

Primordial black holes and numerical stochastic inflation

Paris, Nov 2023

Eemeli Tomberg, Lancaster University

Based on 2012.06551, 2111.07437, 2210.17441,
2205.13540, 2304.10903

Why primordial black holes (PBHs)?

Black holes formed in early Universe

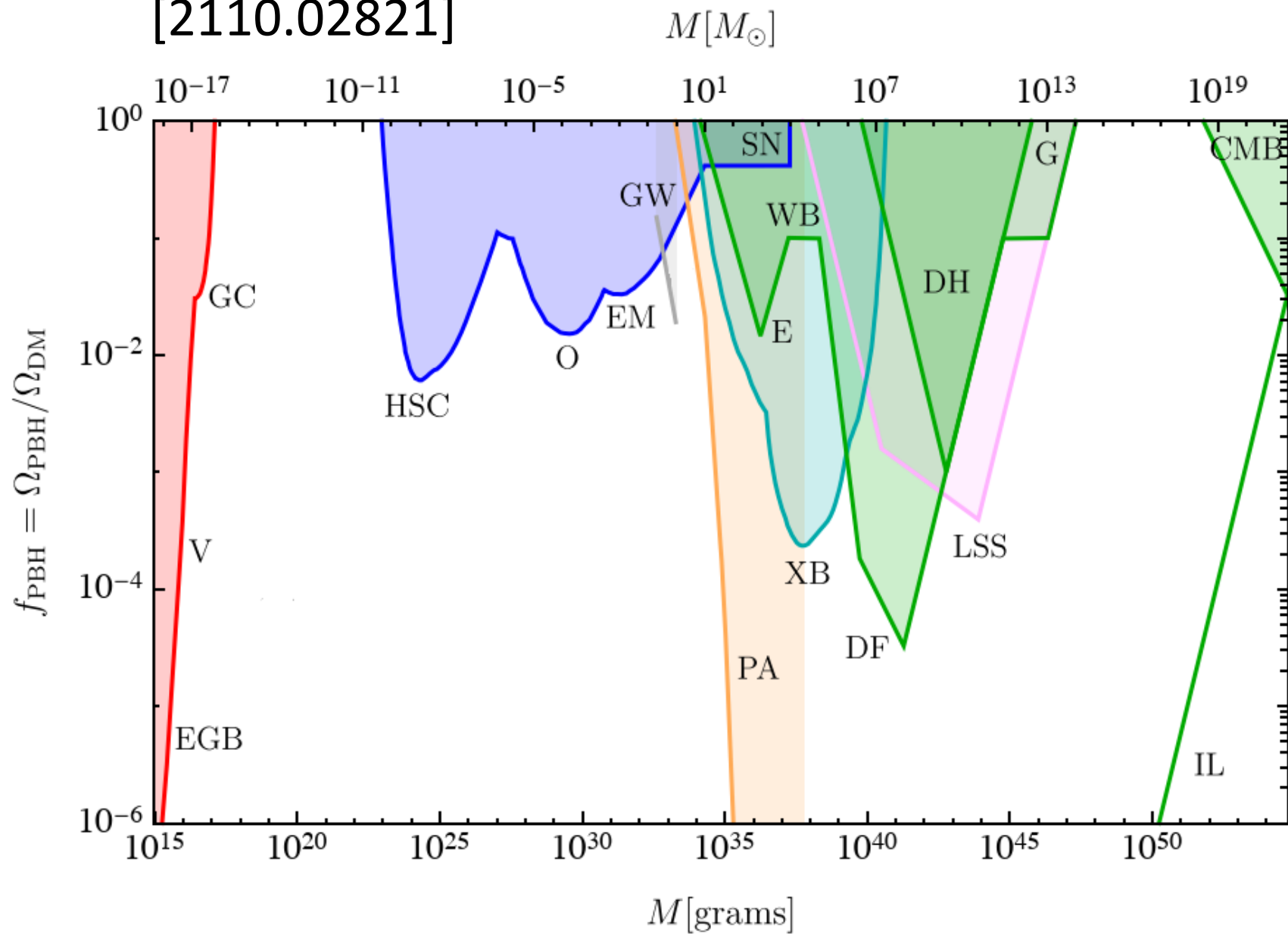
- Carry information of conditions there (small-scale perturbations)
- Any mass (Hawking evaporation?)

Applications in cosmology

- Dark matter candidate
- Seeds of supermassive black holes



[2110.02821]



Primordial black holes are connected to gravitational waves (GWs)

Black hole mergers

Stochastic scalar-induced gravitational waves

Matching stochastic GW signal to PBH statistics?

- GWs: sourced by typical scalar perturbations
- PBHs: sourced by extreme scalar perturbations

Origins of primordial black holes

Cosmological phase transitions

Cosmic strings

Primordial perturbations: cosmic inflation

Origins of primordial black holes

Cosmological phase transitions

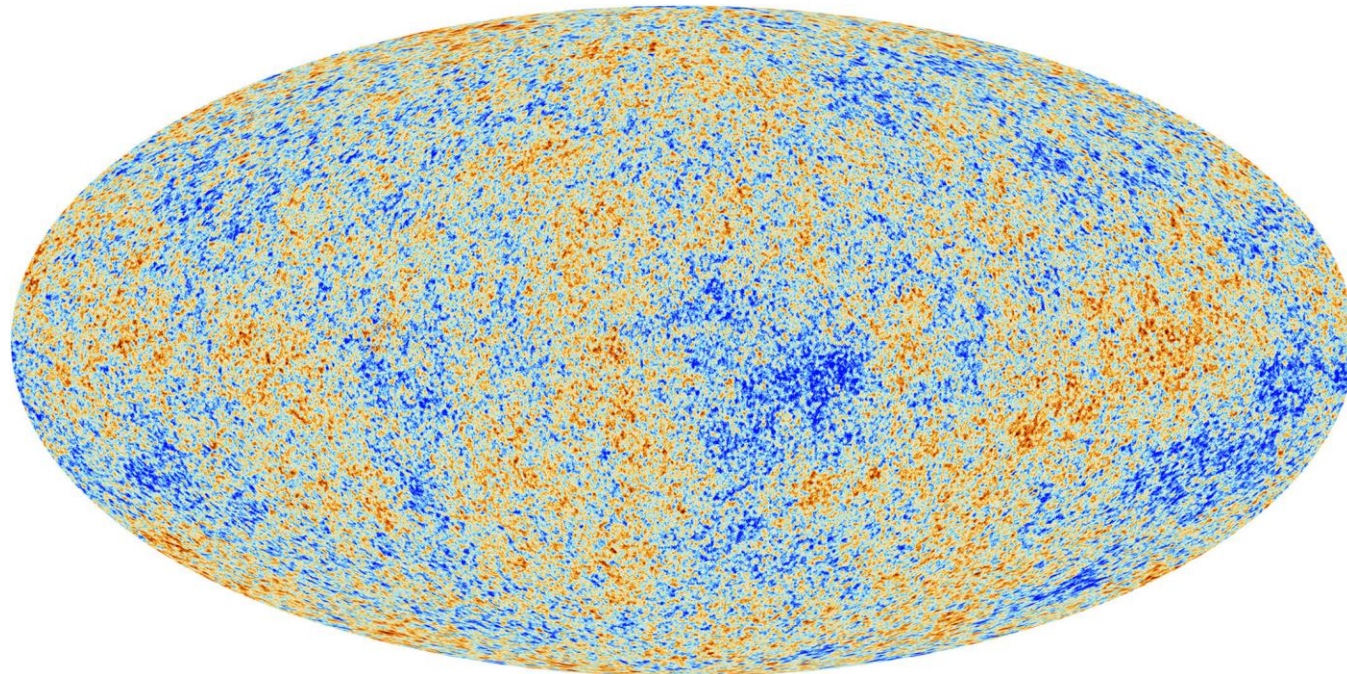
Cosmic strings

Primordial perturbations: cosmic inflation

Black holes from primordial perturbations

Cosmic inflation: quantum fluctuations

Later: strongest collapse into black holes



I. (Semi-)inflection point inflation

II. Stochastic inflation

III. Black hole statistics

I. (Semi-)inflection point inflation

II. Stochastic inflation

III. Black hole statistics

Single-field inflation is simple

Action:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi) \right]$$

Single-field inflation is simple

Action:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi) \right]$$

Background equations of motion:

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \quad 3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

Single-field inflation is simple

Action:

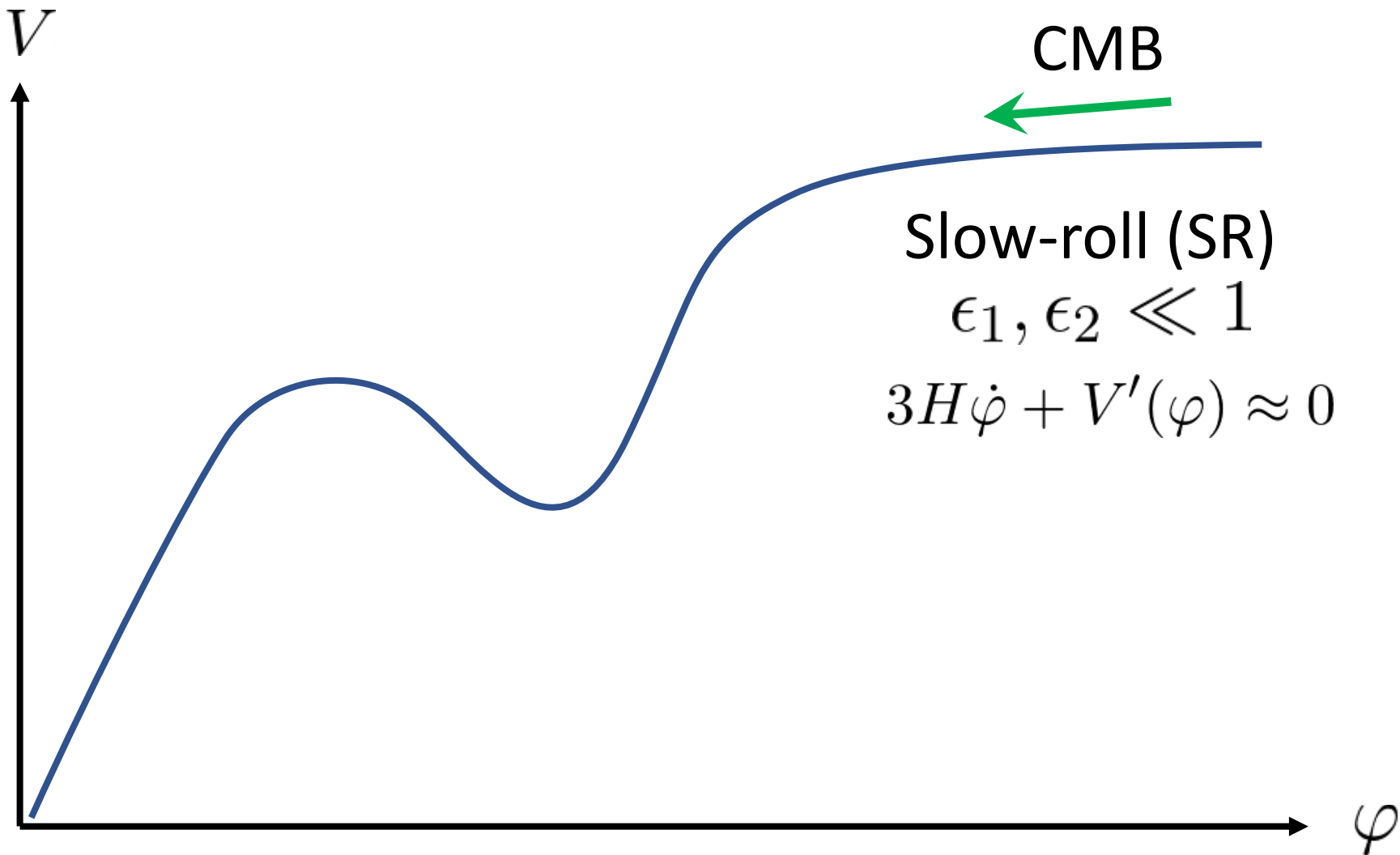
$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi) \right]$$

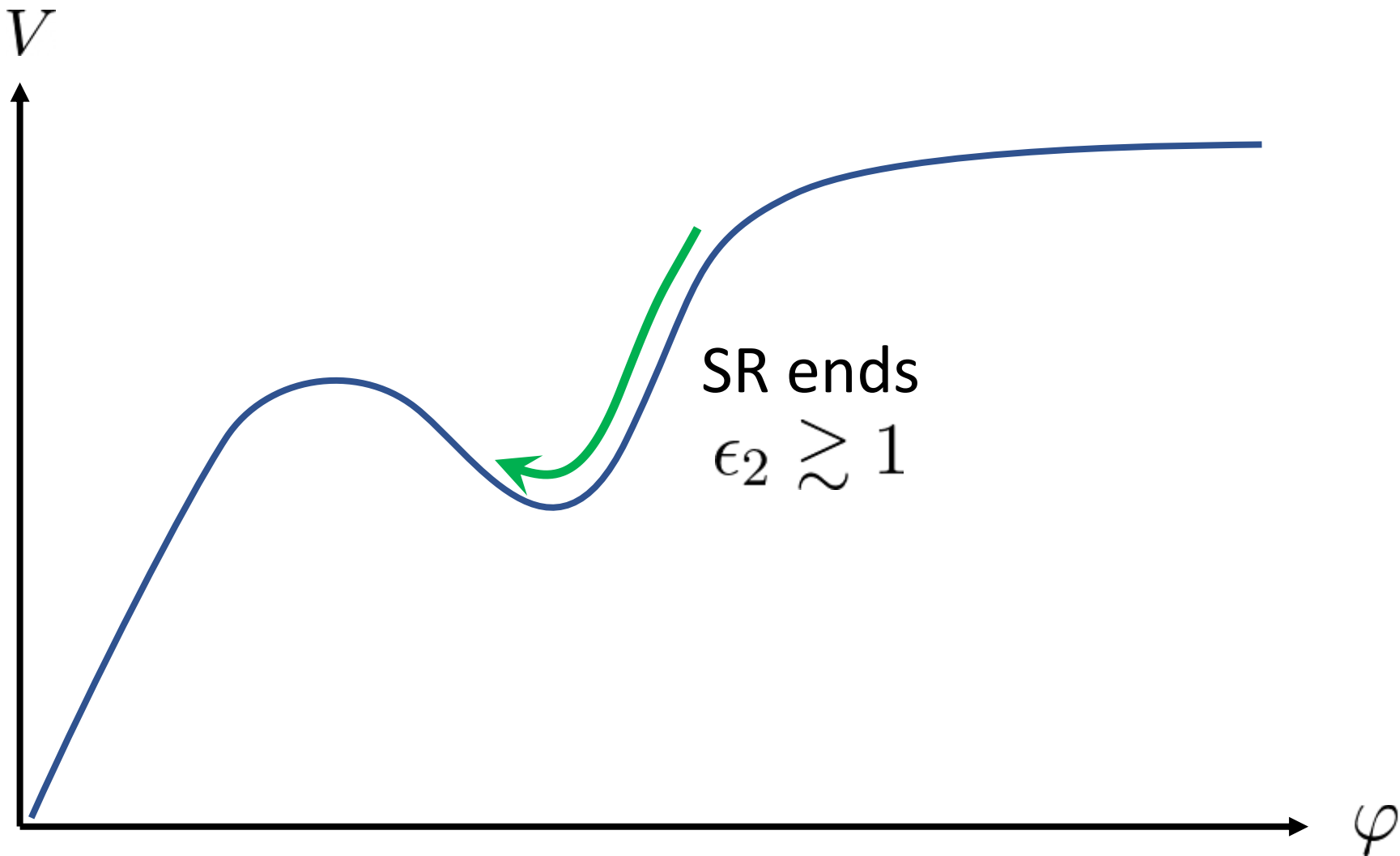
Background equations of motion:

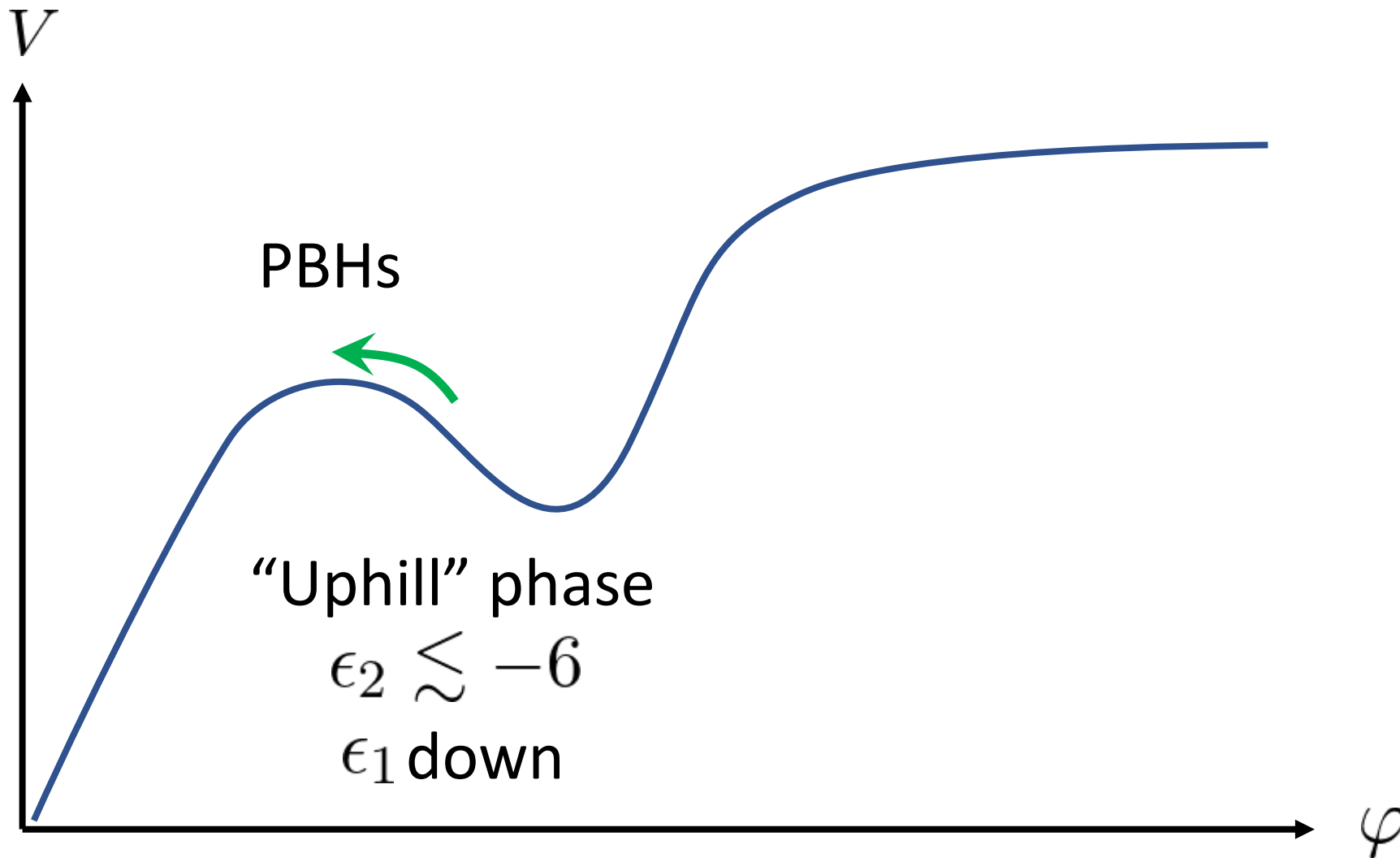
$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \quad 3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

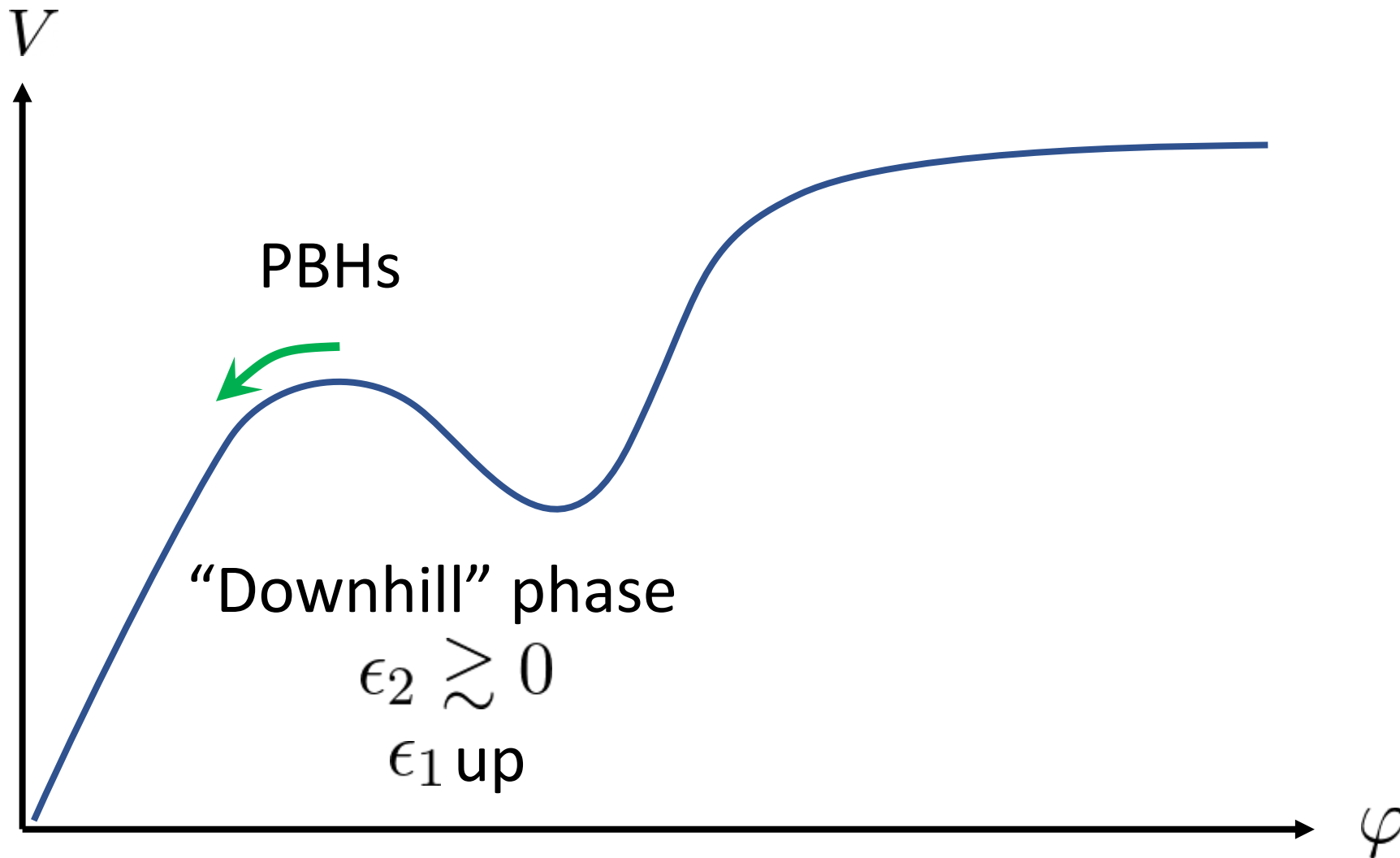
Slow-roll parameters:

$$\epsilon_1 \equiv -\partial_N \ln H, \quad \epsilon_2 \equiv \partial_N \ln \epsilon_1$$









Linear perturbations grow near feature

Comoving curvature perturbation $\mathcal{R} = \frac{\delta\varphi}{\sqrt{2\epsilon_1}}$

$$\ddot{\mathcal{R}}_k + H(3 + \epsilon_2)\dot{\mathcal{R}}_k + \frac{k^2}{a^2}\mathcal{R}_k = 0$$

Linear perturbations grow near feature

Comoving curvature perturbation $\mathcal{R} = \frac{\delta\varphi}{\sqrt{2\epsilon_1}}$

$$\ddot{\mathcal{R}}_k + H(3 + \epsilon_2)\dot{\mathcal{R}}_k + \frac{k^2}{a^2}\mathcal{R}_k = 0$$

Vacuum initial conditions:

$$\mathcal{R}_k = \frac{1}{2a\sqrt{k\epsilon_1}}e^{ik/(aH)}$$

Late times:

$$\mathcal{R}_k \rightarrow \text{const. if } \epsilon_2 > -3$$

Linear perturbations grow near feature

Comoving curvature perturbation $\mathcal{R} = \frac{\delta\varphi}{\sqrt{2\epsilon_1}}$

$$\ddot{\mathcal{R}}_k + H(3 + \epsilon_2)\dot{\mathcal{R}}_k + \frac{k^2}{a^2}\mathcal{R}_k = 0$$

Vacuum initial conditions:

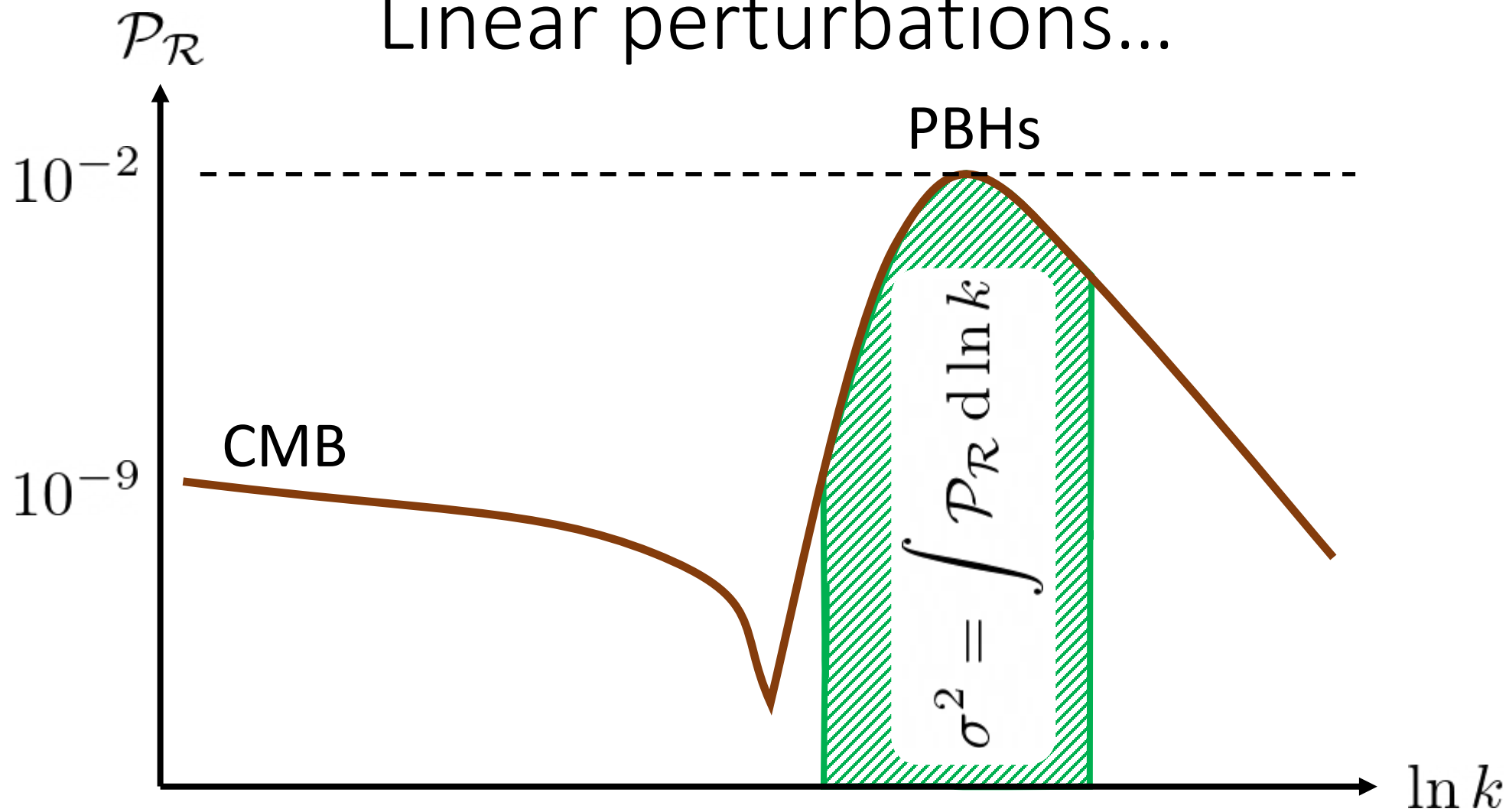
$$\mathcal{R}_k = \frac{1}{2a\sqrt{k\epsilon_1}}e^{ik/(aH)}$$

Late times:

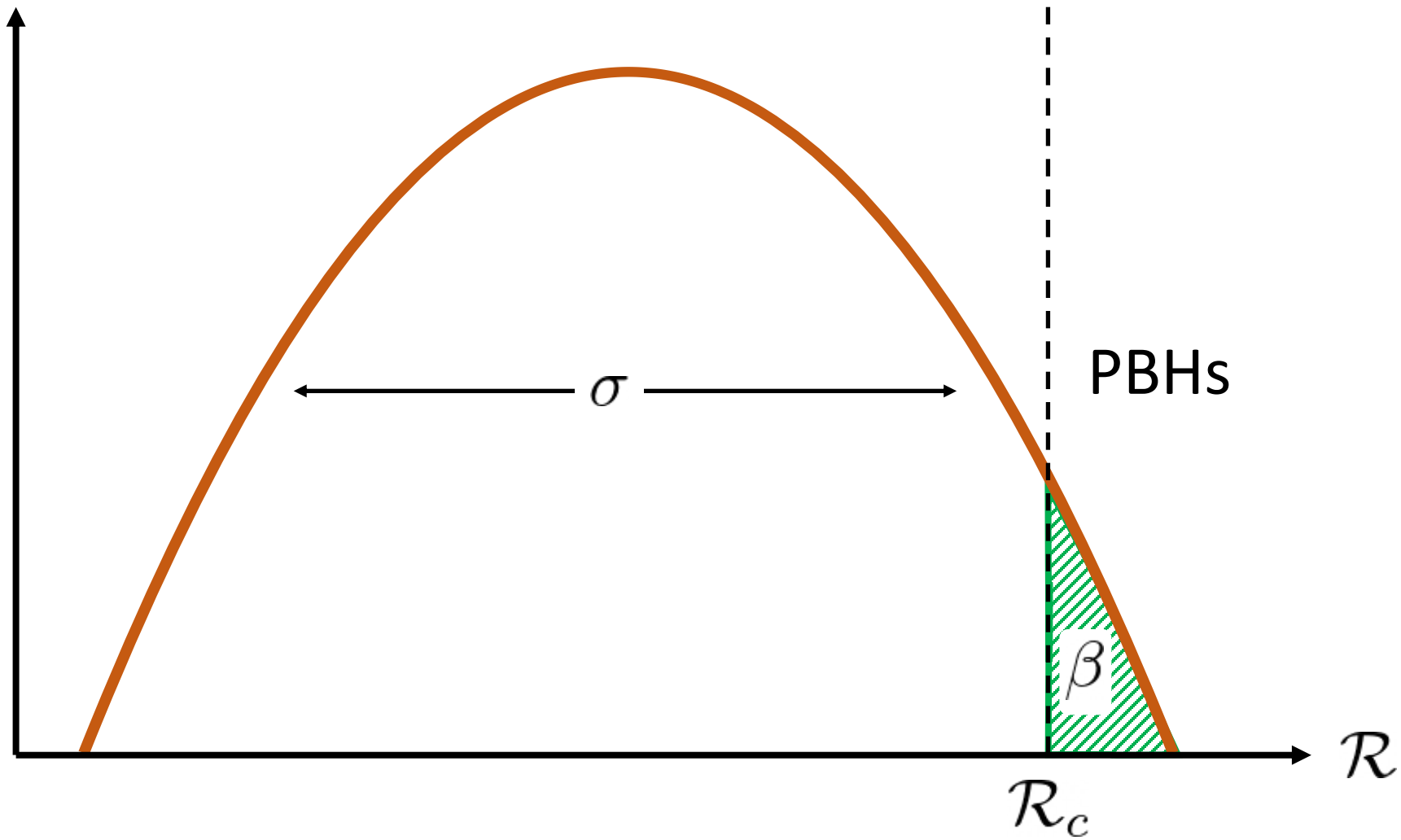
$$\mathcal{R}_k \rightarrow \text{const. if } \epsilon_2 > -3$$

Define power spectrum: $\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2$

Linear perturbations...



$\log p(\mathcal{R})$...Gaussian distribution



Why this picture is wrong

\mathcal{R} is not the correct statistic for PBH formation

Perturbations in the tail are not Gaussian

I. (Semi-)inflection point inflation

II. Stochastic inflation

III. Black hole statistics

Approximations in two regimes

Sub-Hubble scales:

Linear perturbation theory good; neglect mode couplings

$$\delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k + H^2 \left(\frac{k^2}{a^2 H^2} - \frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3 \right) \delta\varphi_k = 0$$

Super-Hubble scales:

Local FLRW equations good; neglect gradient terms

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0$$

Approximations in two regimes

Inflaton field: $\varphi = \phi + \delta\phi$

Coarse-grained:
FLRW

Short-wavelength:
linear perturbation theory

$$\phi \equiv \int_{k < k_\sigma} \frac{d^3k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}} \quad \delta\phi \equiv \int_{k > k_\sigma} \frac{d^3k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}}$$

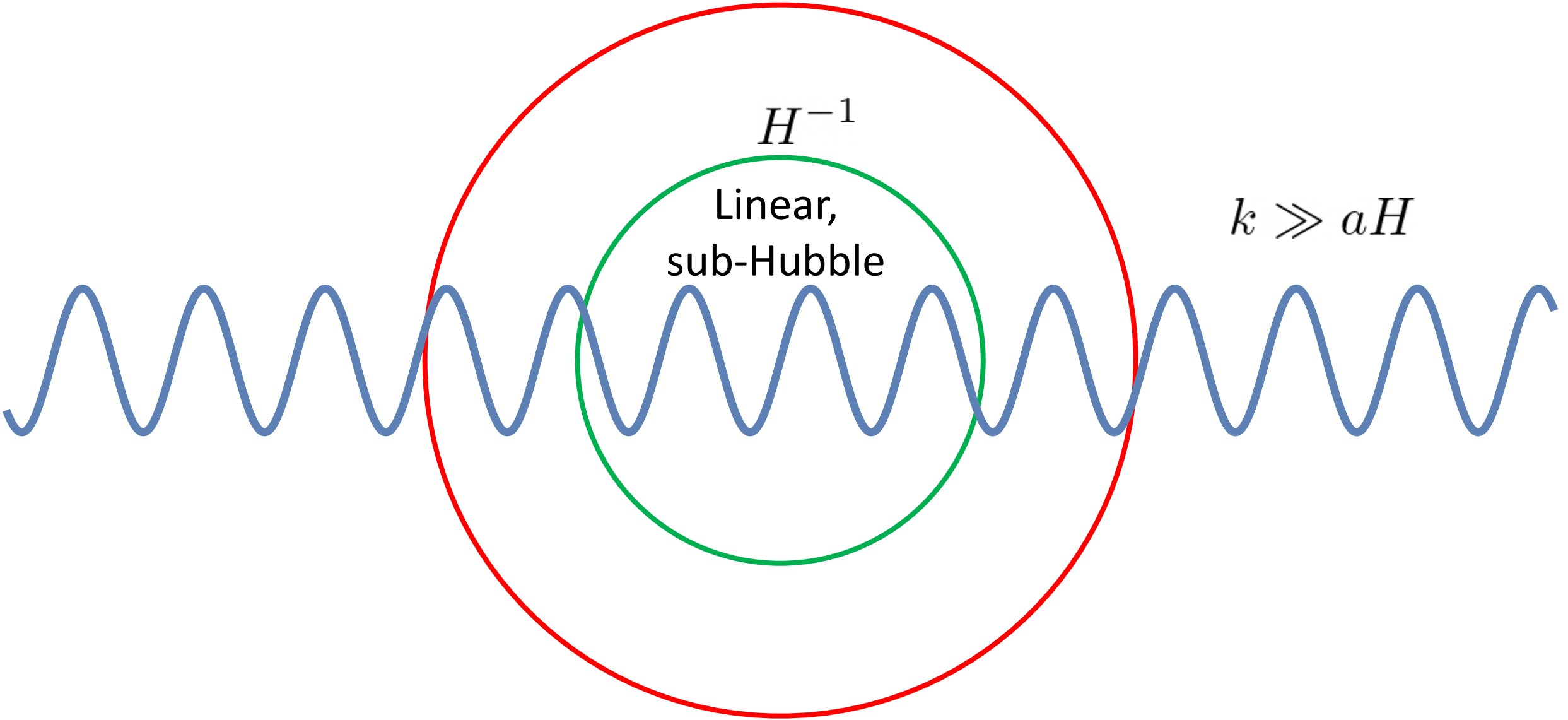
Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma aH$

$$(\sigma H)^{-1}$$

$$H^{-1}$$

Linear,
sub-Hubble

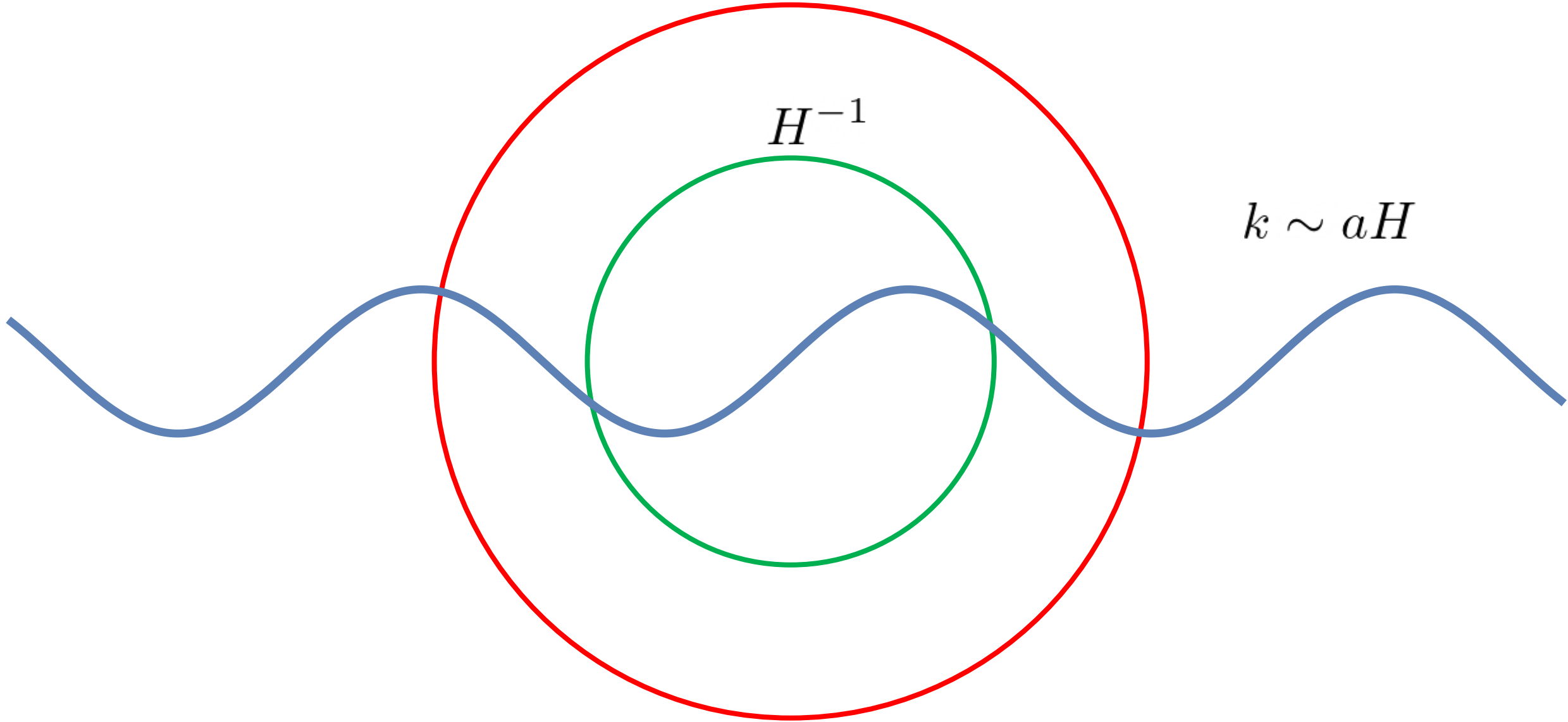
$$k \gg aH$$



$$(\sigma H)^{-1}$$

$$H^{-1}$$

$$k \sim aH$$

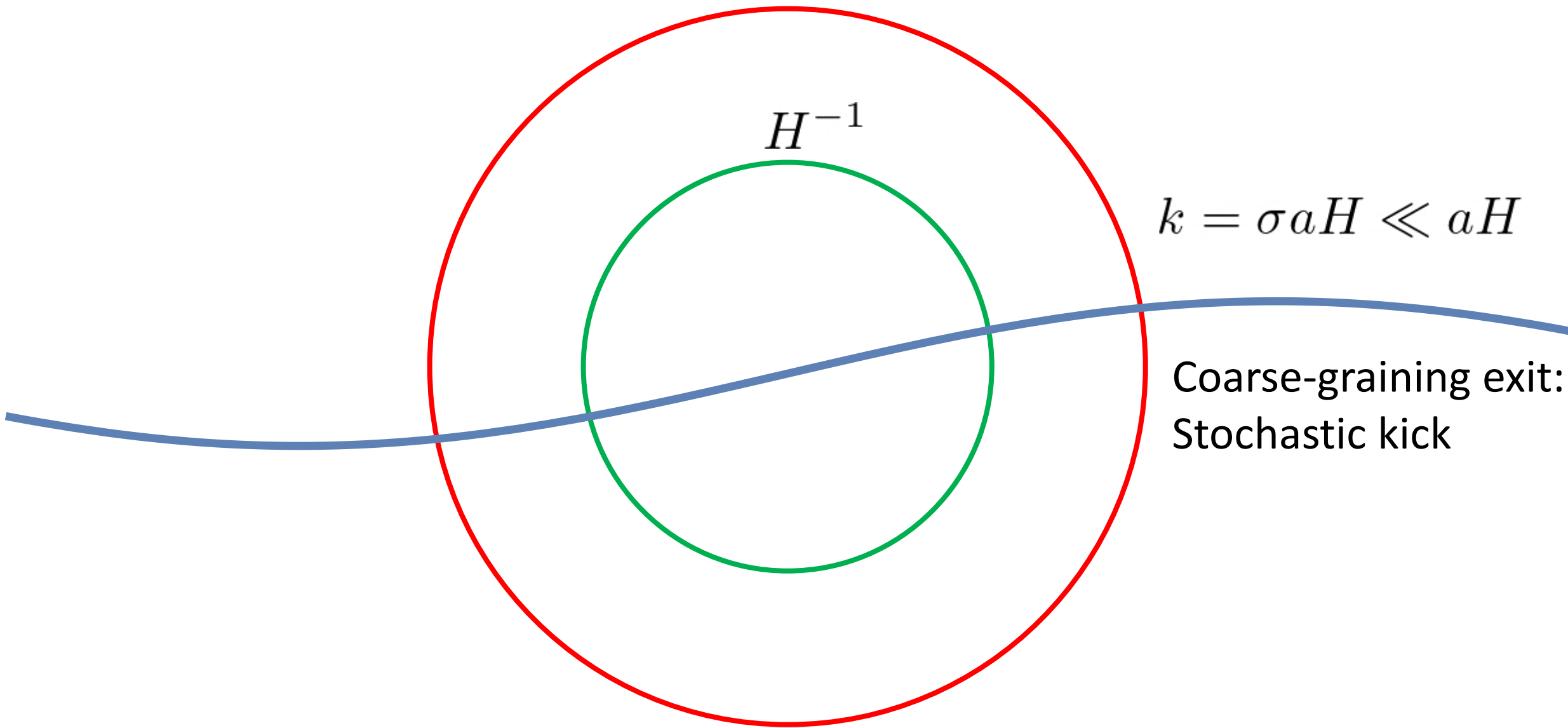


$$(\sigma H)^{-1}$$

$$H^{-1}$$

$$k = \sigma a H \ll a H$$

Coarse-graining exit:
Stochastic kick



Stochastic inflation

$$\phi' = \pi + \xi_\phi, \quad \pi' = - \left(3 - \frac{1}{2}\pi^2 \right) \pi - \frac{V'(\phi)}{H^2} + \xi_\pi, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$
$$\delta\phi_k'' = - \left(3 - \frac{1}{2}\pi^2 \right) \delta\phi_k' - \left[\frac{k^2}{a^2 H^2} + \pi^2 \left(3 - \frac{1}{2}\pi^2 \right) + 2\pi \frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2} \right] \delta\phi_k$$

$$\langle \xi_\phi(N) \xi_\phi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} |\delta\phi_{k_\sigma}(N)|^2 \delta(N - N')$$

$$\langle \xi_\pi(N) \xi_\pi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} |\delta\phi_{k_\sigma}'(N)|^2 \delta(N - N')$$

$$\langle \xi_\phi(N) \xi_\pi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} \delta\phi_{k_\sigma}(N) \delta\phi_{k_\sigma}'^*(N) \delta(N - N')$$

