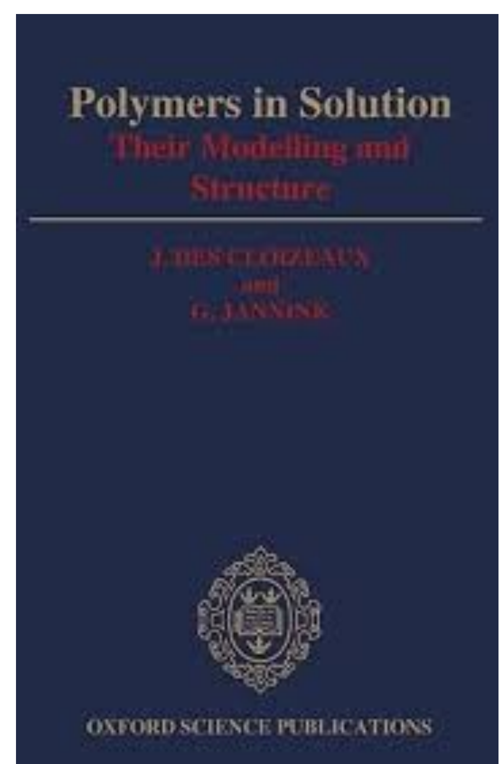
Loop soups and mathematical physics

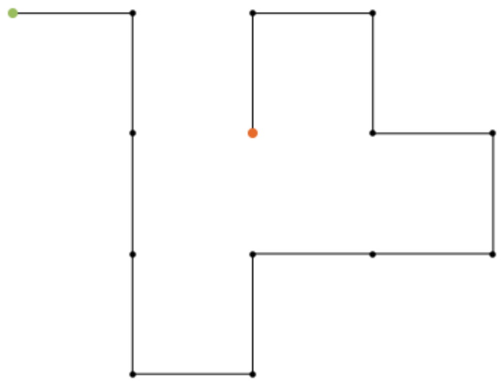
by H. Saleur

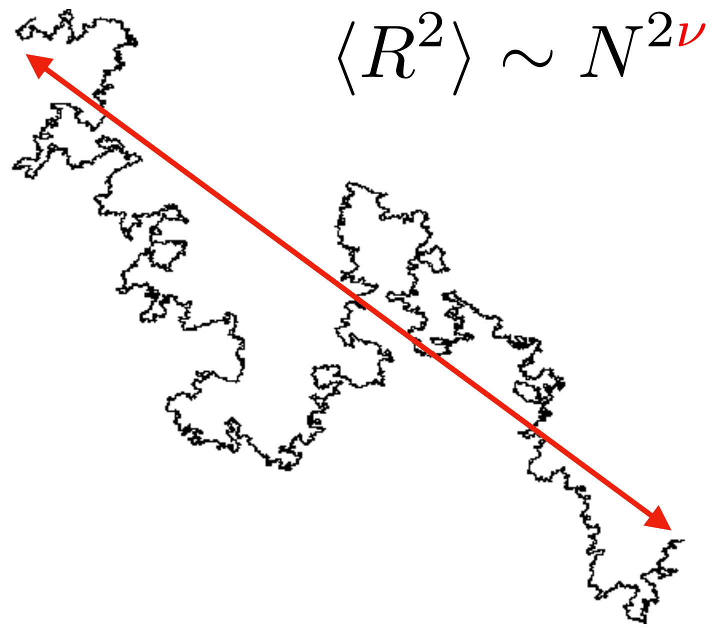
for the 60th anniversary of the IPhT

Dedicated to the memory of J. Des Cloizeaux



- The SAW problem





$$\langle R^2 \rangle \sim N^{2\nu}$$

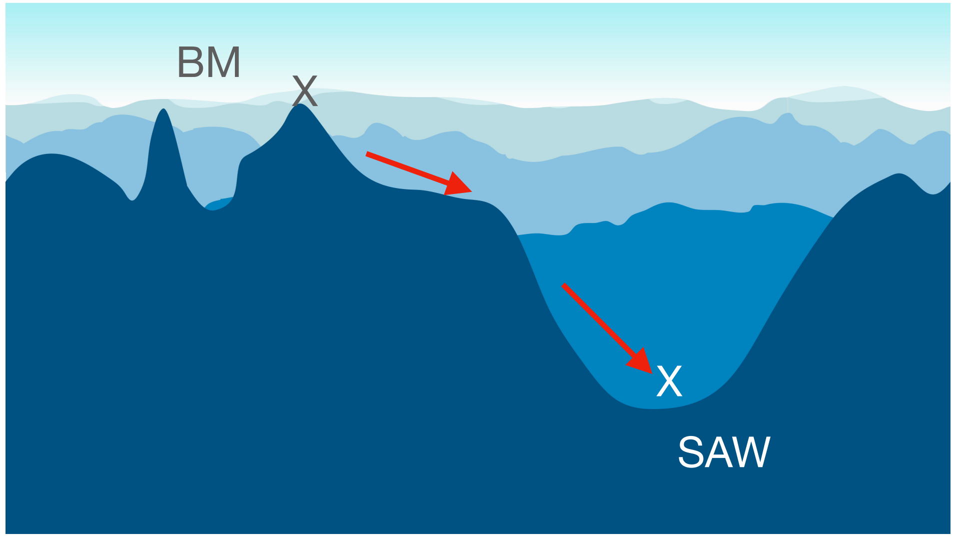
quite generic: Brownian walks + very small repulsive interactions = SAWs at large distances

- limit $N \to \infty$ = follow an RG flow from an unstable to a stable fixed point

Non-perturbative: in 2D $\nu = \frac{3}{4}$

language of criticality
and phase transitions

(DeGennes, Duplantier Descloizeaux)



