Large $|\eta|$ approach to single-field inflation

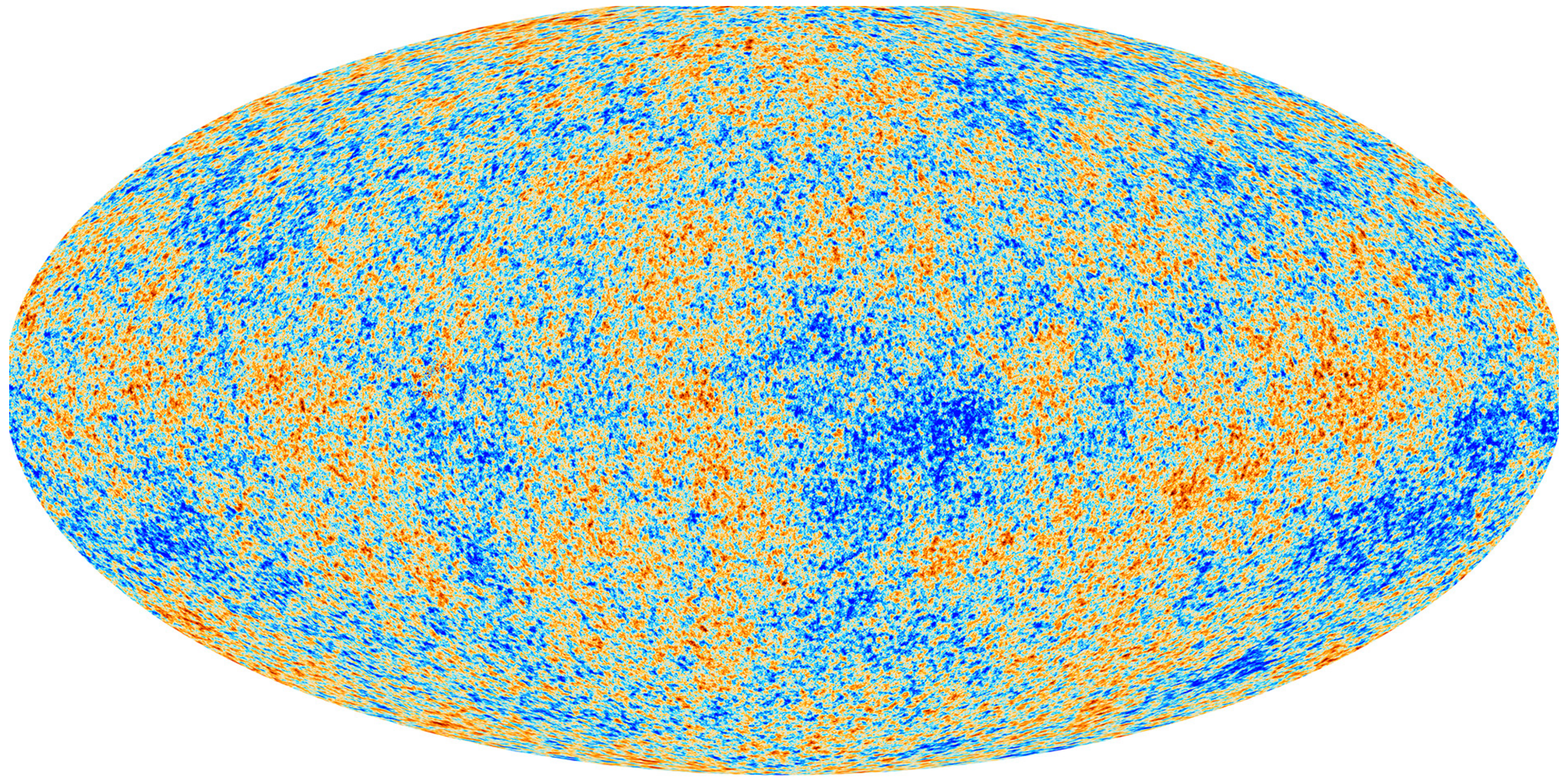
Gianmassimo Tasinato

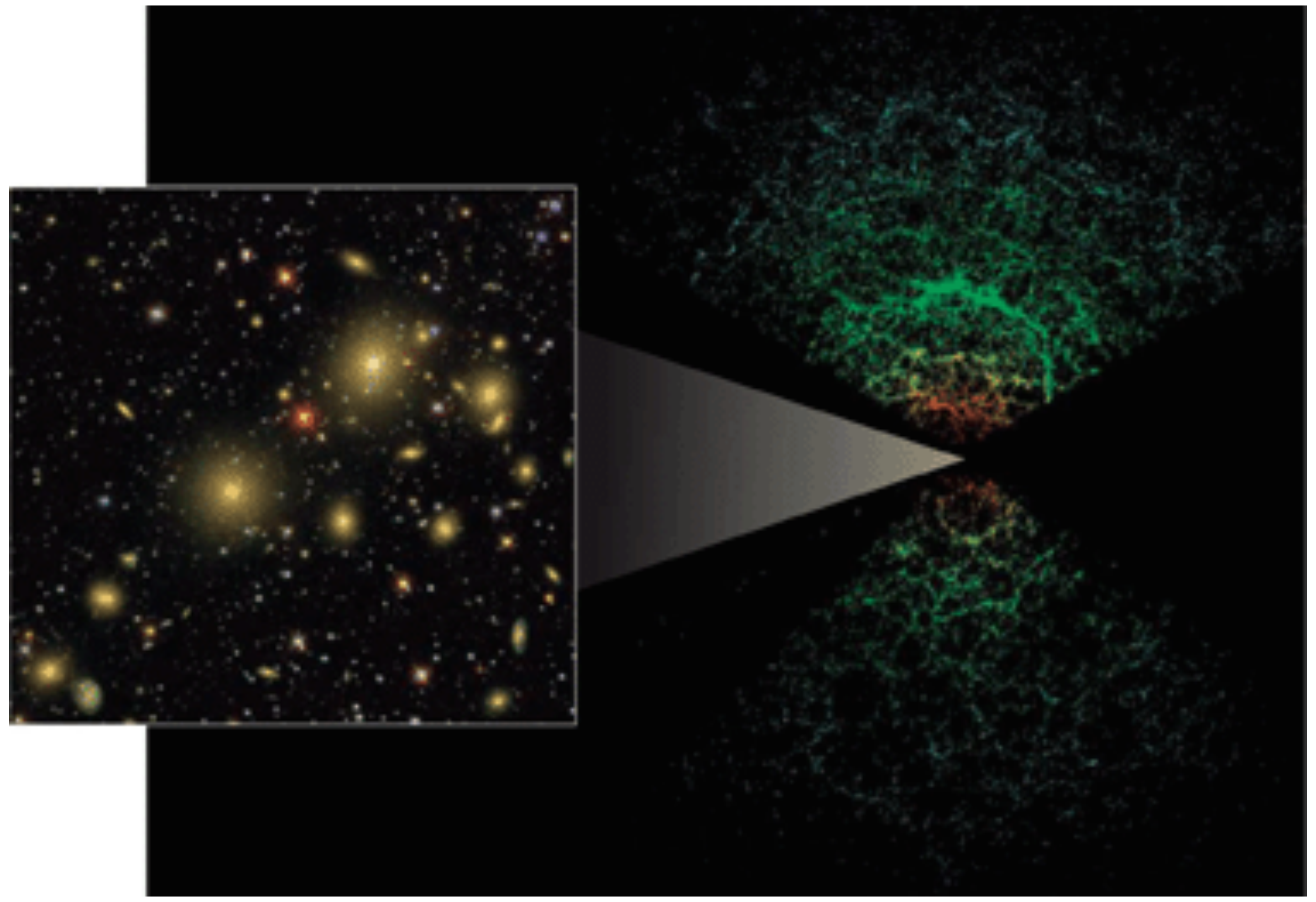
Swansea University and University of Bologna

Based on 2305.11568

Introduction

- **Inflation** is a short period of **superluminal**, accelerated **expansion**, occurred within the first second of our universe life.