$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

ΔN formalism

$$ds^2 = -dt^2 + a^2(t)e^{2\zeta(x,t)}dx^2$$

$$\Delta N \equiv N - \bar{N} = \mathcal{R} = \zeta$$

Stochastic ΔN formalism:

- solve stochastic system many times; include kicks up to scale k
- collect N on each run
- build statistics for coarse-grained curvature perturbation $\mathcal{R}_{<k}$

ΔN formalism

$$ds^2 = -dt^2 + a^2(t)e^{2\zeta(x,t)}dx^2$$

$$\Delta N \equiv N - \bar{N} = \mathcal{R} = \zeta$$

Stochastic ΔN formalism:

- solve stochastic system many times; include kicks up to scale k
- collect N on each run
- build statistics for coarse-grained curvature perturbation $\mathcal{R}_{<k}$

Stochastic inflation

$$\phi' = \pi + \xi_\phi, \quad \pi' = - \left(3 - \frac{1}{2}\pi^2 \right) \pi - \frac{V'(\phi)}{H^2} + \xi_\pi, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$
$$\delta\phi_k'' = - \left(3 - \frac{1}{2}\pi^2 \right) \delta\phi_k' - \left[\frac{k^2}{a^2 H^2} + \pi^2 \left(3 - \frac{1}{2}\pi^2 \right) + 2\pi \frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2} \right] \delta\phi_k$$

$$\langle \xi_\phi(N) \xi_\phi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} |\delta\phi_{k_\sigma}(N)|^2 \delta(N - N')$$

$$\langle \xi_\pi(N) \xi_\pi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} |\delta\phi_{k_\sigma}'(N)|^2 \delta(N - N')$$

$$\langle \xi_\phi(N) \xi_\pi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} \delta\phi_{k_\sigma}(N) \delta\phi_{k_\sigma}'^*(N) \delta(N - N')$$

$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

Stochastic inflation

$$\phi' = \pi + \xi_\phi, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V(\phi)}{H^2} = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -\left(3 - \frac{1}{2}\pi^2\right)\delta\phi_k' - \left[\frac{k^2}{\sigma^2} + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle \xi_\phi(N)\xi_\phi(N') \rangle = \frac{1}{\sigma^2} \delta(N - N')$$

$$\langle \xi_\pi(N)\xi_\pi(N') \rangle = |k_\sigma(N)|^2 \delta(N - N')$$

$$\langle \xi_\phi(N)\xi_\pi(N') \rangle = \frac{v_\sigma}{H^2} \frac{d\phi}{dN} \delta\phi_{k_\sigma}(N) \delta\phi_{k_\sigma}'^*(N) \delta(N - N')$$

$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

COMPLICATED

How to move forward?

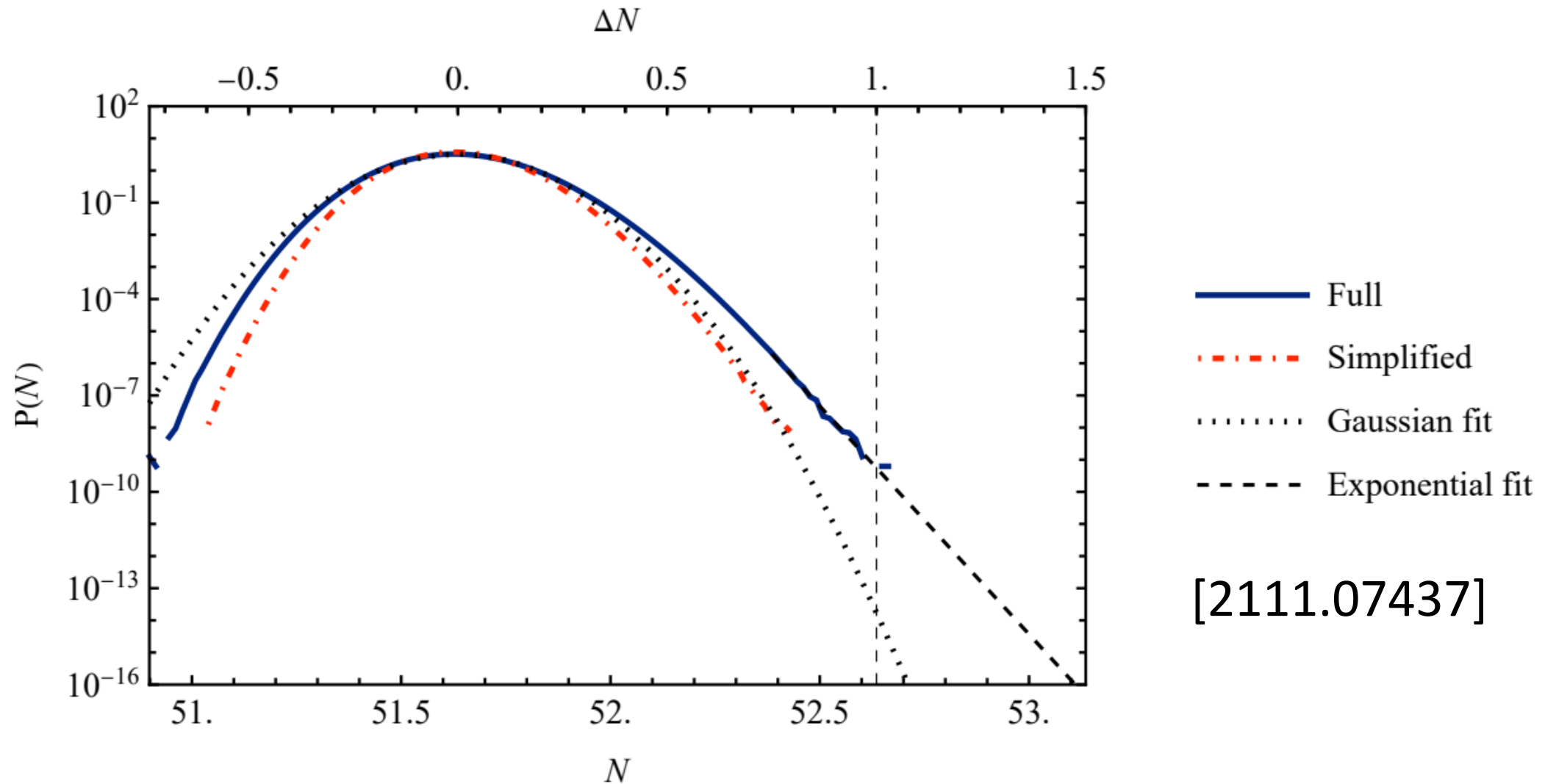
Analytical approximations?

$$\langle \xi_\phi(N) \xi_\phi(N') \rangle \approx \frac{H^2}{4\pi^2} \delta(N - N')$$

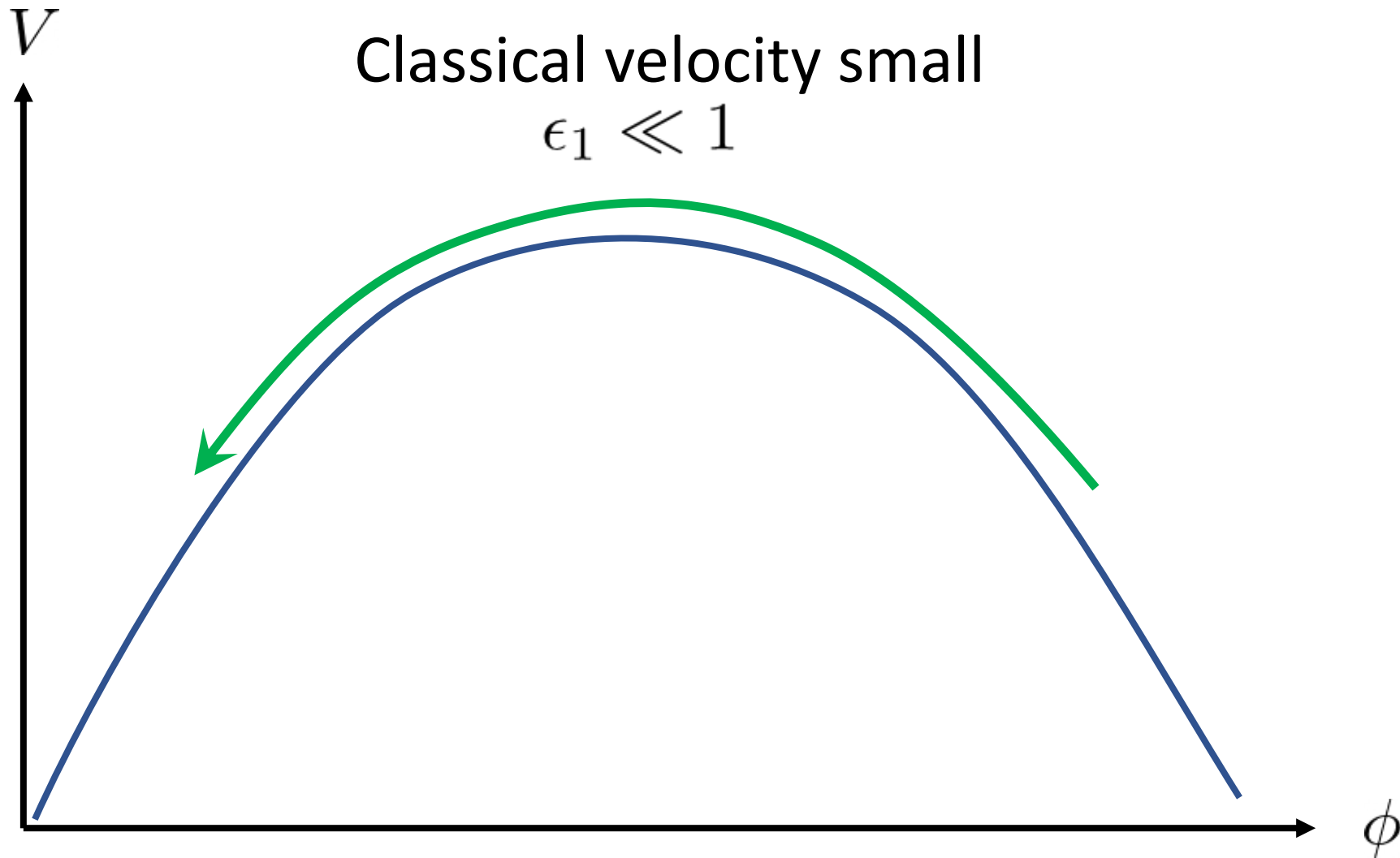
Full numerical computations?