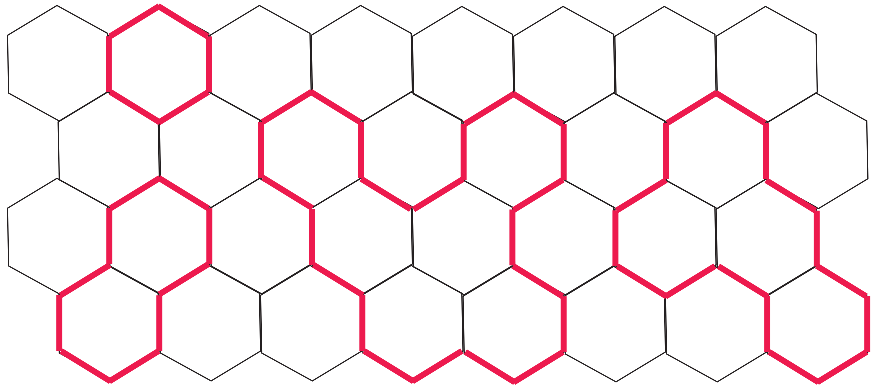
- DeGennes: SAW $= n \to 0$ in the $O(n)$ (vector) Landau-Ginzburg

Lattice model (Affleck, Nienhuis, Schwimmer)

n-component vectors $\vec{S}_i$ with $O(n)$ symmetric $\vec{S}_i.\vec{S}_j$ couplings

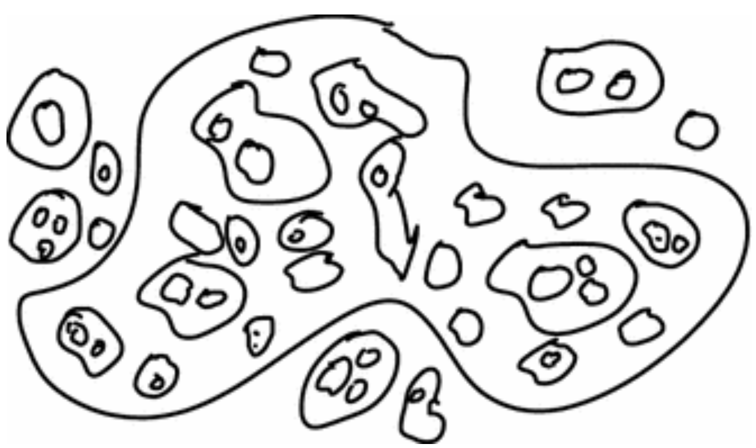
$$\left(Z \propto \int \prod_i d\vec{S}_i \prod_{<ij>} \left(1 + K\vec{S}_i.\vec{S}_j\right) \right)$$

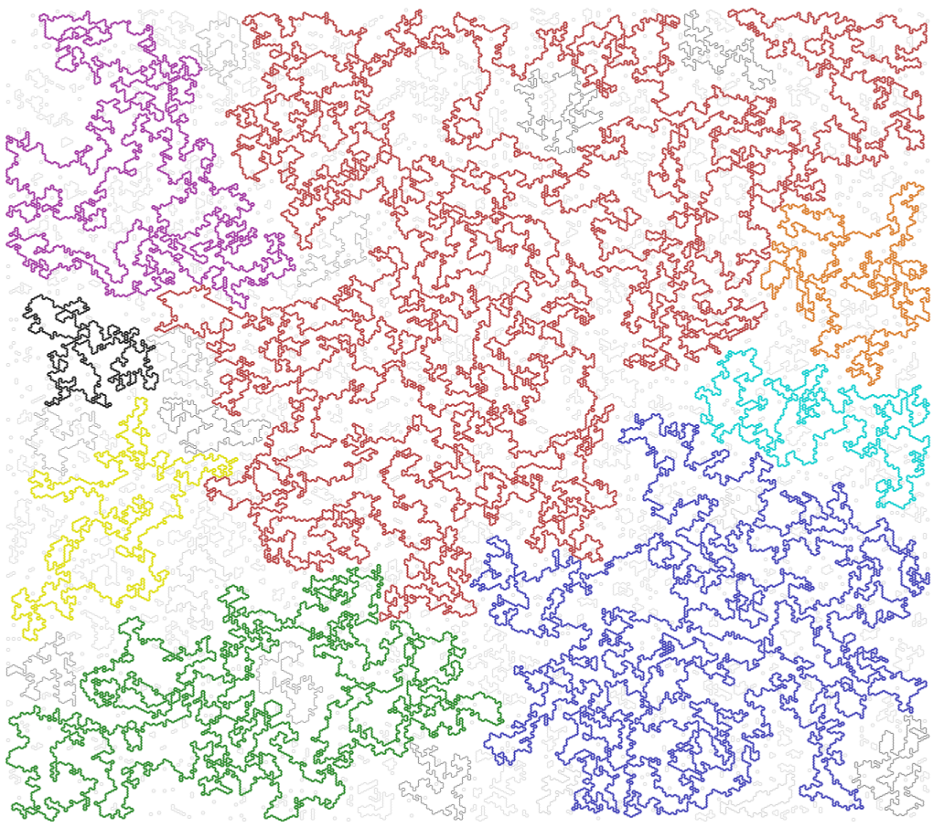
$$Z = \sum_{\text{dilute loop gas}} K^B n^L$$



$n \in \mathbb{C}$. $n = 0$: SAW

loop soups



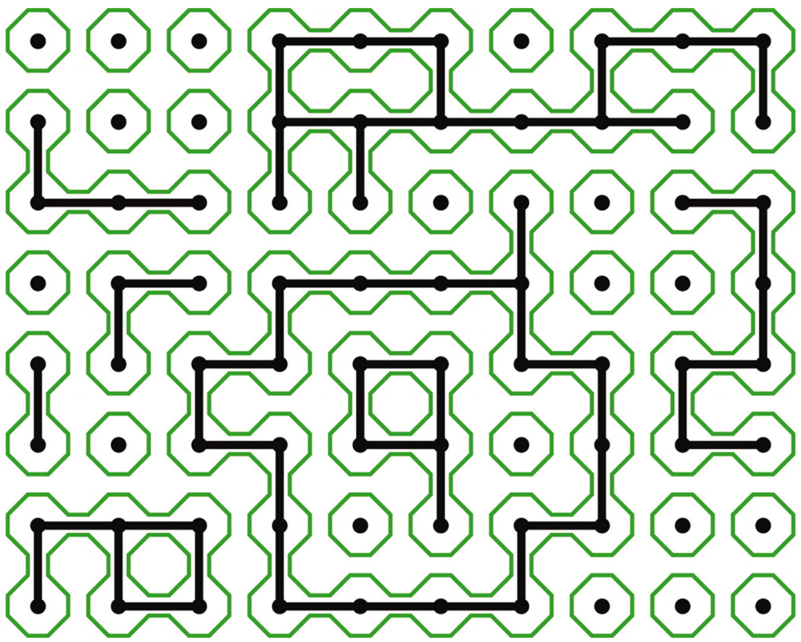


critical at $K = K_c$

- Loops and clusters are related (Potts model)

  $Q = n^2$, $Q = 1$ is percolation

  (Fortuin Kasteleyn, Baxter . . .)