- It solves problems of big bang cosmology: `horizon, flatness, entropy problems`
- Moreover, inflation provides an **elegant mechanism** for generating the **primordial seeds** for the CMB and the LSS





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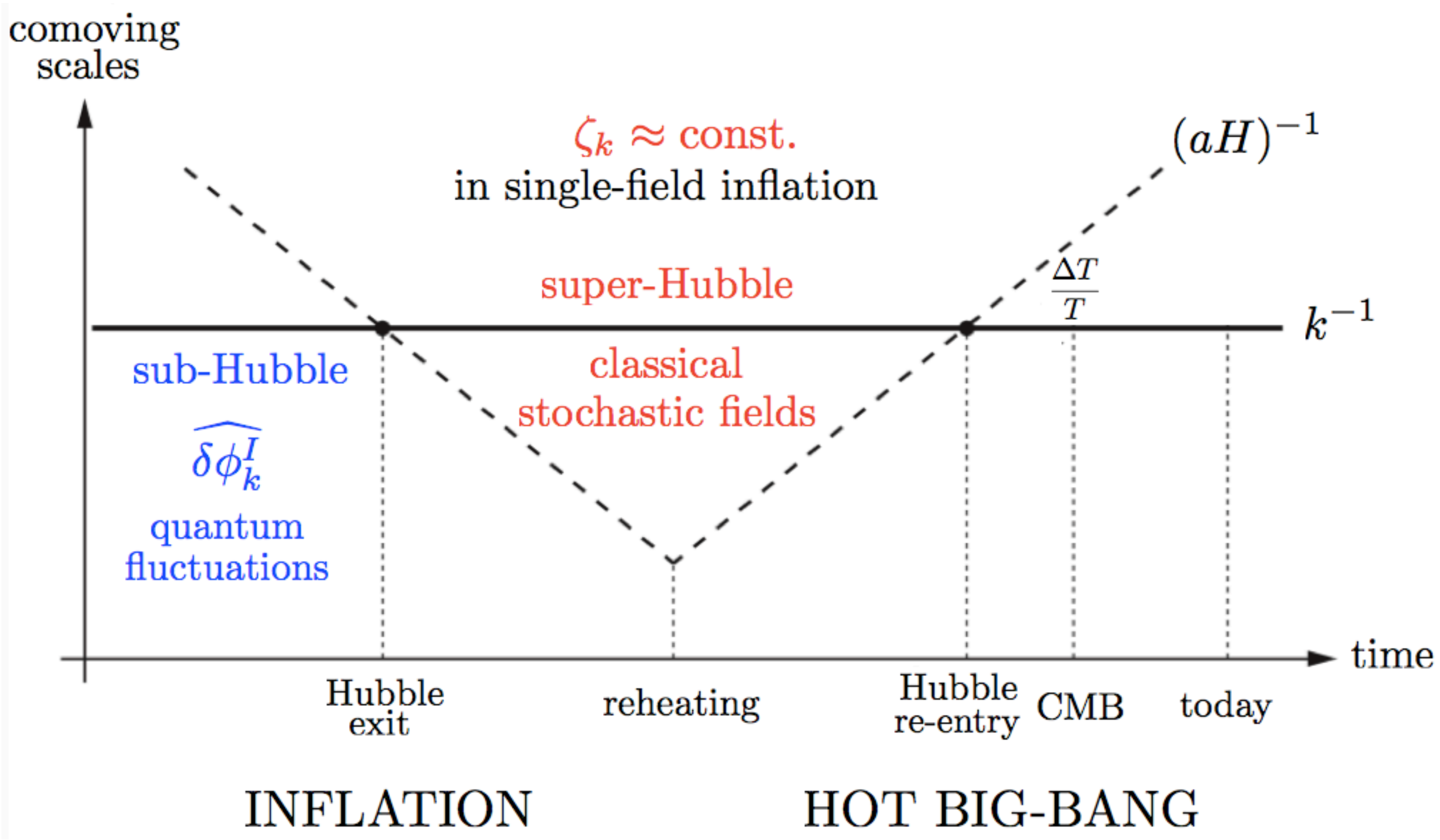


