and they offer useful lattice regularisations of various field theories
- The use of integrability and symmetries allows to obtain non-perturbative results on a host of physical problems
- The richer the integrable deformations the more useful and flexible they are, hence the necessity of expanding our pool of integrable models and of tools to constrain and solve them