- Conformal loop ensembles $\leftrightarrow$ many other physics problems:
  - Polymers at interfaces
  - General Quantum Field Theory (Brydges, Fröhlich, Spencer, Sokal)
  - Plateau transitions in the (2+1 D) integer quantum Hall effect (class A, class C) (Chalker Coddington, Gruzberg, Ludwig, Read)
  - Properties of interfaces in classical spin systems
  - Properties of (generalizations of) toric codes in topological quantum computation (Kitaev, Freedman, Nayak, Wang)

- Central to modern probability theory (Werner, Smirnov, Dominil-Copin)
- Reveal an astonishing depth from a mathematical physics point of view as well

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From diagram algebras to the Virasoro algebra

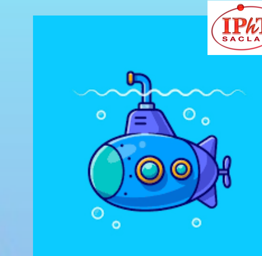
From Hopf algebras (quantum groups) to categorical symmetries

Coulomb gas, Liouville theory, the Bethe-ansatz

SLE evolution, Quantum Gravity

From basic conformal field theory to the bootstrap

⋮



- More than a “stamp collection” for theoretical physicists.

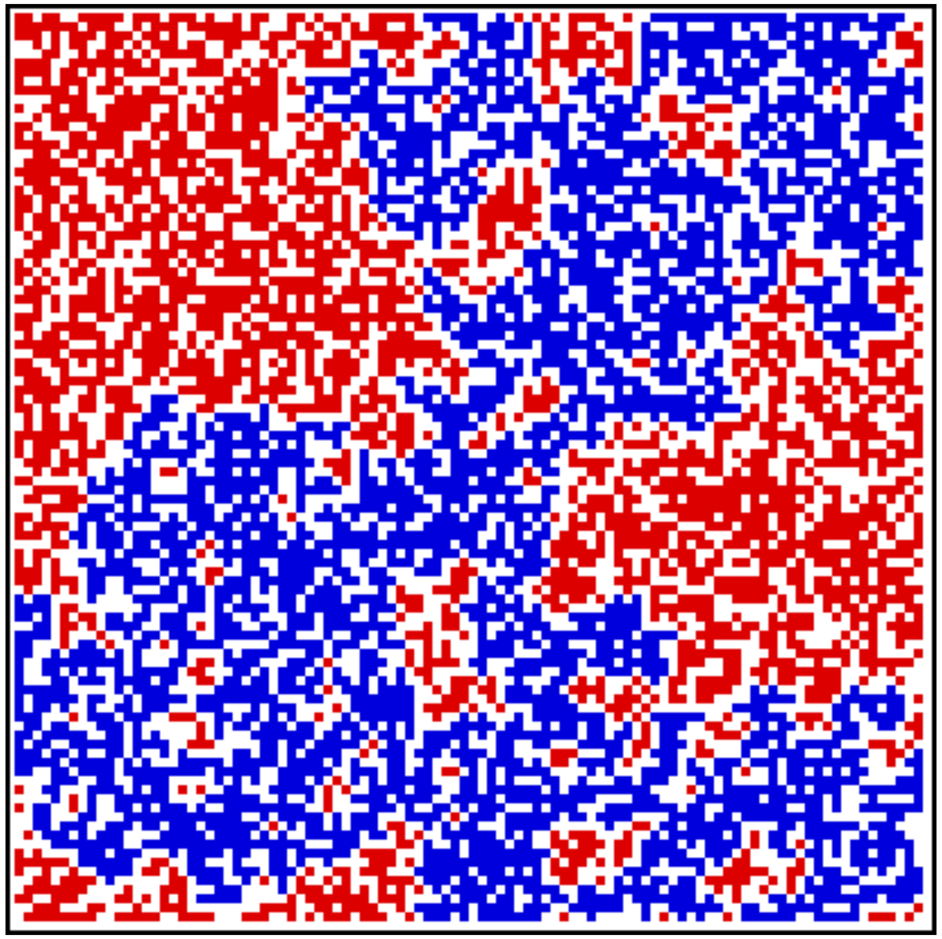
  This armada is needed.

- Surprising since we’ve had integrability and conformal invariance for 40 years.

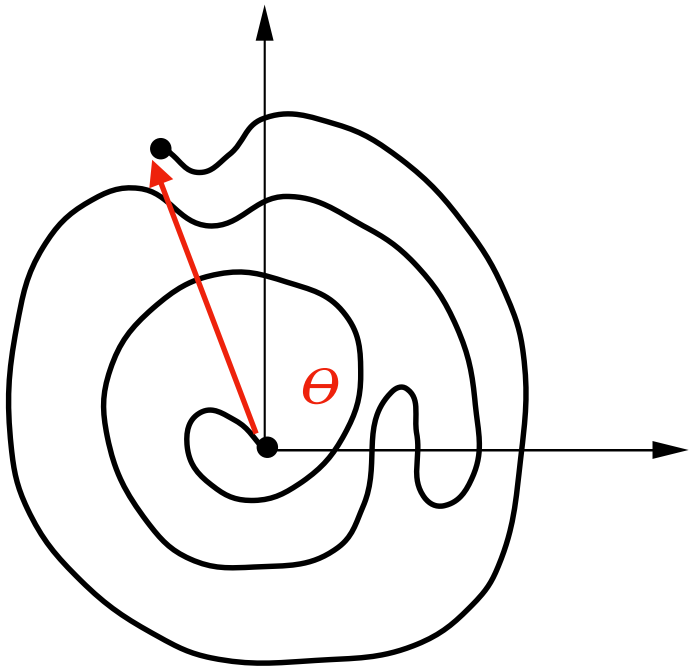
  After all how difficult can $O(n)$ be since $O(1)$ (Ising) is so “easy”?