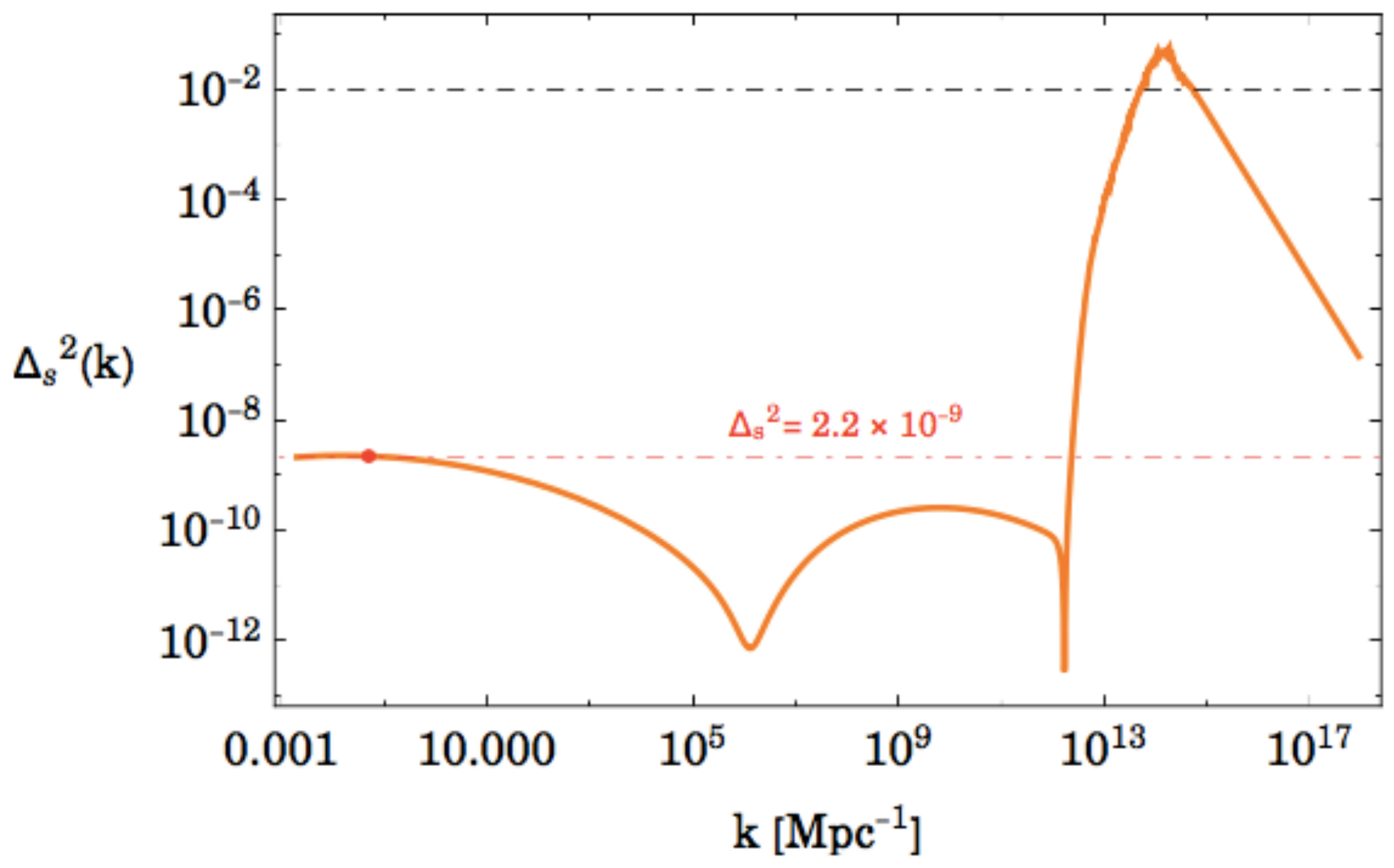
- Cosmological fluctuations are produced by quantum effects at short distances,
- Their wavelength stretched beyond the horizon by the superluminal expansion.
- Then re-enter the horizon after inflation ends

