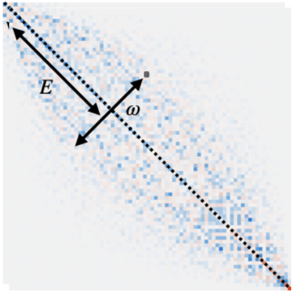
Eigenstate thermalization in quantum many-body systems



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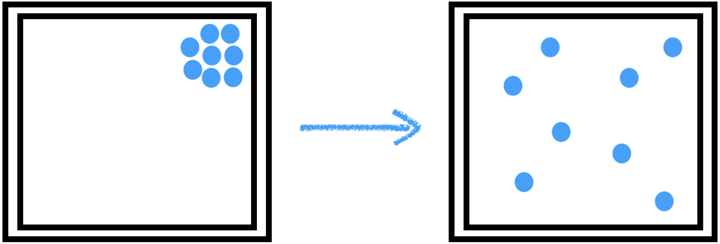
60 ans de l'IPhT

Context

ETH explains equilibration of *isolated quantum many-body* systems



In our work: need to characterise better ETH ansatz, all about dynamics *at equilibrium*

Dynamics and ETH

Heisenberg picture (evolution of the operators):

$$A(t) = e^{iHt} A e^{-iHt} = \sum_{ij} e^{i(E_i - E_j)t} A_{ij} |E_i\rangle\langle E_j|$$

$$A_{ij} = \langle E_i | A | E_j \rangle$$

Look at matrix elements of observables in the basis of the energy

Characterise them "statistically"

Toy ETH

(Diagonal) matrix A (observable) of size $\mathcal{N}$

Matrix elements in a random basis

$$\overline{A_{ii}} = \frac{1}{\mathcal{N}} \sum_i \lambda_i = m_1 = \kappa_1$$

$$\overline{A_{ij}} = 0 \qquad i \neq j$$

$$\overline{A_{ij}^2} = \frac{1}{\mathcal{N}} \left[\frac{1}{\mathcal{N}} \sum_i \lambda_i^2 - \left(\frac{1}{\mathcal{N}} \sum_i \lambda_i \right)^2 \right] = \frac{1}{\mathcal{N}} \left[m_2 - m_1^2 \right] = \frac{1}{\mathcal{N}} \kappa_2$$

Toy ETH: ansatz

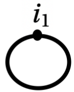
$$A_{ij} = \kappa_1 \delta_{ij} + \sqrt{\frac{\kappa_2}{\mathcal{N}}} R_{ij}$$

$$\overline{R_{ij}} = 0 \qquad \overline{R_{ij}^2} = 1 \tag{1}$$

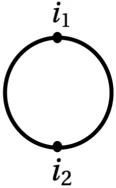
No info about correlations

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_p i_1}} \simeq \frac{1}{\mathcal{N}^{(p-1)}} \kappa_p \qquad i_1 \neq i_2 \neq \dots \neq i_p \tag{2}$$

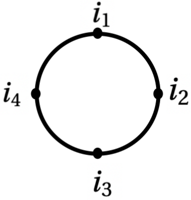
Toy ETH: diagrams



$$\overline{A_{i_1 i_1}} = \kappa_1$$



$$\overline{A_{i_1 i_2}}^2 = \frac{1}{\mathcal{N}}\kappa_2$$



$$\overline{A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{i_4 i_1}} = \frac{1}{\mathcal{N}^{(p-1)}}\kappa_p \qquad p = 4$$

The many-body problem

E.g. H spin chain, physical observable:

$$A = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^z$$

H and A matrices of size $\mathcal{N} = 2^N$

$$\rho(E = Ne) = \sum_{\alpha=1}^{\mathcal{N}} \delta(E - E_i) \propto e^{S(E)} \simeq e^{Ns(e)}$$

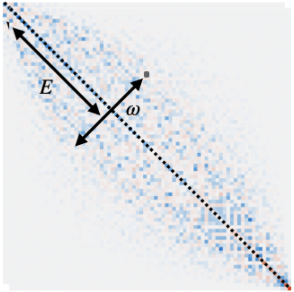
Eigenstate thermalization

Single eigenstates provide equilibrium statistical averages
$\langle E_i|A|E_i\rangle$ varies smoothly with the energy E_i

For dynamics necessary off-diagonal matrix elements

J. Deutsch (1991), M. Srednicki (1994)
Review: D'Alessio, Kafri, Polkovnikov, Rigol (2016)

Eigenstate thermalization ansatz



$$A_{ij} = \mathscr{A}(e)\delta_{ij} + e^{-Ns(e)/2} f_e(\omega) R_{ij}$$

$$E = (E_i + E_j)/2 \quad e = E/N \quad \omega = E_i - E_j$$

R_{ij} (pseudo)-random numbers

$$\overline{R_{ij}} = 0 \;\; \overline{R_{ij}^2} = 1$$

M. Srednicki (1999)

Fictitious ensemble

$$A_{ij} \to \text{random matrix element}$$

Ensemble ?