- Indeed progress in this area was initially very fast on the physics side

  (Den Nijs, Nienhuis, Belavin Polyakov Zamolodchikov, Dotsenko Fateev…)



– Hull of percolating cluster :$D_f = \frac{7}{4}$ (Duplantier Saleur 1987)



– Probability distribution of SAW winding angle (Duplantier Saleur 1988)

$$P\left[x = \frac{\theta}{(4\ln l)^{1/2}}\right] = \frac{e^{-x^2}}{\sqrt{\pi}},\ \ l \to \infty$$

⋮

- Then the field branched into two directions

Quantum gravity, KPZ (Knzihnik, Polyakov and Zamolodchikov)
and SLE (Schramm Loewner)

(Duplantier, Kostov, Bauer, Bernard, David. . .)

Loop models as a genuine CFT

(Cardy, Mussardo, Delfino, Viti, Santachiara)
(Jacobsen, Grans-Samuelsson, He, Nivesvivat, Ribault, HS)
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elegant but indirect and thus limited

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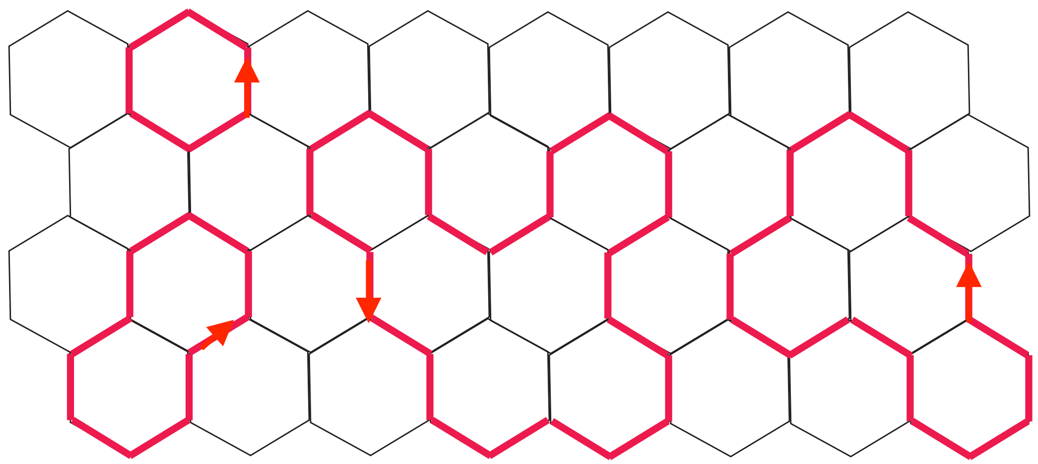
difficult

- So why is it so difficult to solve the loop models CFT?
  - The geometrical definition is obviously non-local
  - The non-locality can be avoided by introducing complex Boltzmann weights



Orient loops and sum over orientations

$|\#\text{left} - \#\text{right}|$ turns $= 6$ (on the plane)

$n = 2\cos 6\alpha$ with weight $e^{\pm i\alpha}$ per turn

The correspondence 2D stat. mech. with 1+1D quantum field theory still holds but

the QFT is not unitary

Typically, the quantum processes have probabilities $p_i >$ or < 0 (but $\sum p_i = 1$)

Physics of nonhermitian degeneracies *)

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Received 1 July 2004

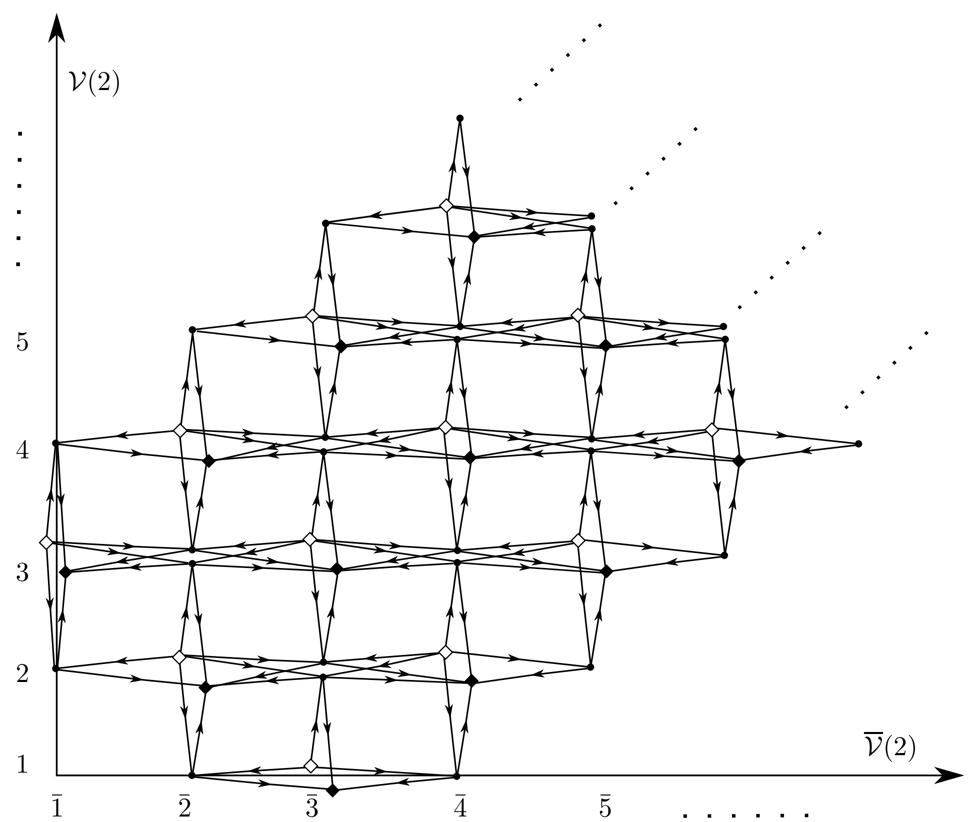
A summary, with references and additional comments, of a talk delivered at the Second International Workshop on Pseudohermitian Hamiltonians in Quantum Physics (Prague, 14–16 June 2004). After explaining some general features of nonhermitian degeneracies ('exceptional points'), several applications are outlined: to multiple reflections in a pile of plates, linewidths of unstable lasers, atom diffraction by light, and crystal optics.