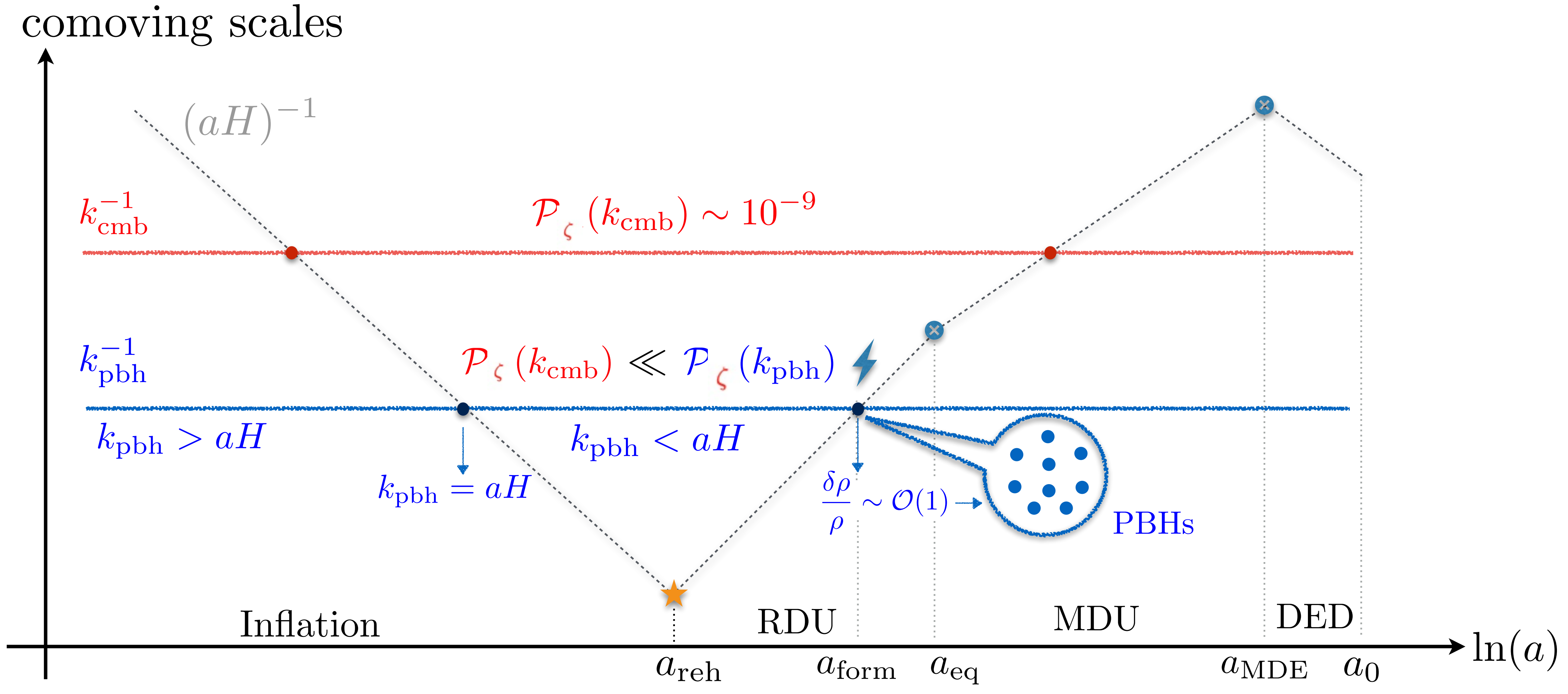
Dark matter and inflation

What about dark matter? Can inflationary fluctuations source it?

Yes if they increase in size at small scales

▷ Primordial black holes





The spectrum of curvature fluctuation ζ increases towards small scales thanks to non-standard inflationary dynamics. When re-entering the horizon during RD, curvature fluctuations source overdensities producing PBH

Slow-roll inflation

The predictions of single-field inflation are very successful at CMB scales:

Fluctuations of ϕ and metric $\Rightarrow$ Curvature perturbation ζ

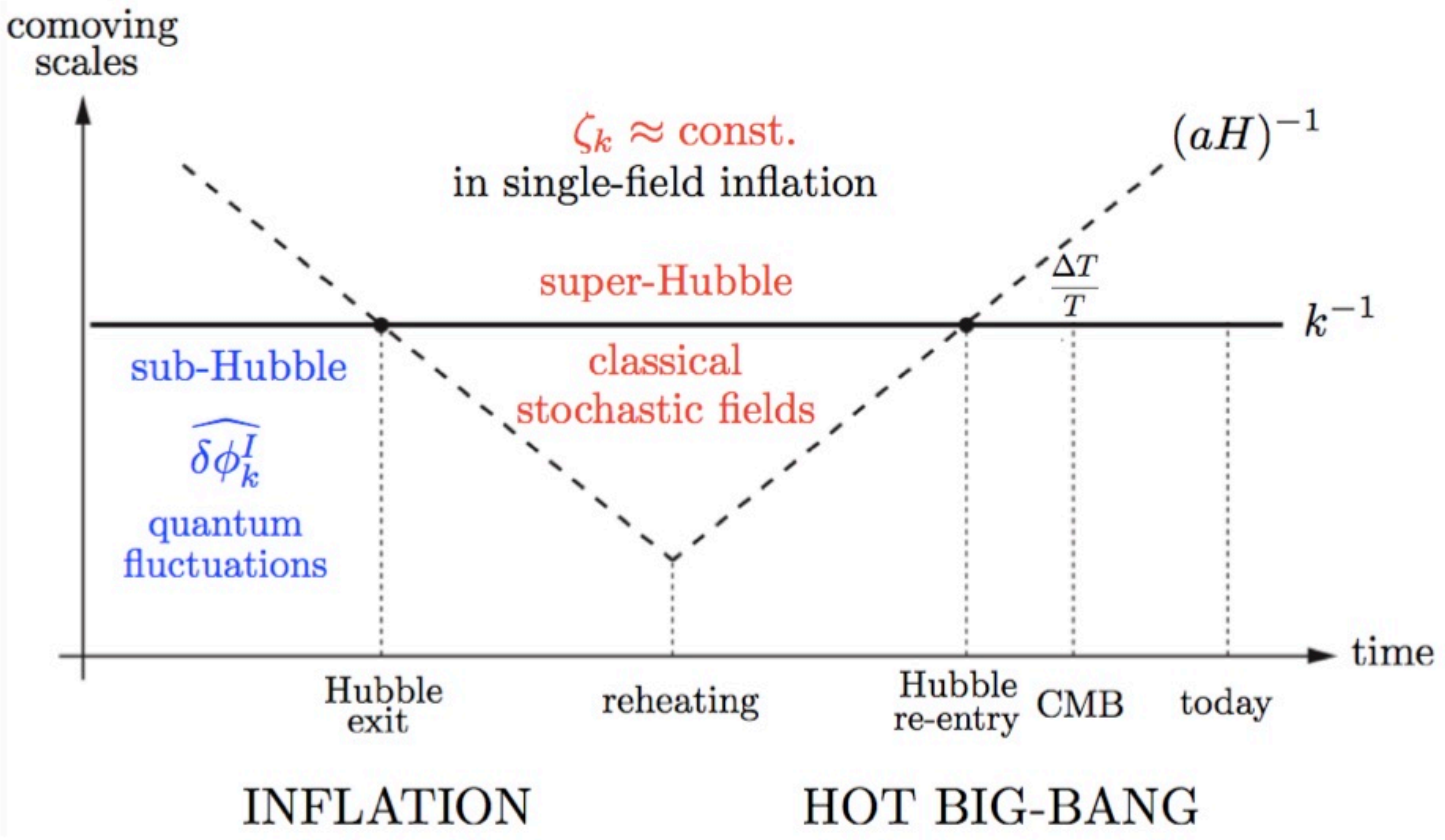
$$\Delta_\zeta = \frac{H^2}{8\pi^2\epsilon}$$

$$n_\zeta - 1 = -2\epsilon - \eta$$

Slow-roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2} \ll 1$$

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} = 2\epsilon + \frac{2\ddot{\phi}}{\dot{\phi} H} \ll 1$$



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- ▷ This is **bad** because the many existing models are degenerate.
- ▷ This is **very good** being a manifestation of EFT of inflation: the slow-roll parameters control the spontaneous breaking of time-reparametrization invariance.

