How the ‘Service de Physique Théorique’ has advanced semiclassical analysis (a personal testimony)

André Voros
Université Paris–Saclay, CNRS, CEA, Institut de Physique Théorique
91191 Gif-sur-Yvette, France

IPhT’s 60-th anniversary
Saclay, Nov. 10, 2023

Reference: A.V., arXiv:2206.14482 [math-ph]

Quantum theory landscape around 1970

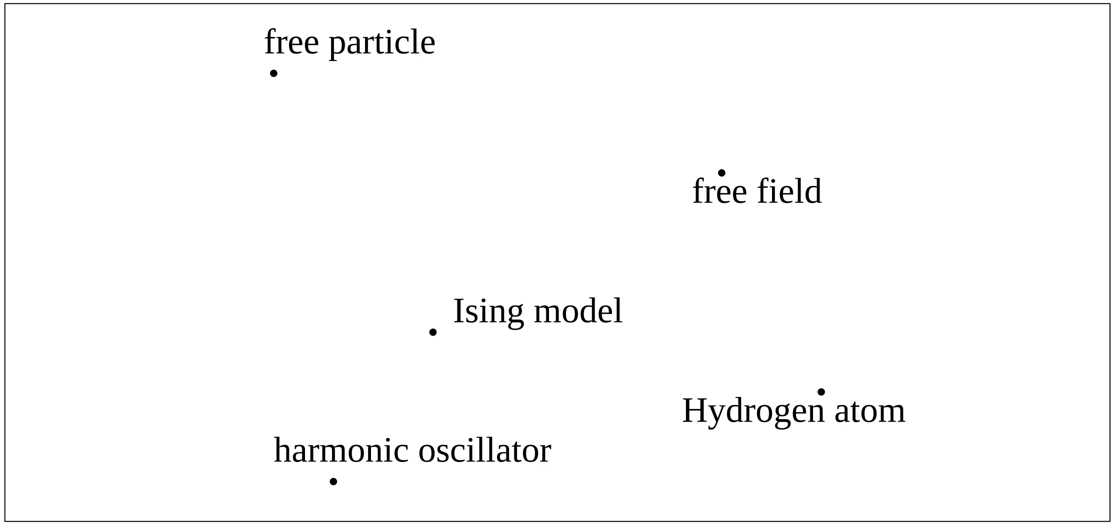


Figure 1: A small isolated set of exceptional solvable cases.

Everything else has to rely on *perturbative* approaches: pick a coupling constant g (with $g = 0$ describing one of the solvable cases), then expand, e.g., an energy eigenvalue $E(g)$, as $\sum_0^\infty E_{(n)} g^n$.

Problem: those series typically *diverge*, resummation techniques (Padé, Borel...) have to be attempted.

Semiclassical approaches

Quantum mechanics based on *Planck's constant* $\hbar > 0$ has to closely follow classical mechanics (which has $\hbar = 0$) when $\hbar$ can be considered negligibly small (and $\hbar \approx 10^{-34}$ [SI]).

In first approximation, quantum mechanics may then be expressible using classical-mechanical solutions.

In one perspective this gives but another perturbative scheme: classical mechanics is the known (solved) case, and $\hbar$ is the expansion parameter.

Still, it is more *generic* (not confined to isolated dynamical models) and *flexible* (schemes tailored to the type of quantity under study).

Early semiclassical successes at ‘SPhT’

They initially focused on *nuclear physics*: atomic nuclei are many-body systems, too complex for plain perturbative approaches.

Nuclear physicists at SPhT brought significant contributions in:
- Hartree–Fock methods (stationary and time-dependent)
- mean-field theories
- generating-coordinate methods
- coherent-state techniques
- Thomas–Fermi calculations,

all of which broadly qualify as generalized semiclassical schemes.

Then, in quantum mechanics and field theory, SPhT shone in:
- large-order perturbation theory and instanton calculus
- large-N limit in quantum field theories
- analytic study of amplitudes and Landau singularities,

all likewise relating to semiclassical philosophies (sometimes in the complex domain).