PACS: 03.65.Yz
Key words: matrices, degeneracies, quantum

1 Introduction

Nonhermitian hamiltonians usually enter physics as a description of part of a system, as a result of a decision not to incorporate all freedoms — for example those describing dissipation. Examples are complex refractive indices in optics, and complex potentials describing the scattering of electrons or X-rays, or by nuclei ('cloudy crystal ball'). Traditionally, the nonhermiticity has been regarded as a perturbation, with the physics essentially unchanged from the hermitian case, except for an exponential decay (for example during propagation through a crystal). But nonhermitian physics differs radically from hermitian physics in the presence of degeneracies, that is coalescences of eigenvalues. My aim here is to illustrate this essentially nonhermitian behaviour with a series of examples, drawn from several areas of physics, that I have encountered over the past decade (Sections 3–7), after some general remarks (Section 2).

- What happens when we lose unitarity?
  - Most approaches are algebraic and thus rely on representation theory which becomes non semi-simple when unitarity is lost
    Modules are not fully reducible. In terms of simple sub-modules they can take any shape (wilderness)



$n = -2$

(Gainutdinov, Read, Saleur)

  - Not only can norm-squares be negative, but zero norm-square states can be non-zero (In fact, for SAW, all the physics is in the sector zero-norm square sector, and $c = 0$)
    No BPZ equations

So it's all fairly complicated

There is no royal road to geometry (Euclid)

but the end is in sight

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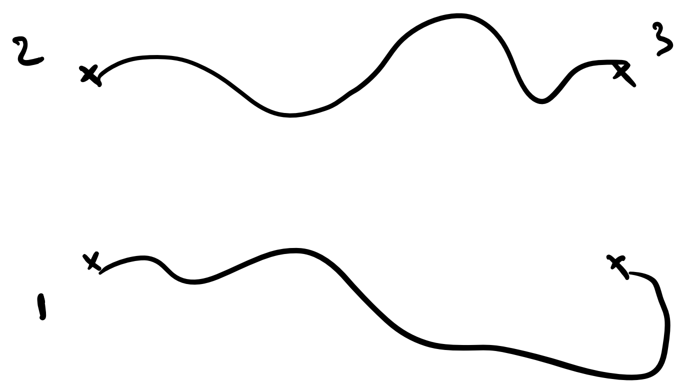
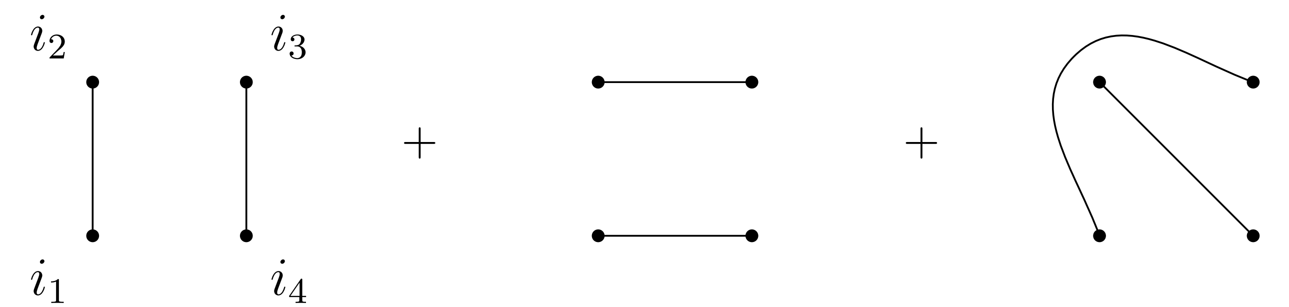
Many outstanding questions have finally been answered in the last couple of years, and here is an example of what we know

- Four-point function of the one-leg operator
  - Order operator $\vec{S}$ transforms in [1] and creates an extra open line in the lattice model



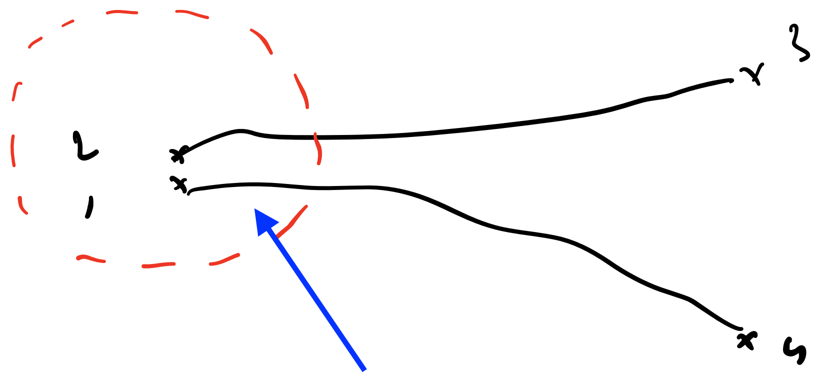
  - $[1]^{\otimes 2} = [] \oplus [11] \oplus [2]$: (tensor) structure

$$\langle S^{i_1} S^{i_2} S^{i_3} S^{i_4} \rangle \quad =$$

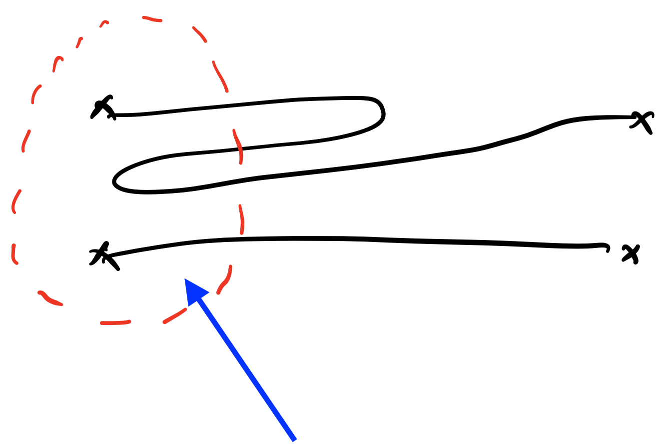


$\delta_{i_1 i_2}\delta_{i_3 i_4} C_1$ $\qquad$ $\delta_{i_2 i_3}\delta_{i_1 i_4} C_2$ $\qquad$ $\delta_{i_1 i_3}\delta_{i_2 i_4} C_3$