$$t \mapsto t - \pi(\boldsymbol{x}, t)$$

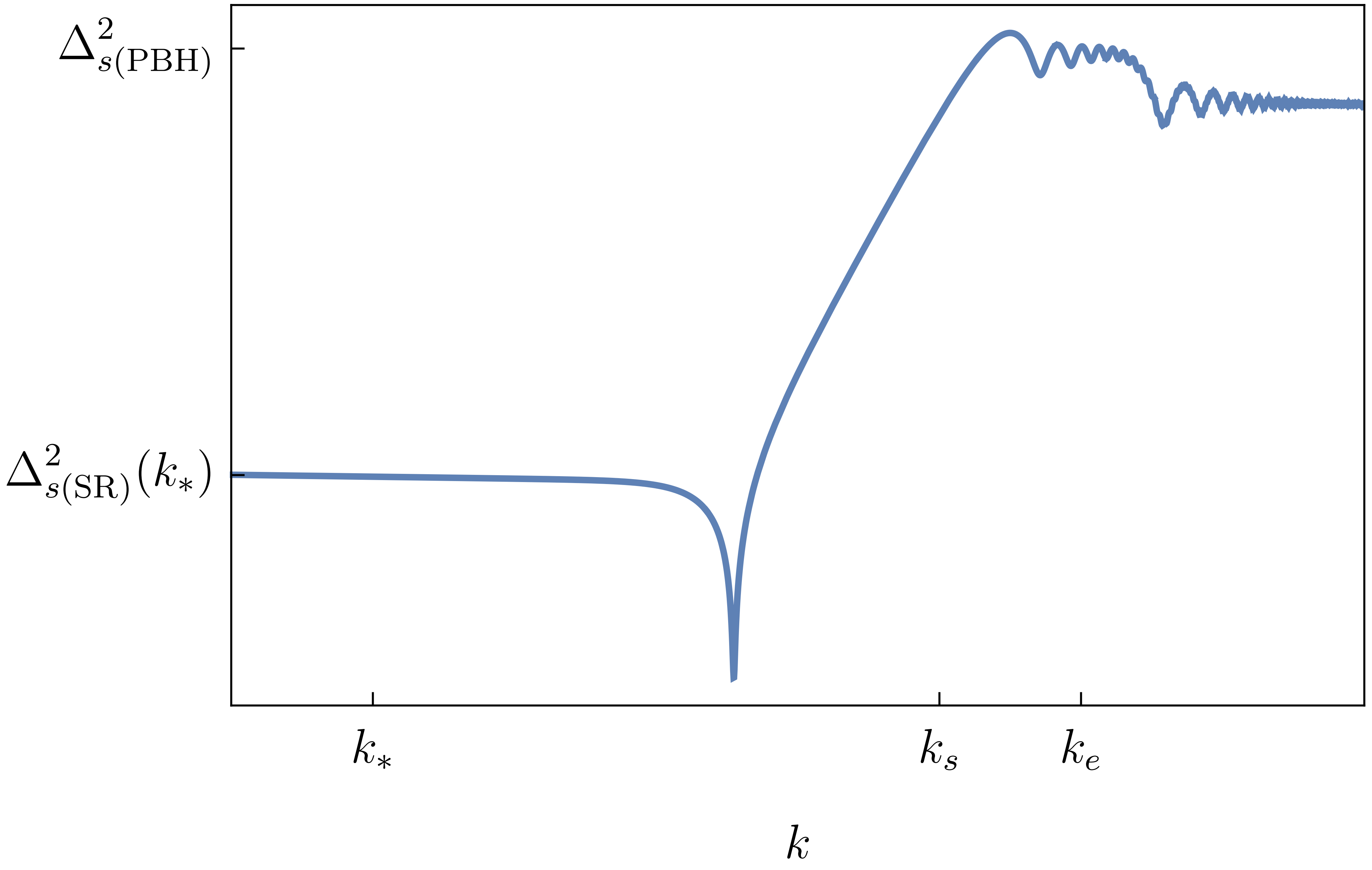
Observables depend on few parameters, controlling pattern of symmetry breaking