The WKB - Wentzel–Kramers–Brillouin - method (the primary 1D semiclassical method): from approximate to fully exact

Figure 2: My walk to and from high school took me every day - unknowingly at the time - alongside the house where L. Brillouin was born, in Sèvres (France).

Starting point: $[-\hbar^2 \mathrm{d}^2/\mathrm{d}q^2 + V(q)]\psi(q,\hbar) = E\,\psi(q,\hbar)$ (1D stationary Schrödinger equation) has formal approximate small-$\hbar$ solutions

$$\psi(q,\hbar) \sim u(q,\hbar)\,\mathrm{e}^{\mathrm{i}S(q)/\hbar}, \quad u(q,\hbar) \sim \sum_n u_n(q)\hbar^n \quad \text{(WKB Ansatz).}$$

Toward making WKB exact

Exact WKB analysis, assuming 1D potentials $V(q)$ *analytic in the complex q-plane*, deciphers the standard complex-WKB equations for the Schrödinger wave function $\psi(q, \hbar)$ as a code for its analytical continuation in q - with this being a *fully exact* operation.

It grew around 1980 upon important earlier works:

- microlocal analysis in the *analytic* category: ramified Cauchy problem (Leray 1957), analytic pseudodifferential operators (Boutet de Monvel–Krée 1967), hyperfunctions and microfunctions (Sato–Kawai–Kashiwara, Pham, Bros–Iagolnitzer $\sim$ 1970);

- Borel-transform approaches to perturbative series (Bender–Wu 1973, Zinn-Justin 1977–) and specially semiclassical ones (Balian–Bloch 1974);

- Dingle's (1973) treatment of asymptotic series using terminants;

- Sibuya's (1975) direct approach yielding exact functional equations for Stokes multipliers.

First came partial analyticity results in the Borel plane for the pure quartic potential (Balian–Parisi–AV 1978–1979), triggered by work of Knoll–Schaeffer 1976.

1D Schrödinger operators: $\hat{H}_N = -\mathrm{d}^2/\mathrm{d}q^2 + |q|^N, \quad q \in \mathbb{R}, \quad N = 1, 2, \ldots$

- Spectral zeta functions $Z_N^{\#}(s)$ (convergent for Re $s > \frac{1}{2} + \frac{1}{N}$):

$$Z_N(s) = \sum_{k=0}^{\infty} \left(E_k^{[N]}\right)^{-s}, \qquad Z_N^{\mathrm{P}}(s) = \sum_{k=0}^{\infty} (-1)^k \left(E_k^{[N]}\right)^{-s}.$$

- Spectral determinant $D_N(\lambda) \stackrel{\mathrm{def}}{=} \det(\hat{H}_N + \lambda)$ (zeta-regularized): defined through its logarithm:

$$\log\det(\hat{H}_N + \lambda) \stackrel{\mathrm{def}}{=} -\partial_s Z_N(s,\lambda)|_{s=0}, \qquad Z_N(s,\lambda) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty} \left(E_k^{[N]} + \lambda\right)^{-s}.$$

Fundamental (convergent) expansion:

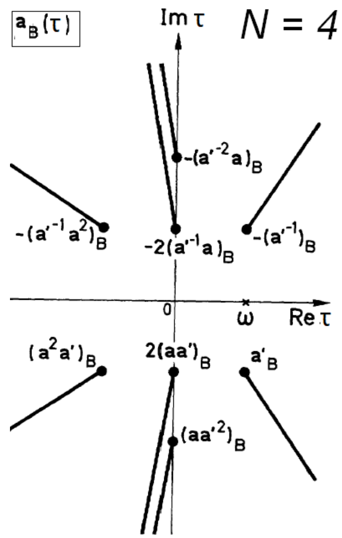
$$\log D_N(\lambda) \equiv -{Z_N}'(0) - \sum_{n=1}^{\infty} \frac{Z_N(n)}{n}(-\lambda)^n.$$