– OPEs are expected to be complicated



appears as the two-leg (watermelon/fuseau) operator



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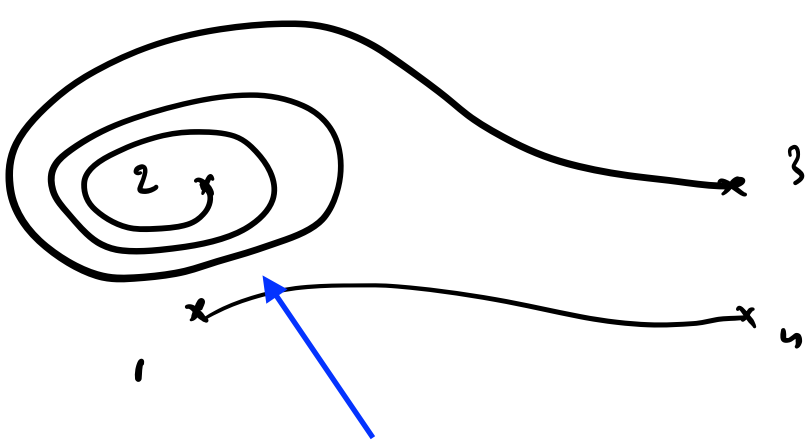


and then there's the winding

(DiFrancesco, HS, Zuber)

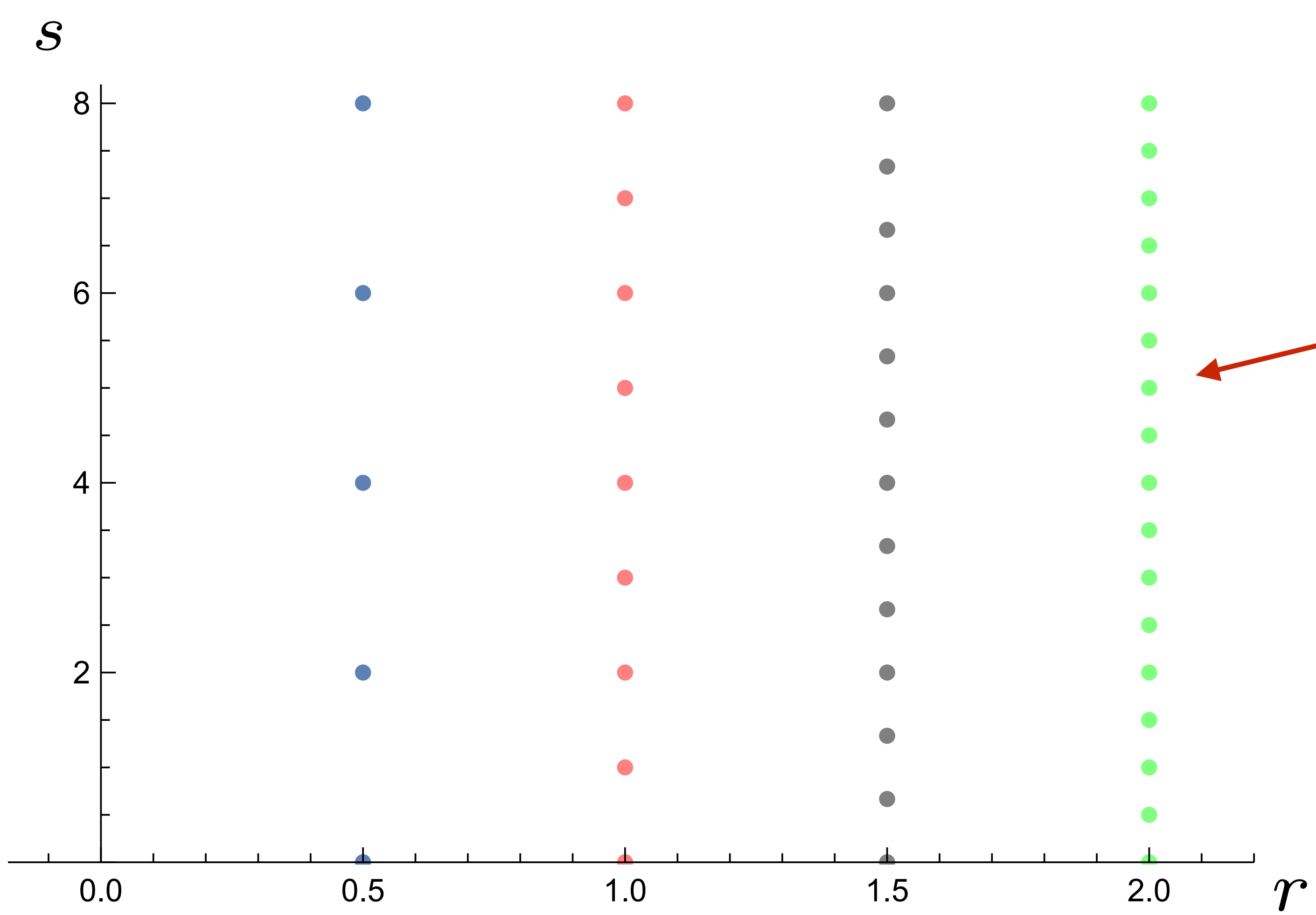
- The spectrum in the s-channel for SAW ($n = 0$, $c = 0$)

$$\mathcal{S} = \{(r,s)^N; r \in \mathbb{N}^*/2, s \in \mathbb{Z}/r\} \cup \{\langle 1, 1+2\mathbb{N}\rangle^D\}$$

$$x_{(r,s)} = \frac{3}{4}r^2 + \frac{1}{3}s^2 - \frac{1}{12}$$

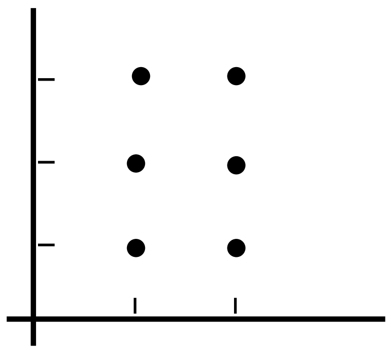
$$spin = rs$$

Physically: $\frac{r}{2}$ = number of legs while s is conjugate to the winding



The non diagonal part has s values dense on the axis

Compare with Ising:

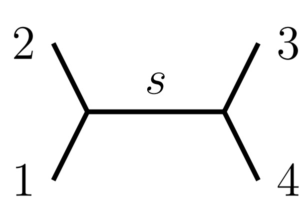


$$V^{[1]}_{(\frac{1}{2},0)}(z,\bar{z})V^{[1]}_{(\frac{1}{2},0)}(0,0) \overset{\text{OPE}}{=} \text{sum over all } \mathcal{S}$$

we're dealing with non rational non unitary non quasi-rational CFTs

(but the exponents for SAW are all rational)

- Correlation functions require a mix of techniques
  - Algebraic (interchiral algebra and affine Temperley-Lieb) (Gainutdinov, Read, HS)
  - Numerical (exact results on lattice) (Jacobsen, HS)
  - Bootstrap (Jacobsen, Grans-Samuelsson, He, Nivesvivat, Ribault, HS)



$$= \sum_{\mathcal{S}} \mathcal{D} \times \mathcal{G}(\xi,\bar{\xi}), \quad \xi = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

The $\mathcal{G}$ are conformal blocks determined by general conformal symmetry

we now know D!!!

Set $\mathcal{D} = C^{\rm ref} C^{\rm ref} d$ where

$$C^{\rm ref}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\tfrac{\beta+\beta^{-1}}{2} + \tfrac{\beta}{2}\left|\textstyle\sum_i \epsilon_i r_i\right| + \tfrac{\beta^{-1}}{2}\textstyle\sum_i \epsilon_i s_i\right)$$

Γ_β is Barnes double Gamma function familiar from Liouville theory (Teschner)

$$\Gamma_\beta(x+\beta) = \sqrt{2\pi}\frac{\beta^{\beta x - \frac{1}{2}}}{\Gamma(\beta x)}\Gamma_\beta(x) \quad , \quad \Gamma_\beta(x+\beta^{-1}) = \sqrt{2\pi}\frac{\beta^{-\beta^{-1}x + \frac{1}{2}}}{\Gamma(\beta^{-1}x)}\Gamma_\beta(x)$$

Then the d are (still somewhat mysterious) polynomials in n

$$\begin{aligned} d^{(s)}_{\rm diag} &= 1 \ , \\ d^{(s)}_{(2,0)} &= n^2 - 4 \ , \\ d^{(s)}_{(2,1)} &= -(n^2-4) \ , \\ 3d^{(s)}_{(3,0)} &= -8n^2(n-2)^2(n+2) \ , \\ 3d^{(s)}_{(3,\frac{2}{3})} &= 4(n^2-1)(n^2-3)(n-2) \ , \end{aligned}$$

$$\begin{aligned} 2d^{(s)}_{(4,0)} &= n^2(n-2)^3(n+1)^2(n+2) \\ &\quad \times \ [n(n+2)^2(n-1)^2 + 2n^4 - 6n^2 - 8n + 16] \end{aligned}$$

$\ldots$

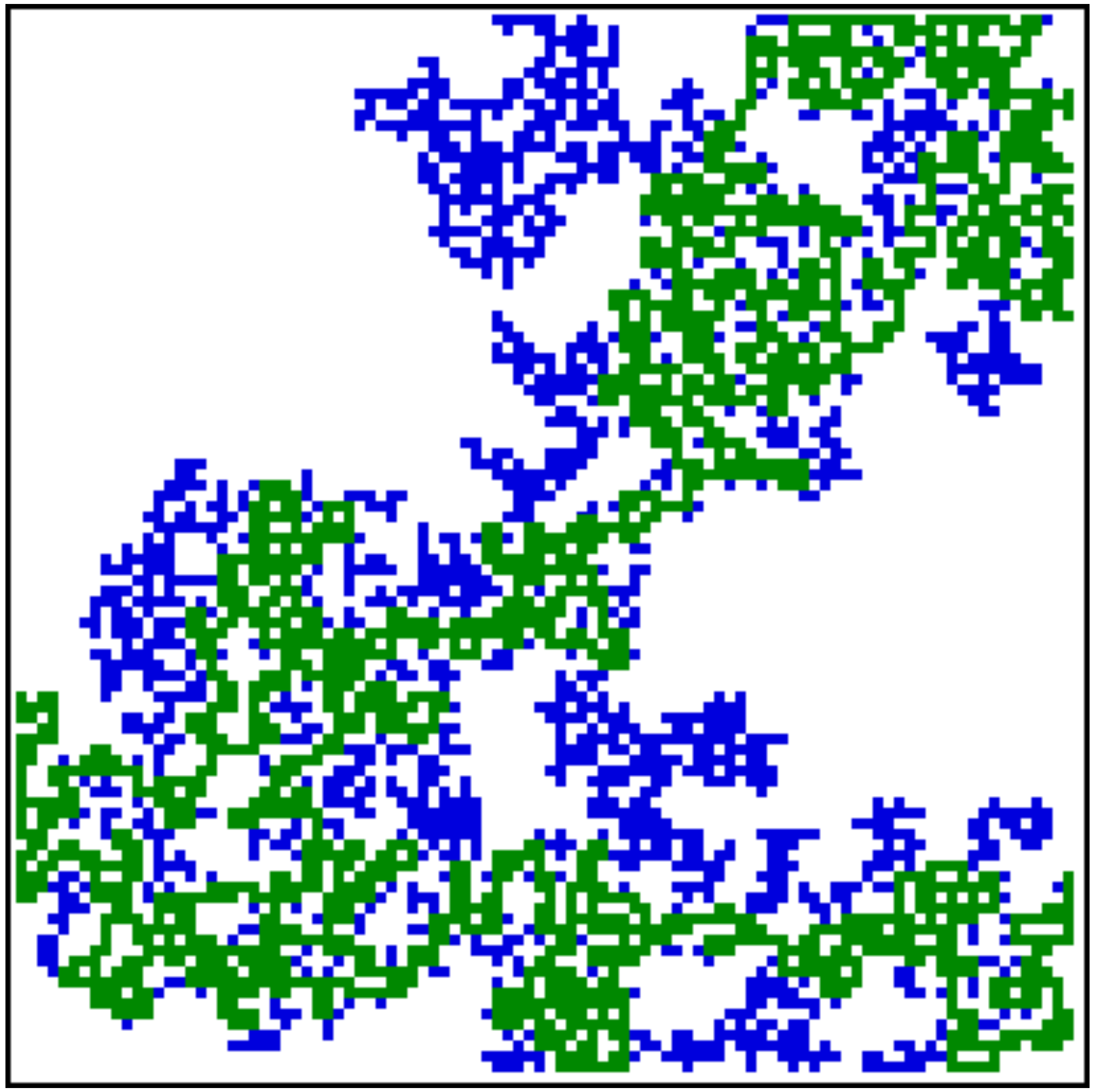
this is almost the end, even though challenges remain