- Small energy windows
- Perturb with small Hamiltonian $H \to H + \epsilon V$ (Deutsch (1991)). Nearby eigenvectors extremely sensitive even to small perturbations. Physics unchanged

One and two-time correlation functions

$$\langle A \rangle_\beta \xrightarrow{N\to\infty} \mathscr{A}(e_\beta)$$

$$\langle A(t)A(0)\rangle_\beta - \langle A \rangle_\beta^2 \xrightarrow{N\to\infty} \int \mathrm{d}\omega \; e^{-\beta\omega/2} e^{i\omega t} |f_{e_\beta}(\omega)|^2$$

One and two-point function independent of correlations between different matrix elements

Multi-point correlation functions

$$C_4^{\beta}(t_1, t_2, t_3) = \mathrm{Tr}\left[\rho_\beta A(t_1) A(t_2) A(t_3) A(0)\right]$$

$C_4^{\beta}(t, 0, t)$ Out-of-Time-Order Correlator

"quantum Lyapunov exponent"

Larkin and Ovchinikov (1969)
Kitaev (2015)
Maldacena, Shenker and Stanford (2016)

Multi-point correlation functions

In the energy eigenbasis

$$C_p^\beta(t_1, \ldots, t_{p-1}) = \sum_{i_1,\ldots,i_p} \left[\frac{e^{-\beta E_{i_1}}}{Z} A_{i_1 i_2}(t_1) A_{i_2 i_3}(t_2) \ldots A_{i_p i_1}(0) \right]$$

For any $p > 2$ products of different matrix elements!

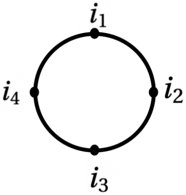
Argument for correlations

$|f_e(\omega)|^2$ Fourier transform of $C_2^\beta(t)$

A_{ij} independent variables $\rightarrow$ all multi-point functions determined solely by $f_e(\omega)$, i.e. by $C_2^\beta(t)$

Unreasonable in general

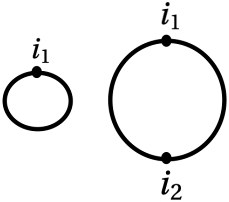
Beyond independent matrix elements



Multipoint functions

$$\overline{A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{i_4 i_1}} \quad \text{for} \quad i_1 \neq i_2 \ldots \neq i_n$$

$$\propto f_e^{(4)}(\omega_1, \omega_2, \omega_3) \to C_4^{\beta}(t_1, t_2, t_3)$$



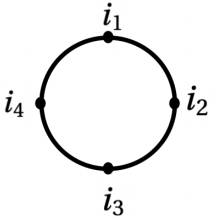
Same spirit as usual ETH (and toy ETH)

$$f_e^{(1)} = \mathscr{A}(e) \qquad f_e^{(2)}(\omega) = |f_e(\omega)|^2$$

Generalized ETH

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_n i_1}} \simeq e^{-(n-1)Ns(e)} f_e^{(n)}(\omega_1, \ldots, \omega_{n-1})$$

for $i_1 \neq i_2 \ldots \neq i_n$



$$e = \frac{1}{n} \sum_{k=1}^{n} e_{i_k} \qquad \omega_k = E_{i_k} - E_{i_{k+1}}$$

+ other assumptions

Foini and Kurchan (2019)

Free probability

Random matrix elements of one operator at different times

One matrix vs Infinitely many matrices!

$$\kappa_n^{\beta}(t_1,\dots,t_{n-1},0) = \frac{1}{Z}\sum_{i_1\neq i_2\neq \dots \neq i_n} e^{-\beta E_{i_1}} A(t_1)_{i_1 i_2}\dots A(0)_{i_n i_1}$$

$$= \int \mathrm{d}\omega_1 \dots \mathrm{d}\omega_{n-1} e^{i\vec{\omega}\cdot\vec{t}} e^{-\beta\vec{\omega}\cdot\vec{l}_n} f^{(n)}_{\epsilon_\beta}(\omega_1,\dots,\omega_{n-1})$$

$\vec{l}_n = \left(\frac{n-1}{n},\dots,\frac{1}{n}\right)$

Pappalardi, Foini and Kurchan (2022)

In summary ...

- Propose a (simple) ansatz able to account for correlations between matrix elements. Relevant for multi-point functions
- Recognise importance of free probability in connection with quantum statistical mechanics

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