Initial (1981) concrete outcomes of the exact WKB method were *exact identities on these spectral-zeta values at* $n = 0, 1, 2, \ldots$, themselves following from *exact functional equations for the spectral determinants.*

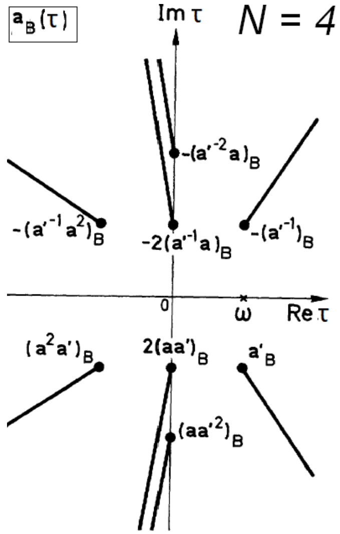
Perturbative WKB results (large variable: $x \equiv \lambda^{\frac{1}{2}+\frac{1}{N}} \propto 1/\hbar \to +\infty$):

$$\log D_N(\lambda) - a_0\, x \stackrel{\text{def}}{=} \log a(x) \sim \sum_{m=1}^{\infty} a_m\, x^{1-2m}$$

Then, with Borel transform specified as $(x^{-\alpha})_{\mathrm{B}} \stackrel{\text{def}}{=} \tau^{\alpha}/\alpha!\,$, *exact WKB* analysis predicts (quartic example):



This exact structure falls into the very general framework of *resurgent* functions (Écalle 1981). Exact-WKB analysis got further developed in that direction by Pham et al. 1985–1997, Aoki–Kawai–Takei 1991–.



Borel resummation: $a(x) = x \int_0^\infty a_{\mathrm{B}}(\tau)\, \mathrm{e}^{-x\tau}\, \mathrm{d}\tau$ converts that information for $N = 4$ into a *functional equation* for $D_4(\lambda) \equiv \mathrm{e}^{a_0 x}\, a(x)$, $x \equiv \lambda^{3/4}$:

$$D_4(\lambda) D_4(\mathrm{j}\lambda) D_4(\mathrm{j}^2\lambda) \equiv D_4(\lambda) + D_4(\mathrm{j}\lambda) + D_4(\mathrm{j}^2\lambda) + 2, \qquad \mathrm{j} \stackrel{\mathrm{def}}{=} \mathrm{e}^{2\pi\mathrm{i}/3},$$

later extended to a bilinear functional relation (with $\nu \stackrel{\mathrm{def}}{=} \frac{1}{N+2}$),

$$\mathrm{e}^{\mathrm{i}\nu\pi}\, D_N^+(\lambda) D_N^-(\mathrm{e}^{4\mathrm{i}\nu\pi}\,\lambda) - \mathrm{e}^{-\mathrm{i}\nu\pi}\, D_N^+(\mathrm{e}^{4\mathrm{i}\nu\pi}\,\lambda) D_N^-(\lambda) = 2\mathrm{i} \qquad (N \neq 2).$$

Enforcing the compatibility of such functional relations with the above expansions for $\log D_N^\#(\lambda)$ at $\lambda = 0$ generates *polynomial identities* on the zeta-values $Z_N'(0), Z_N^\#(n)$ $(n = 1, 2, \ldots)$: "exact-WKB sum rules".

Exact-WKB sum rules for the potentials $|q|^N$ ($N = 1, 2, \ldots$):

countably many *exact identities* for the spectral zeta functions $Z_N^{\#}(s)$, specifically for the values at nonnegative integers $Z'_N(0)$ and $Z_N^{\#}(n)$ $(n = 1, 2, \ldots)$:

$$(n=0:) \qquad Z'_N(0) = \log\sin\nu\pi \qquad \left[\nu \overset{\text{def}}{=} \frac{1}{N+2}\right]$$