(Nolin, Qian, Sun, Zhuang, Sept. 2023)

Theorem 1.1. *The backbone exponent* ξ *is the unique solution in the interval* $(\frac{1}{4}, \frac{2}{3})$ *to the equation*

$$\frac{\sqrt{36\xi+3}}{4} + \sin\left(\frac{2\pi\sqrt{12\xi+1}}{3}\right) = 0. \tag{1.2}$$

$\xi = 0.35666683671288\ldots.$
$D_f = 2 - \xi$



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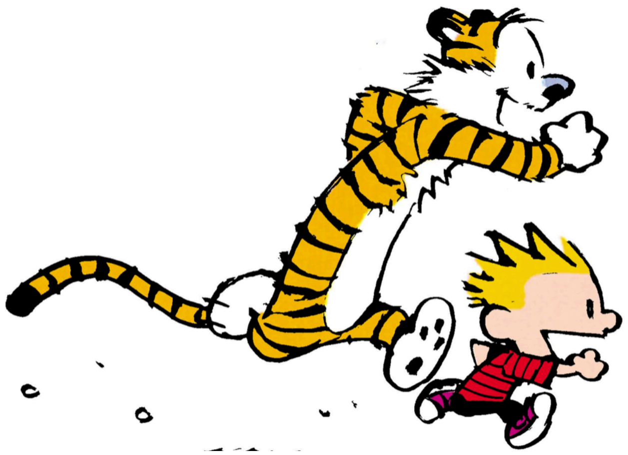
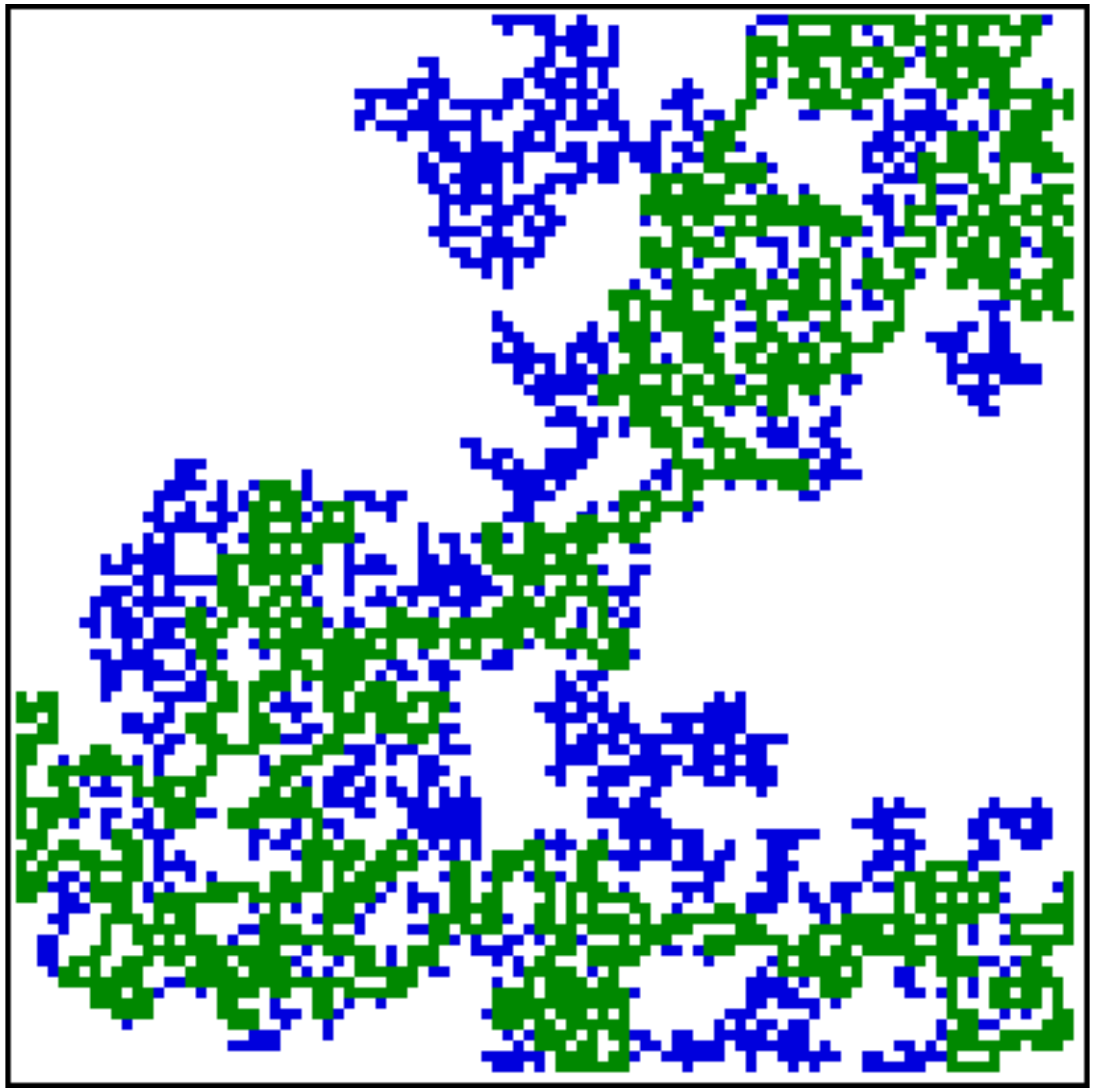
and so the race goes on





Thank you IPhT

and all whom I have not mentioned, including De Dominicis,
Derrida, De Seze, Orland - and E. Brezin and C.Itzykson