Full numerical computations

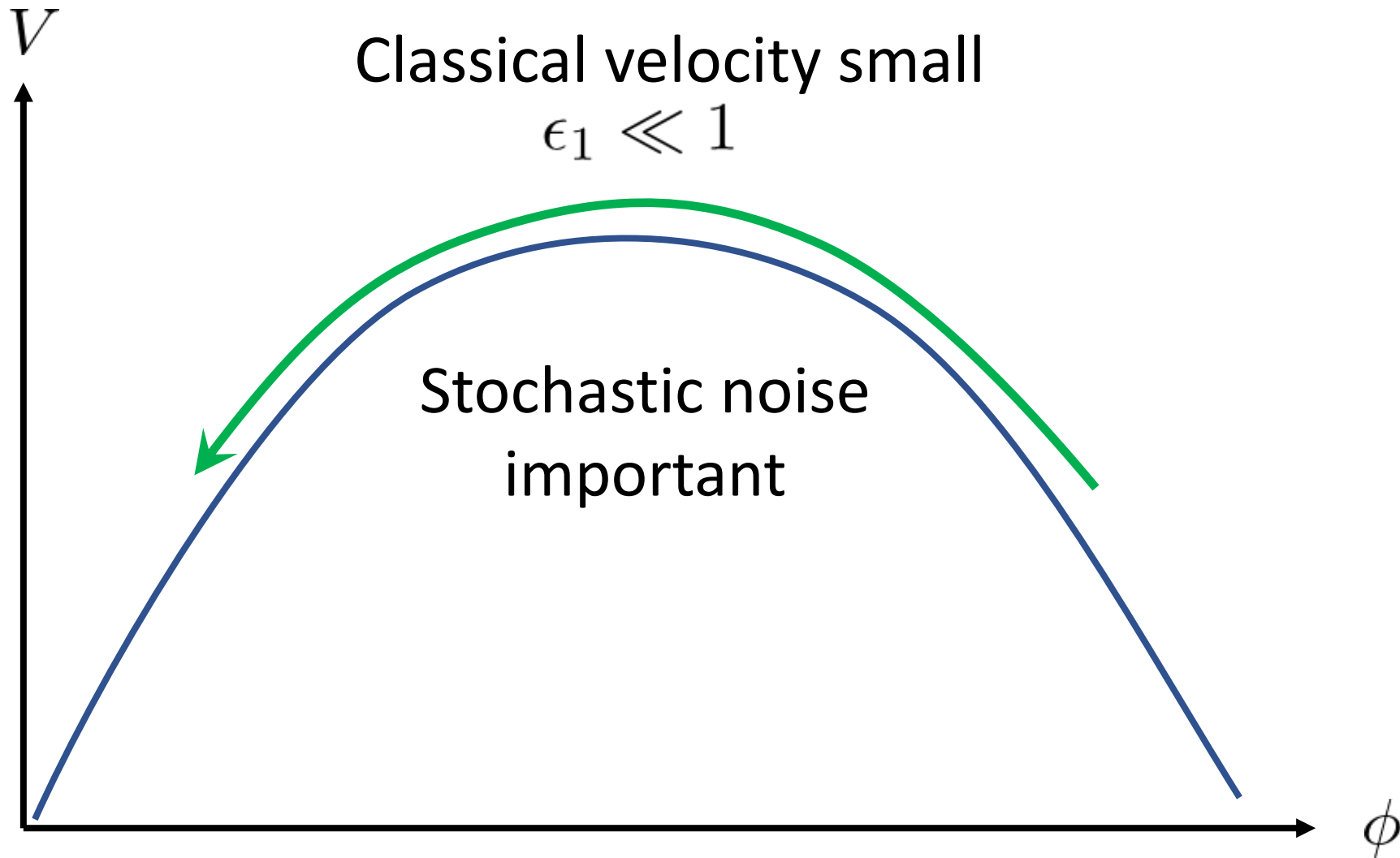
One million CPU hours



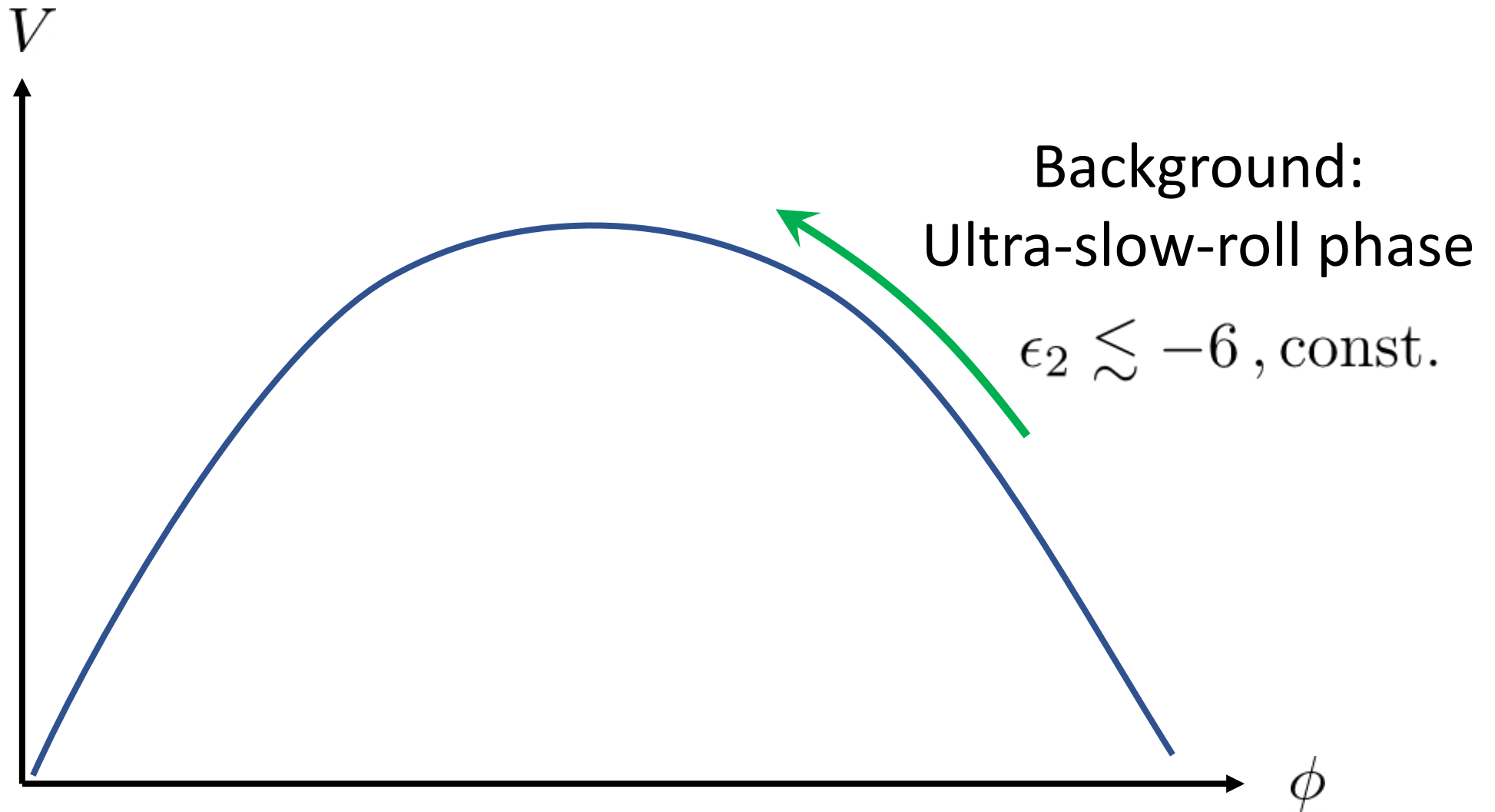
Zoom into the hilltop



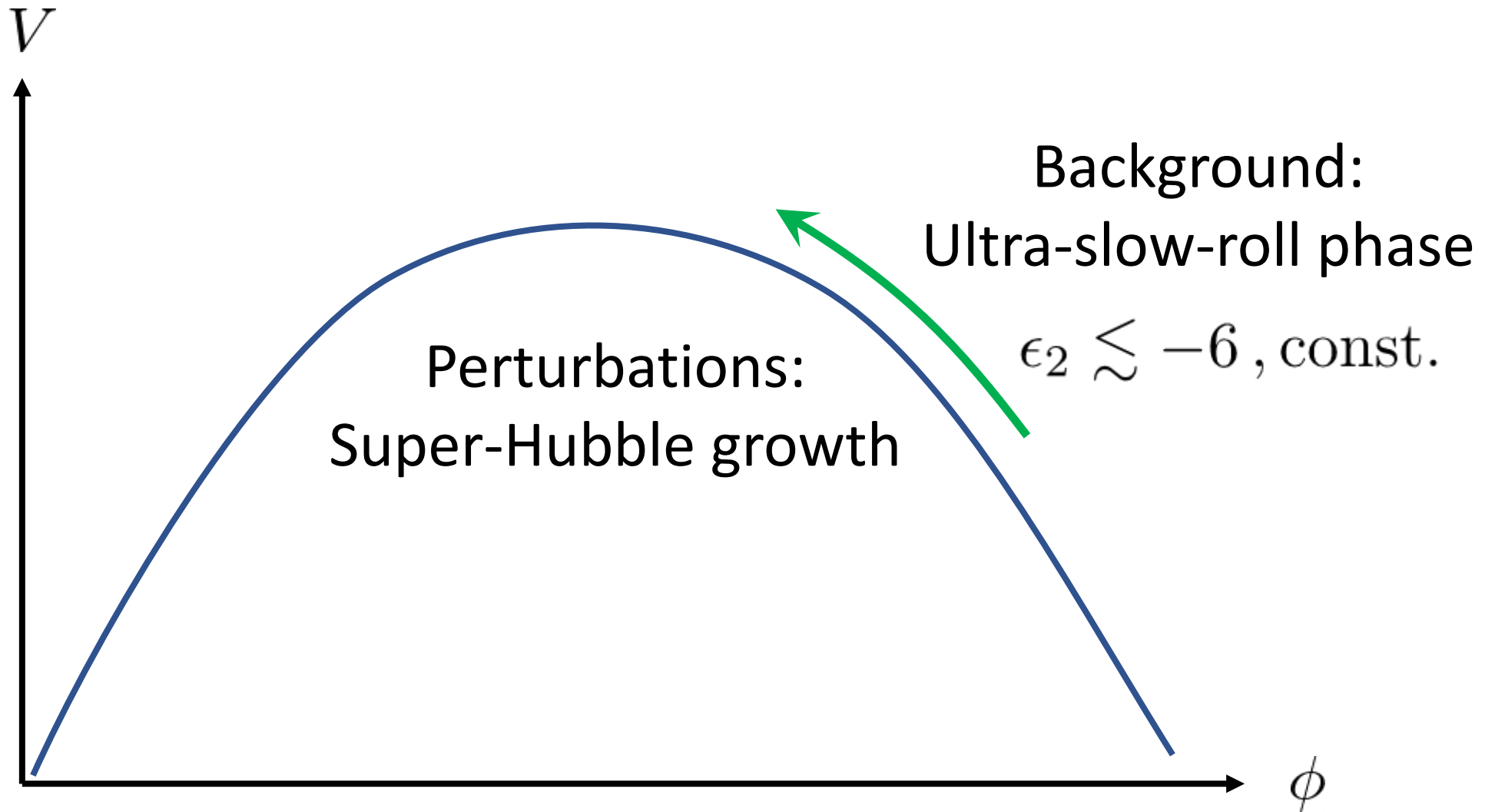
Zoom into the hilltop



Zoom into the hilltop



Zoom into the hilltop

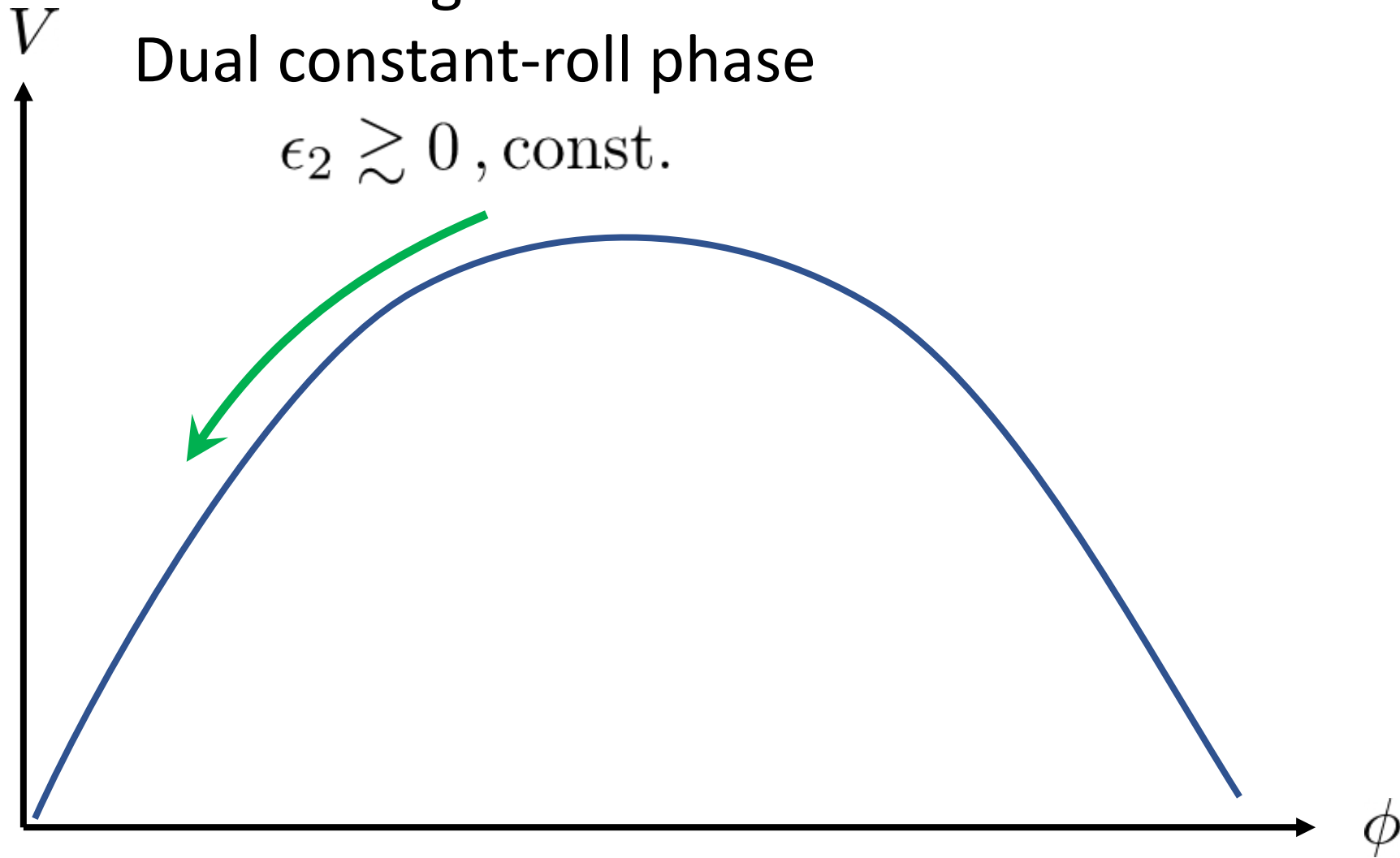


Zoom into the hilltop

Background:

Dual constant-roll phase

$$\epsilon_2 \gtrsim 0, \text{ const.}$$

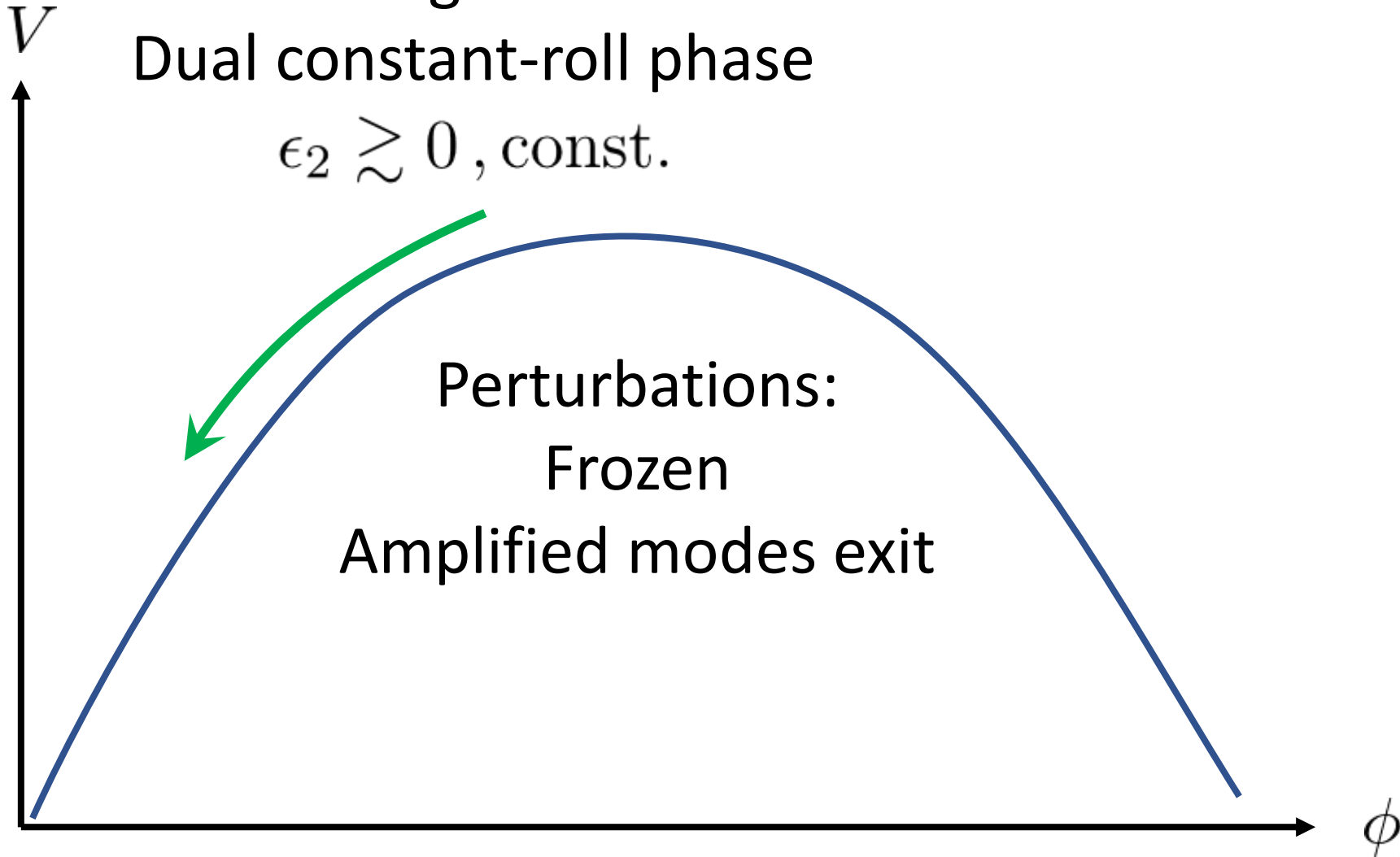


Zoom into the hilltop

Background:

Dual constant-roll phase

$$\epsilon_2 \gtrsim 0, \text{ const.}$$



Equations simplify in dual constant-roll phase

Adiabatic perturbations:


motion along classical trajectory only

Noise independent of background stochasticity:

pre-compute power spectrum

Simplified stochastic equation:


$$d\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)dN + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} \sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)} dN \hat{\xi}_N$$


$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

Simplified stochastic equation:

$$d\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)dN + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} \sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)}dN \hat{\xi}_N$$

$$\phi(N) = \phi_0 \left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X_{<k_\sigma}$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$


$$X_{<k} \equiv \sum_{\tilde{k}=k_{\text{ini}}}^k \sqrt{\mathcal{P}_{\mathcal{R}}(\tilde{k})} d \ln k \hat{\xi}_{\tilde{k}}$$

ΔN distribution

$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}, \quad \sigma_k^2 \equiv \int_{k_{\text{ini}}}^k \mathcal{P}_{\mathcal{R}}(\tilde{k}) d \ln \tilde{k}$$

ΔN distribution

$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}, \quad \sigma_k^2 \equiv \int_{k_{\text{ini}}}^k \mathcal{P}_{\mathcal{R}}(\tilde{k}) d \ln \tilde{k}$$


$$X_{<k} = \frac{2}{\epsilon_2} \left(1 - e^{-\frac{\epsilon_2}{2} \Delta N_{<k}} \right)$$

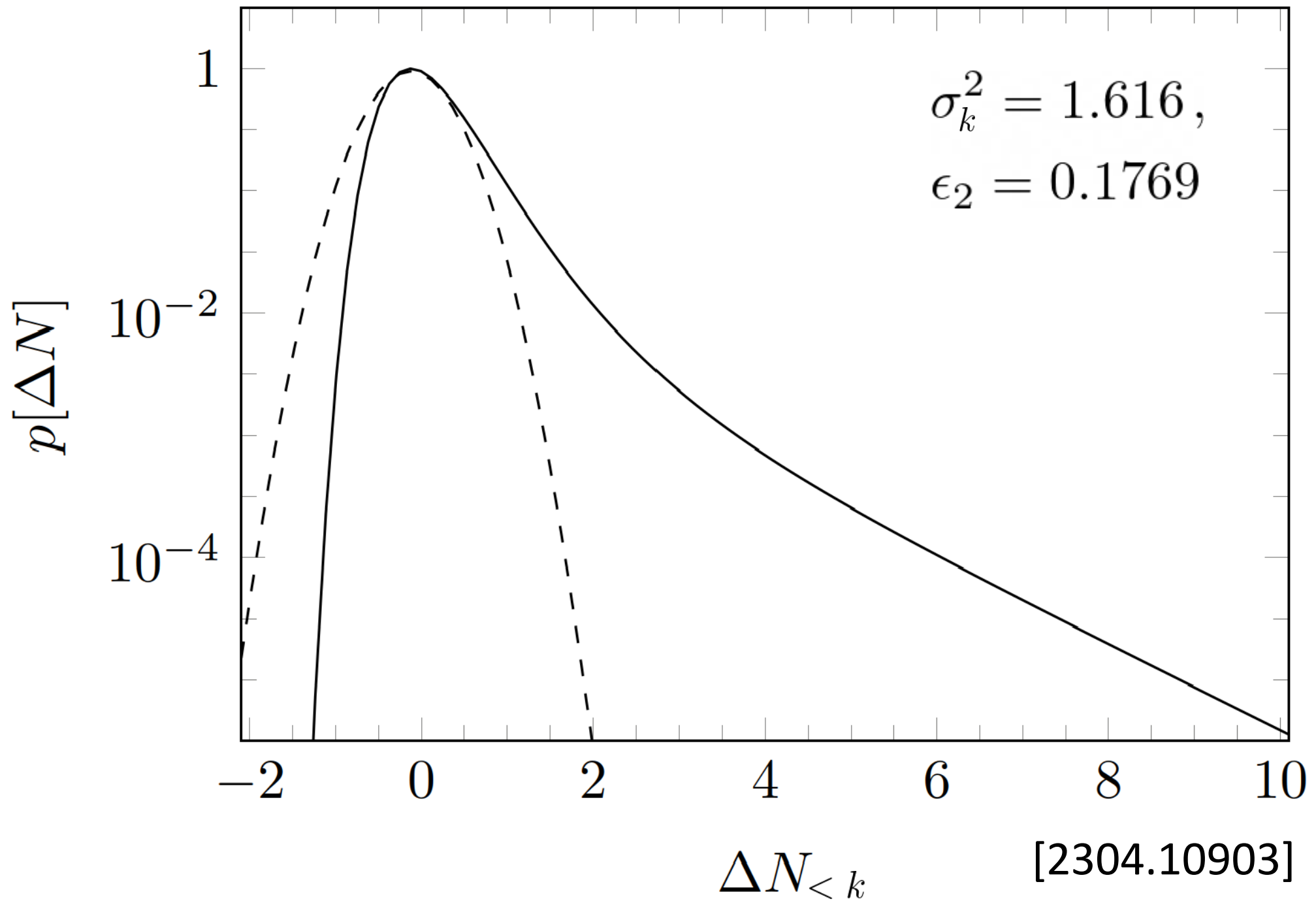
ΔN distribution

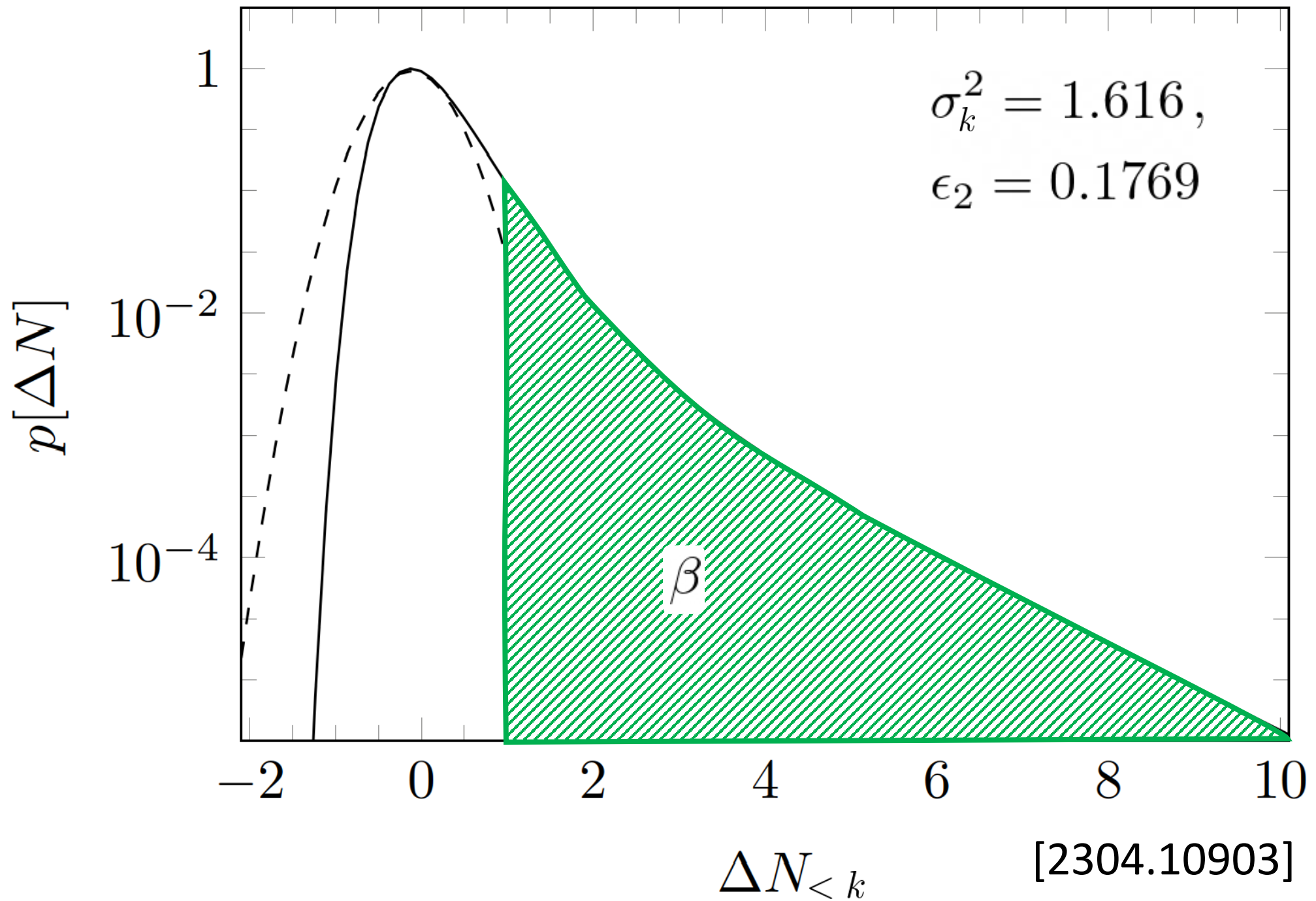
$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}, \quad \sigma_k^2 \equiv \int_{k_{\text{ini}}}^k \mathcal{P}_{\mathcal{R}}(\tilde{k}) d \ln \tilde{k}$$

$$X_{<k} = \frac{2}{\epsilon_2} \left(1 - e^{-\frac{\epsilon_2}{2} \Delta N_{<k}} \right)$$

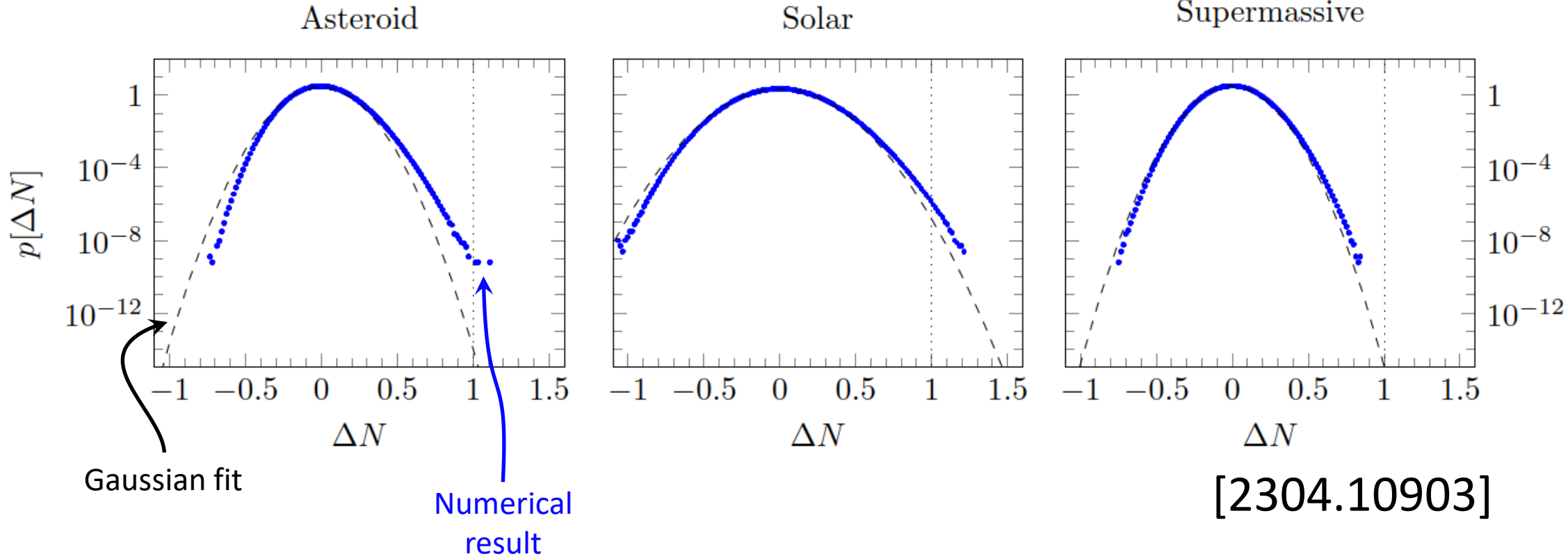
$$p(\Delta N_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp \left[-\frac{2}{\sigma_k^2 \epsilon_2^2} \left(1 - e^{-\frac{\epsilon_2}{2} \Delta N_{<k}} \right)^2 - \frac{\epsilon_2}{2} \Delta N_{<k} \right]$$