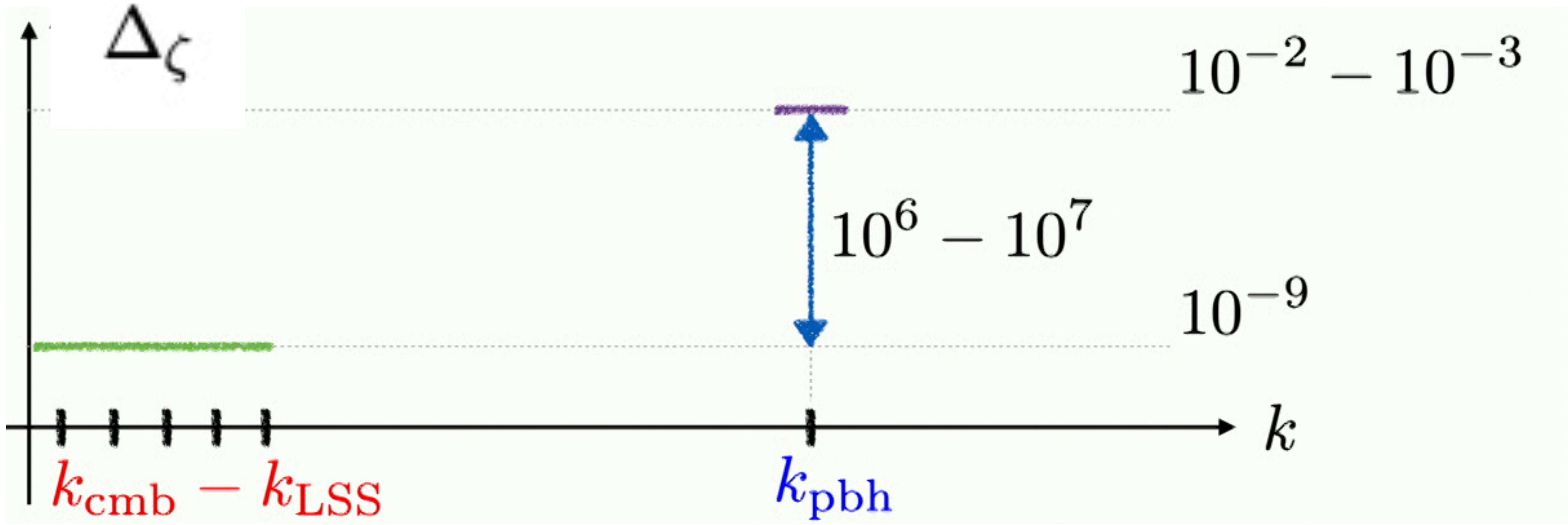
Inflation and PBH

We need to abandon slow-roll regime

The parameter ϵ changes
by several orders of magnitude in few e-folds

$$\Delta_\zeta = \frac{H^2}{8\pi^2\epsilon}$$





Inflation and PBH

We **need to abandon** slow-roll regime

The parameter ϵ changes
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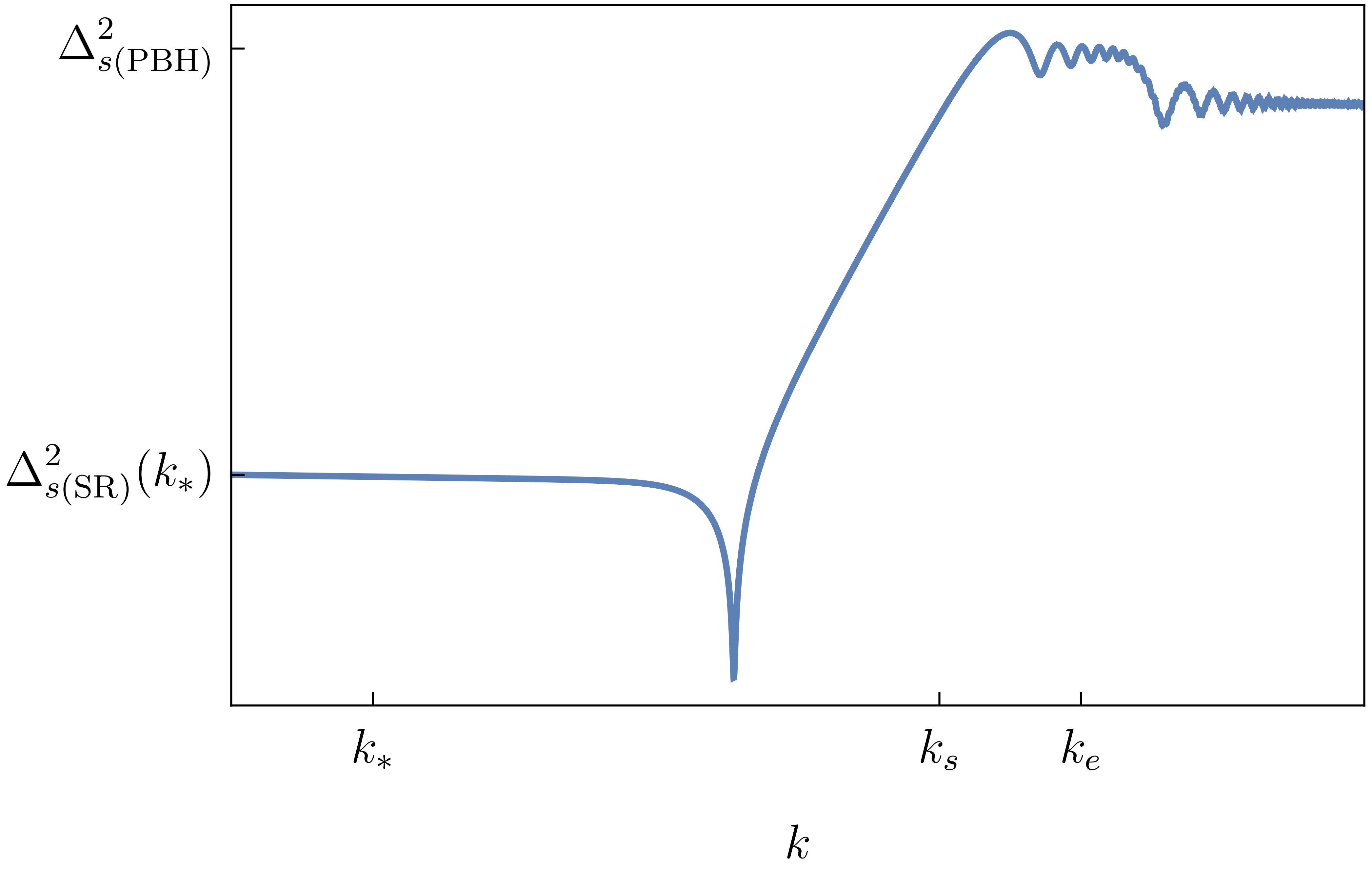
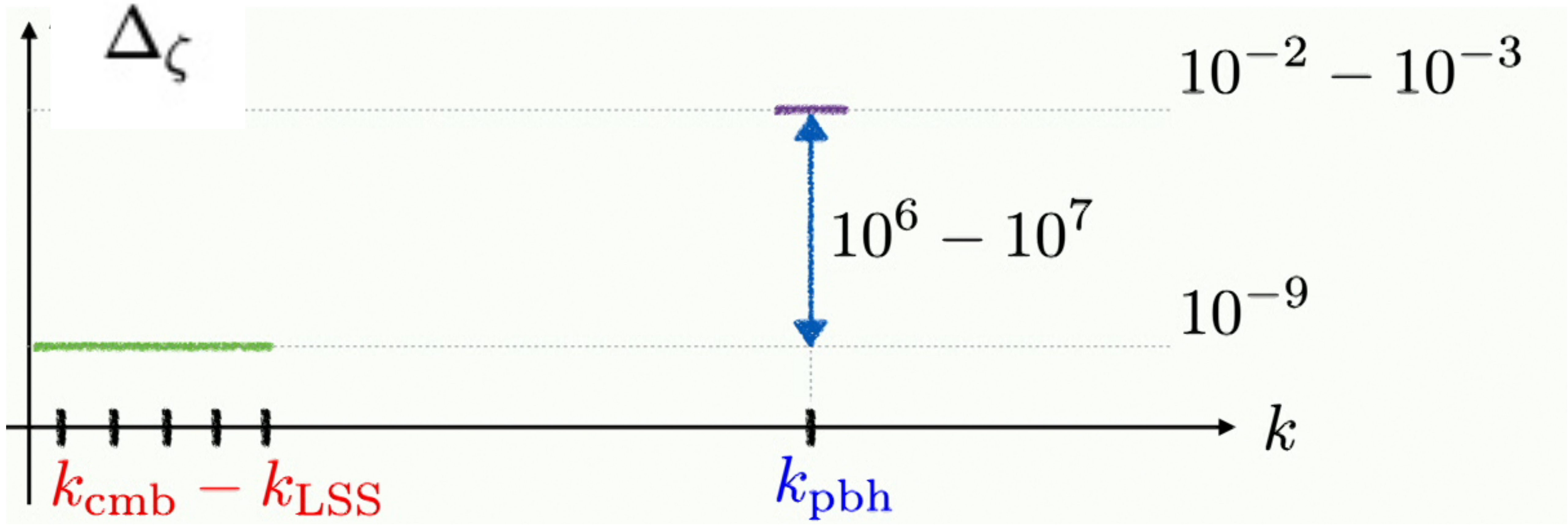
$$\Delta_\zeta = \frac{H^2}{8\pi^2\epsilon}$$