$$\begin{aligned}
-\cot\nu\pi\sin 2\nu\pi\, Z_N^{\mathrm{P}}(1) + \cos 2\nu\pi\, Z_N(1) &= 0 \quad [\text{indeterminacy for } N=2] \\
-\cot\nu\pi\sin 4\nu\pi\, Z_N^{\mathrm{P}}(2) + \cos 4\nu\pi\, Z_N(2) &= -4\cos^2\nu\pi\, Z_N^{\mathrm{P}}(1)^2 \\
-\cot\nu\pi\sin 6\nu\pi\, Z_N^{\mathrm{P}}(3) + \cos 6\nu\pi\, Z_N(3) &= 4\cos^2\nu\pi\big[2\cos^2\nu\pi\, Z_N^{\mathrm{P}}(1)^3 \\
&\qquad - 3\cos 2\nu\pi\, Z_N^{\mathrm{P}}(1) Z_N^{\mathrm{P}}(2)\big] \\
&\ \vdots \\
-\cot\nu\pi\sin 2n\nu\pi\, Z_N^{\mathrm{P}}(n) + \cos 2n\nu\pi\, Z_N(n) &= \mathcal{P}_{N,n}\{Z_N^{\mathrm{P}}(m)\}_{1\le m<n} \\
&\ \vdots
\end{aligned}$$

with $\mathcal{P}_{N,n}$: homogeneous polynomials of degree n if $\deg[Z_N^{\mathrm{P}}(m)] \overset{\text{def}}{=} m$.

Exceptionally explicit examples occur at $n = 2$:

$$\cot \nu\pi \sin 4\nu\pi\, Z_N^{\mathrm{P}}(2) - \cos 4\nu\pi\, Z_N(2) = \frac{\pi (2\nu)^{4N\nu}}{4} \left[\frac{\Gamma(\nu)\Gamma(3\nu)}{\Gamma(1-2\nu)\Gamma(2\nu+\frac{1}{2})} \right]^2$$

(with $\nu \stackrel{\mathrm{def}}{=} \frac{1}{N+2}$).

Specially:

- harmonic oscillator q^2 : $\sum_0^\infty E_k^{-2} = \pi^2/8 \iff \zeta(2) = \pi^2/6 \quad (E_k \equiv 2k+1)$
- cubic oscillator $|q|^3$: $\sum_0^\infty E_{2k}^{-2} = (\frac{2}{5})^{2/5} \frac{2}{\sqrt{5}+1}\, \pi\, [\Gamma(\frac{6}{5})/\Gamma(\frac{9}{10})]^2$
- sextic oscillator q^6 : $\sum_0^\infty (-1)^k E_k^{-2} = \frac{1}{8}\, [\pi\, \Gamma(\frac{5}{4})]^2/\Gamma(\frac{7}{8})^4$.

Subsequently, all by itself, the key functional equation reached above,

$$e^{i\nu\pi} D_N^+(\lambda) D_N^-(e^{4i\nu\pi}\lambda) - e^{-i\nu\pi} D_N^+(e^{4i\nu\pi}\lambda) D_N^-(\lambda) = 2i \qquad (N \neq 2)$$

gave rise to:
1) exact-WKB solutions for the eigen/values/functions (AV 1994–1999)
2) the ODE/IM correspondence, initiated by Dorey–Tateo 1999, linking the exact-WKB method to the *Bethe Ansatz for integrable models* in 2D statistical mechanics.

Ongoing applications of resurgent functions and exact WKB theory appear to include:

- conformal field theories
- Chern–Simons theories
- topological quantum field theories
- SUSY gauge theories
- topological recursion
- Painlevé functions
- wall-crossing theory
- cluster algebras

Quantum chaos

The semiclassical description of $D > 1$ strongly coupled (= far from integrable) systems remains much less understood analytically.

Still, the SPhT has scored active contributions:

- to *quantum maps* (Balazs, Saraceno 1986–1994)
- and to phase-space descriptions of *eigenvectors* by the *Husimi-stellar representations* (Leboeuf, Tualle, Nonnenmacher 1990–).