$$\Delta N_{<k} = \mathcal{R}_{<k}$$

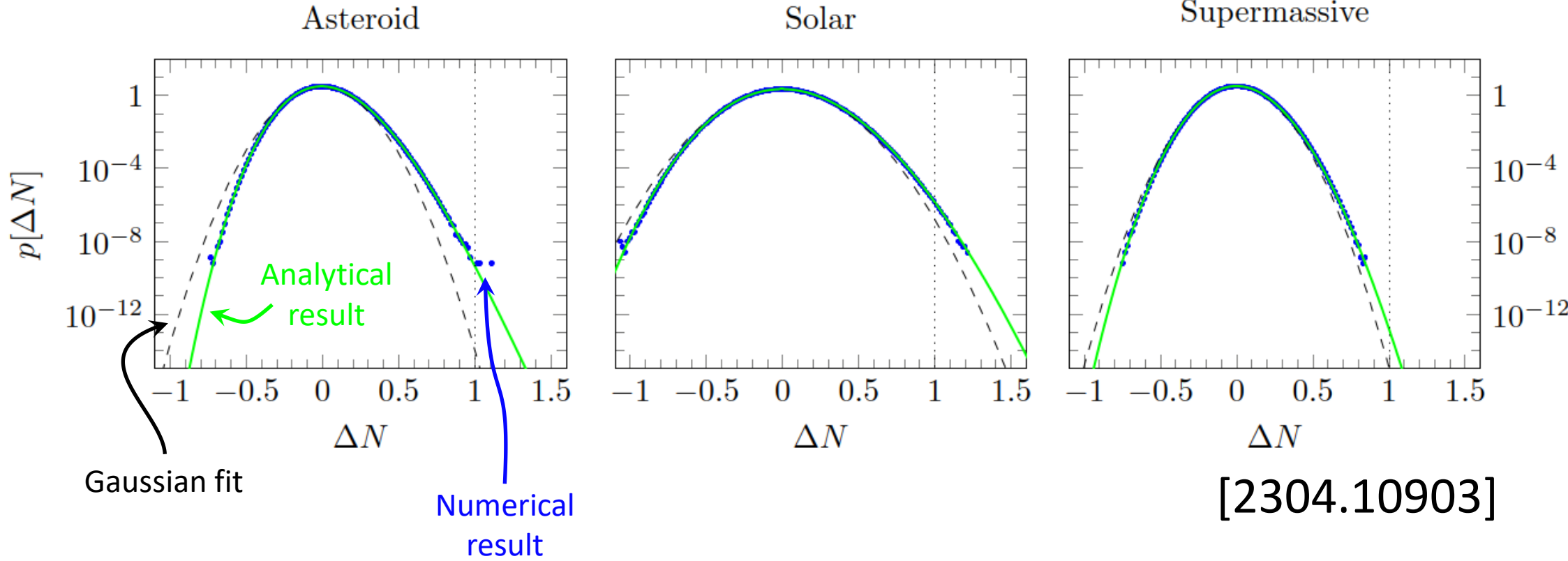




Comparison to numerics



Comparison to numerics



I. (Semi-)inflection point inflation

II. Stochastic inflation

III. Black hole statistics

Compaction function: right tool for determining the collapse threshold

$$\mathcal{C} \equiv 2 \frac{M_{\text{MS}} - M_{\text{bg}}}{R}$$

Collapse: $\mathcal{C}_{\text{max}} > \mathcal{C}_c \approx 0.4$

Compaction function: right tool for determining the collapse threshold

$$\mathcal{C} \equiv 2 \frac{M_{\text{MS}} - M_{\text{bg}}}{R}$$

Collapse: $\mathcal{C}_{\text{max}} > \mathcal{C}_c \approx 0.4$

In inflationary variables:

$$\mathcal{C}(r) = \frac{2}{3} (1 - [1 + r\zeta'(r)]^2)$$

Assume spherical symmetry

$$r\zeta'(r) = \sum_k \frac{2k^2 dk}{\sqrt{2\pi}} \zeta_k \left[\cos(kr) - \frac{\sin(kr)}{kr} \right]$$

$$\zeta_k = \frac{\sqrt{2\pi}}{2k^3} \frac{d\zeta_{<k}}{d \ln k}$$

Assume spherical symmetry

$$r\zeta'(r) = \sum_k \frac{2k^2 dk}{\sqrt{2\pi}} \zeta_k \left[\cos(kr) - \frac{\sin(kr)}{kr} \right]$$

$$\zeta_k = \frac{\sqrt{2\pi}}{2k^3} \frac{d\zeta_{<k}}{d \ln k}$$

Vary k:
Full profile
in one patch of space!



Recall: in the stochastic picture,

$$\zeta_{<k} = \Delta N_{<k} = -\frac{2}{\epsilon_2} \ln \left(1 - \frac{\epsilon_2}{2} X_{<k} \right) = -\frac{2}{\epsilon_2} \ln \left(1 - \frac{\epsilon_2}{2} \sum_{\tilde{k}=k_{\text{ini}}}^k \sqrt{\mathcal{P}_{\mathcal{R}}(\tilde{k})} d \ln k \hat{\xi}_{\tilde{k}} \right)$$

Master formula

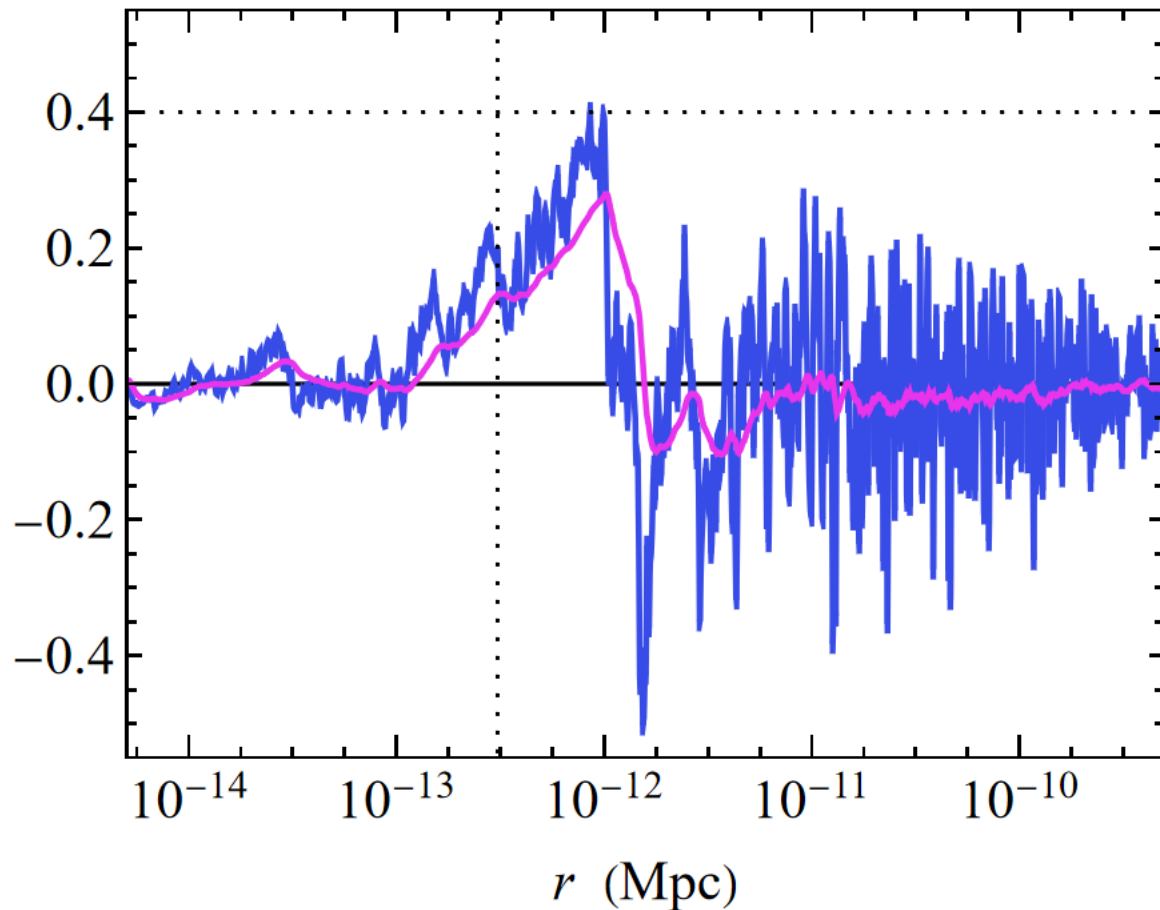
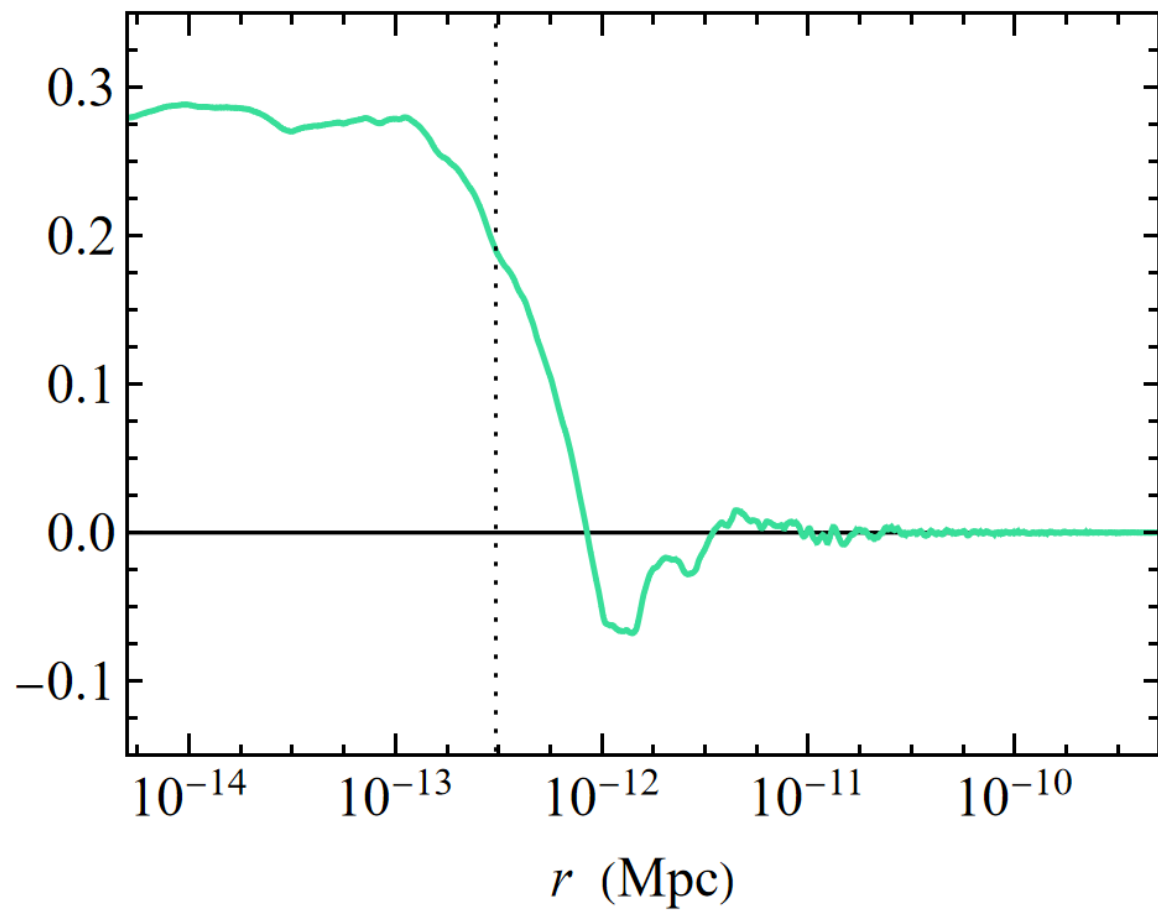
$$r\zeta'(r) = \sum_k \left[- \frac{\hat{\xi}_k}{1 - \frac{\epsilon_2}{2} X_{<k}} \sqrt{\mathcal{P}_\zeta(k)} d \ln k \right. \\ \left. + \frac{\epsilon_2}{4 \left(1 - \frac{\epsilon_2}{2} X_{<k}\right)^2} \mathcal{P}_\zeta(k) d \ln k \right] \\ \times \left[\cos(kr) - \frac{\sin(kr)}{kr} \right]$$

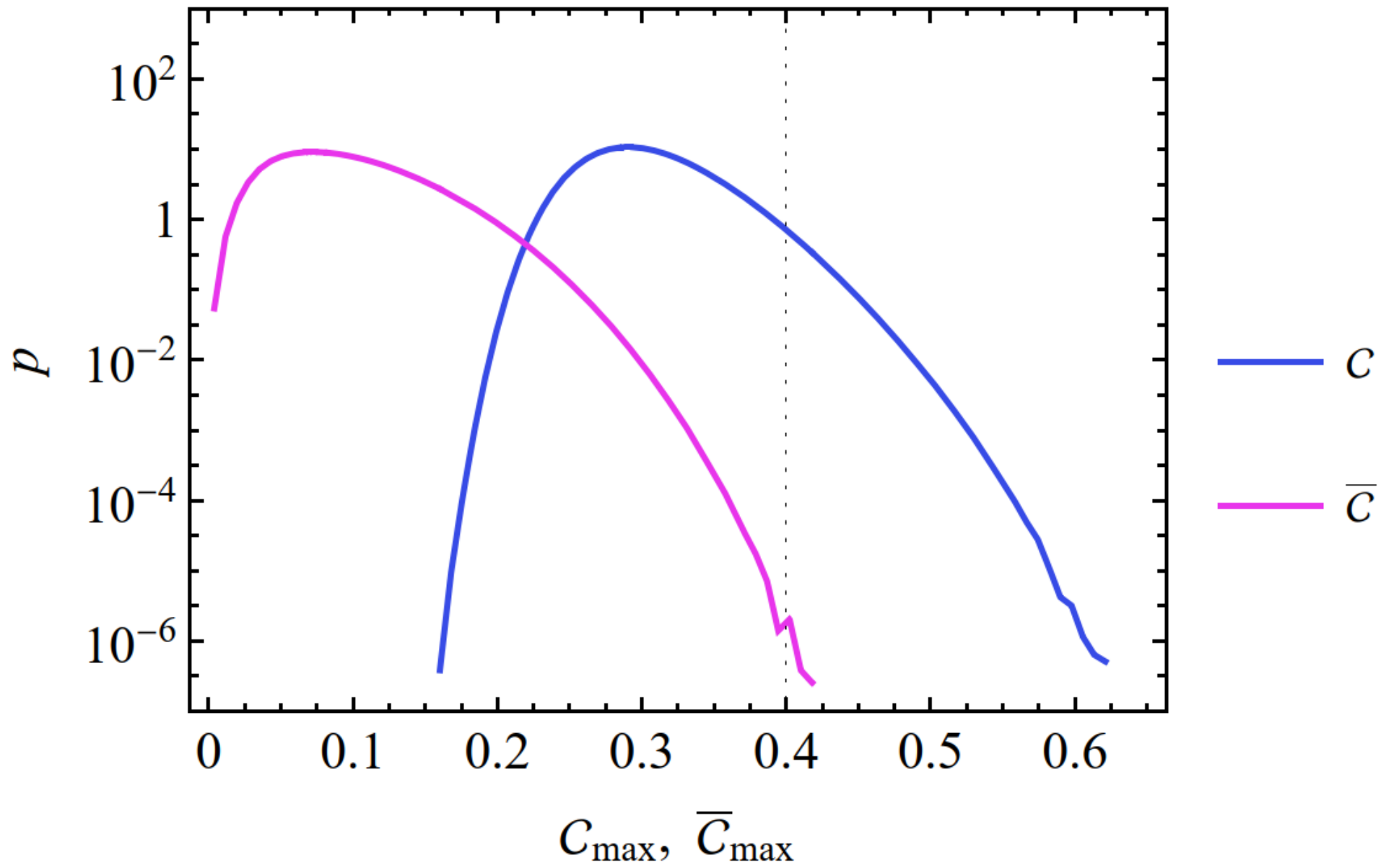
Alternative collapse measure:
averaged compaction function

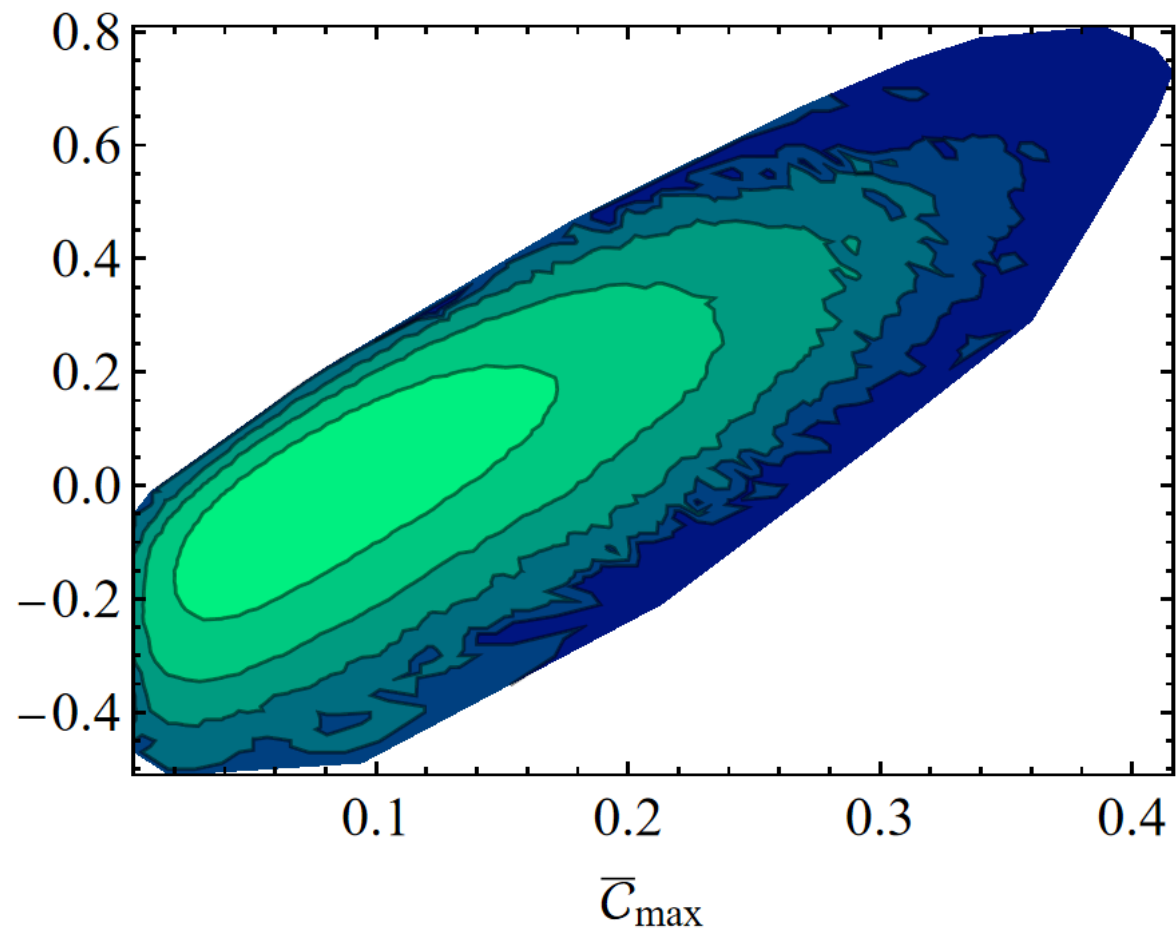
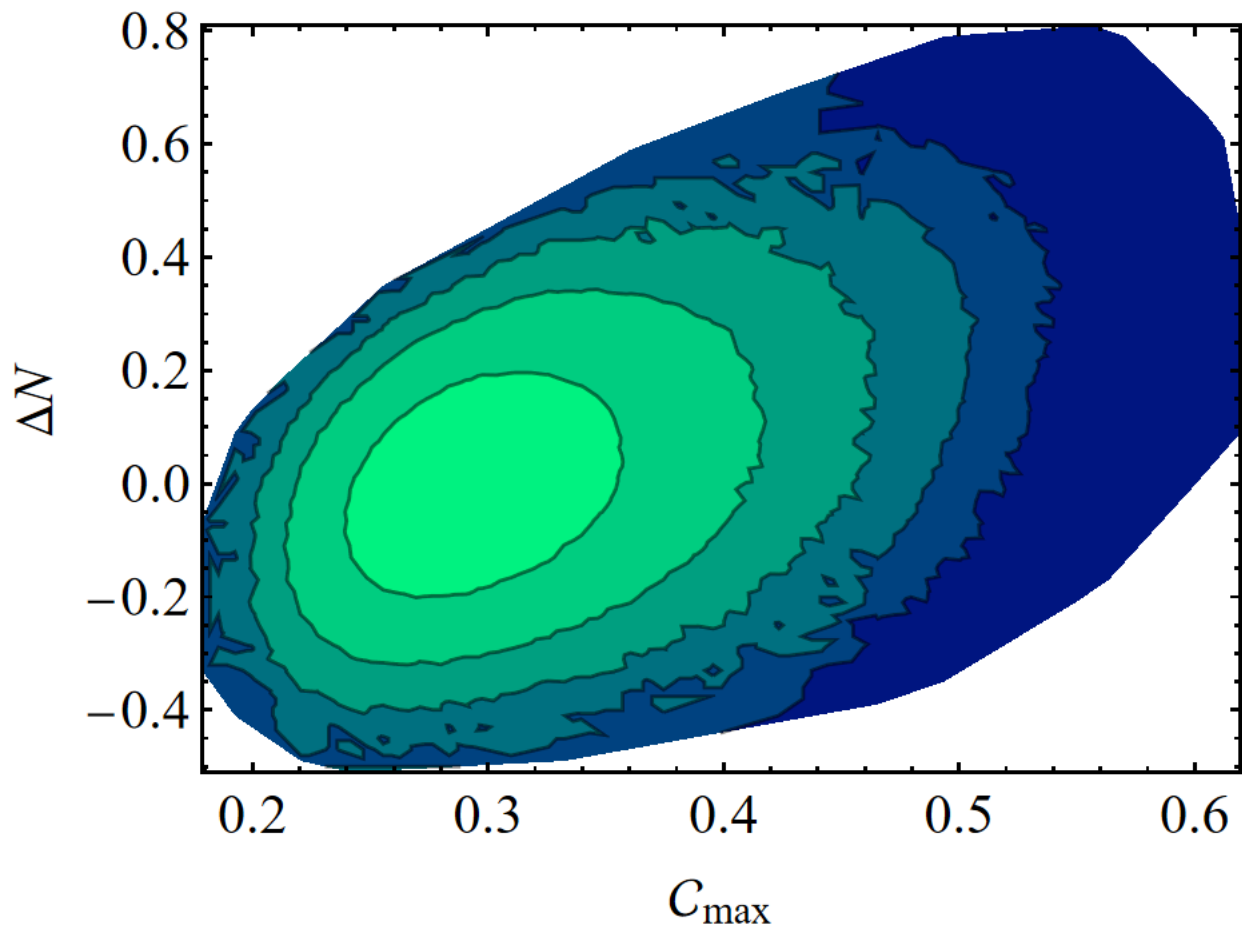
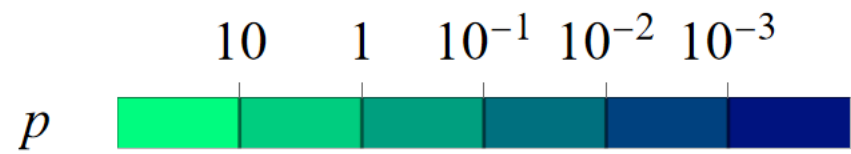
$$\begin{aligned}\bar{\mathcal{C}}(r) &\equiv \frac{3}{R(r)^3} \int_0^{R(r)} d\tilde{R} \tilde{R}^2 \mathcal{C} \\ &= -\frac{2}{r^3 e^{3\zeta(r)}} \int_0^r d\tilde{r} \tilde{r}^2 e^{3\zeta} [2\tilde{r}\zeta' + 3(\tilde{r}\zeta')^2 + (\tilde{r}\zeta')^3]\end{aligned}$$

$R = a r e^\zeta$

— ζ — c — \bar{c}







Initial PBH fractions

Gaussian approximation, $\mathcal{R}_{<k} > 1$, fixed k : $\beta \approx 5 \times 10^{-16}$

Non-Gaussian statistics, $\mathcal{R}_{<k} > 1$, fixed k : $\beta \approx 2.2 \times 10^{-11}$

$\bar{\mathcal{C}}_{\max} > 0.4$: $\beta \approx 1.4 \times 10^{-8}$

$\mathcal{C}_{\max} > 0.4$: $\beta \approx 0.016$

Problems

Collapse simulations have smooth peaks.

Us: Stochastic peaks?

- Numeric converge: ok
- Physics? Smoothing? Window functions?

Multiple peaks?

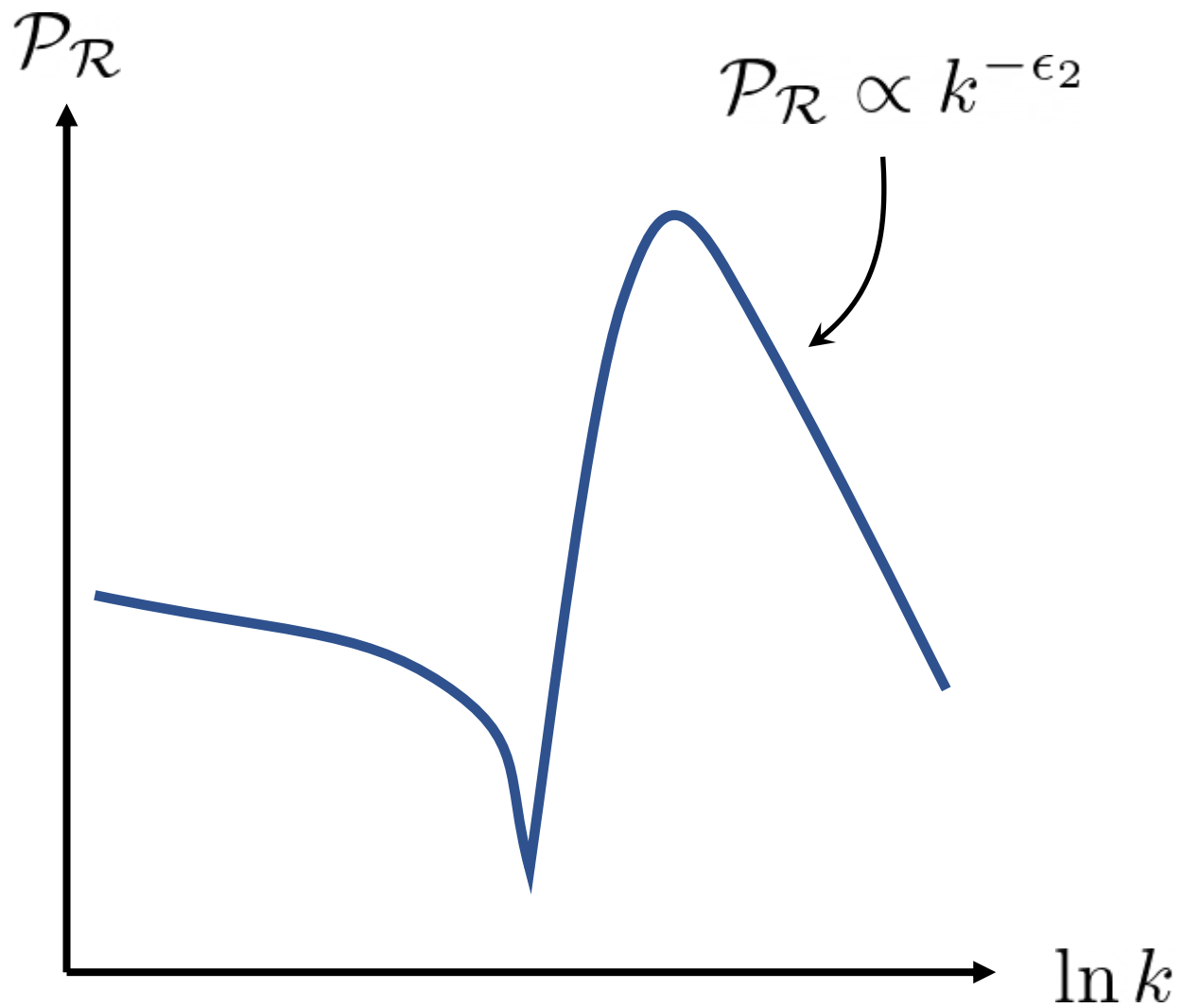
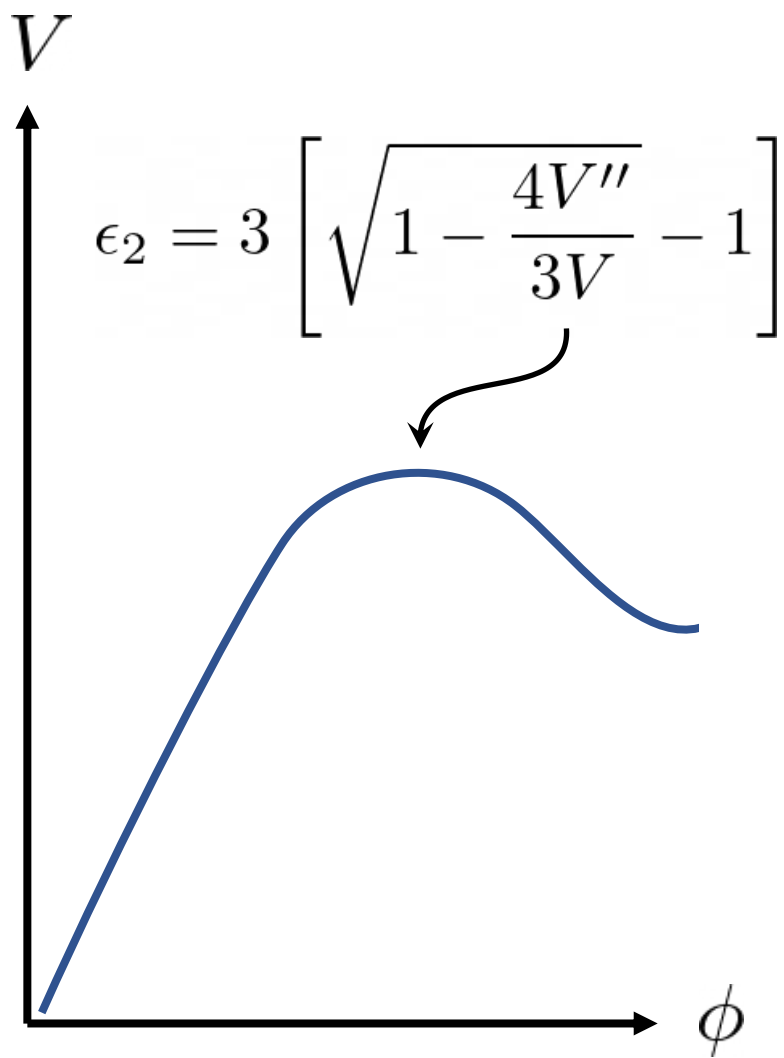
- “Outermost peak” gives final collapse?
- Overlapping peaks?

Conclusions

Stochastic inflation introduces non-Gaussian corrections to PBH statistics

Compaction function formalism needed for accurate results

Spiked radial profiles: what to do?



[2205.13540]