η must become large and negative

$$\eta = \frac{\dot\epsilon}{\epsilon H} = 2\epsilon + \frac{2\ddot\phi}{\dot\phi H}$$





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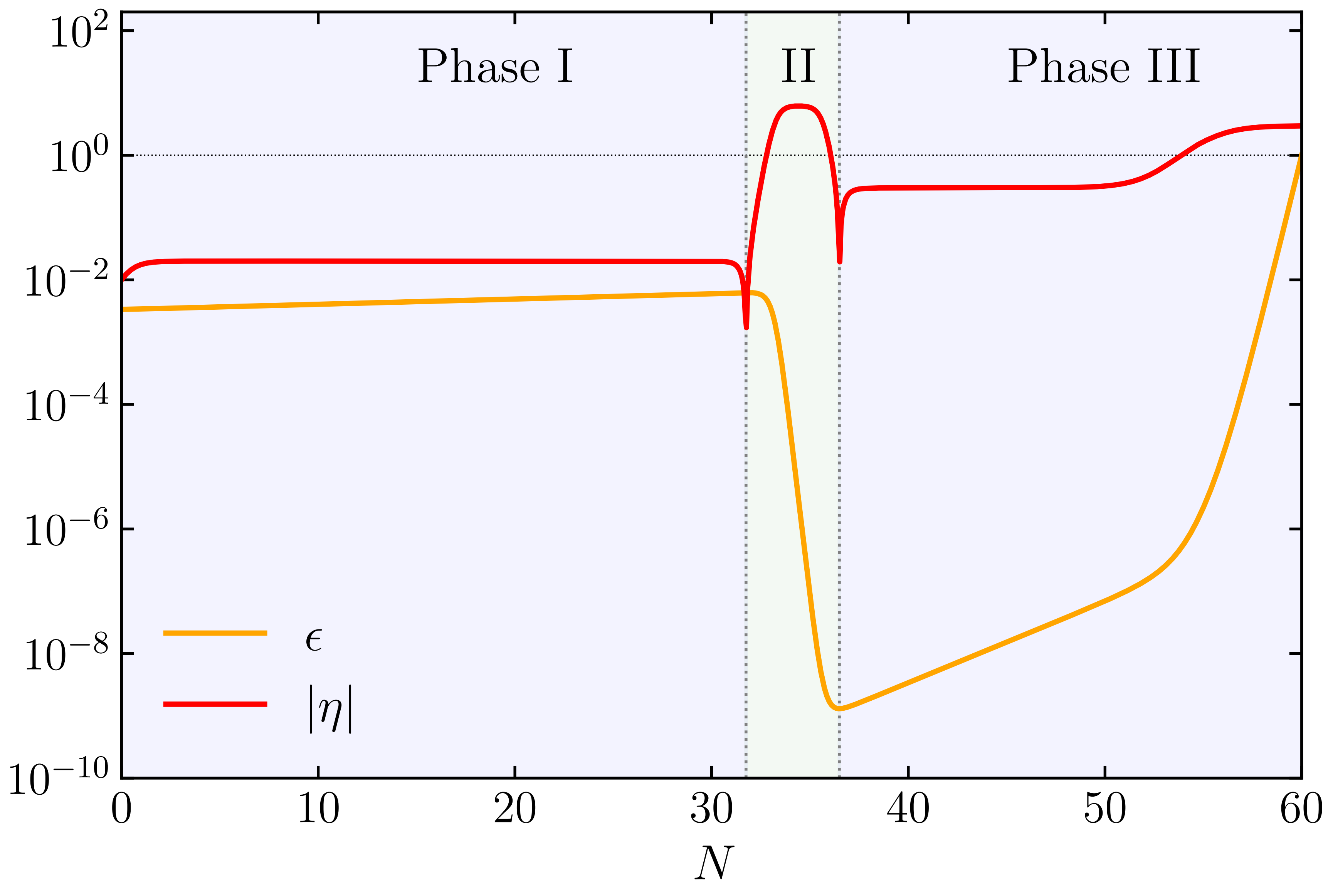

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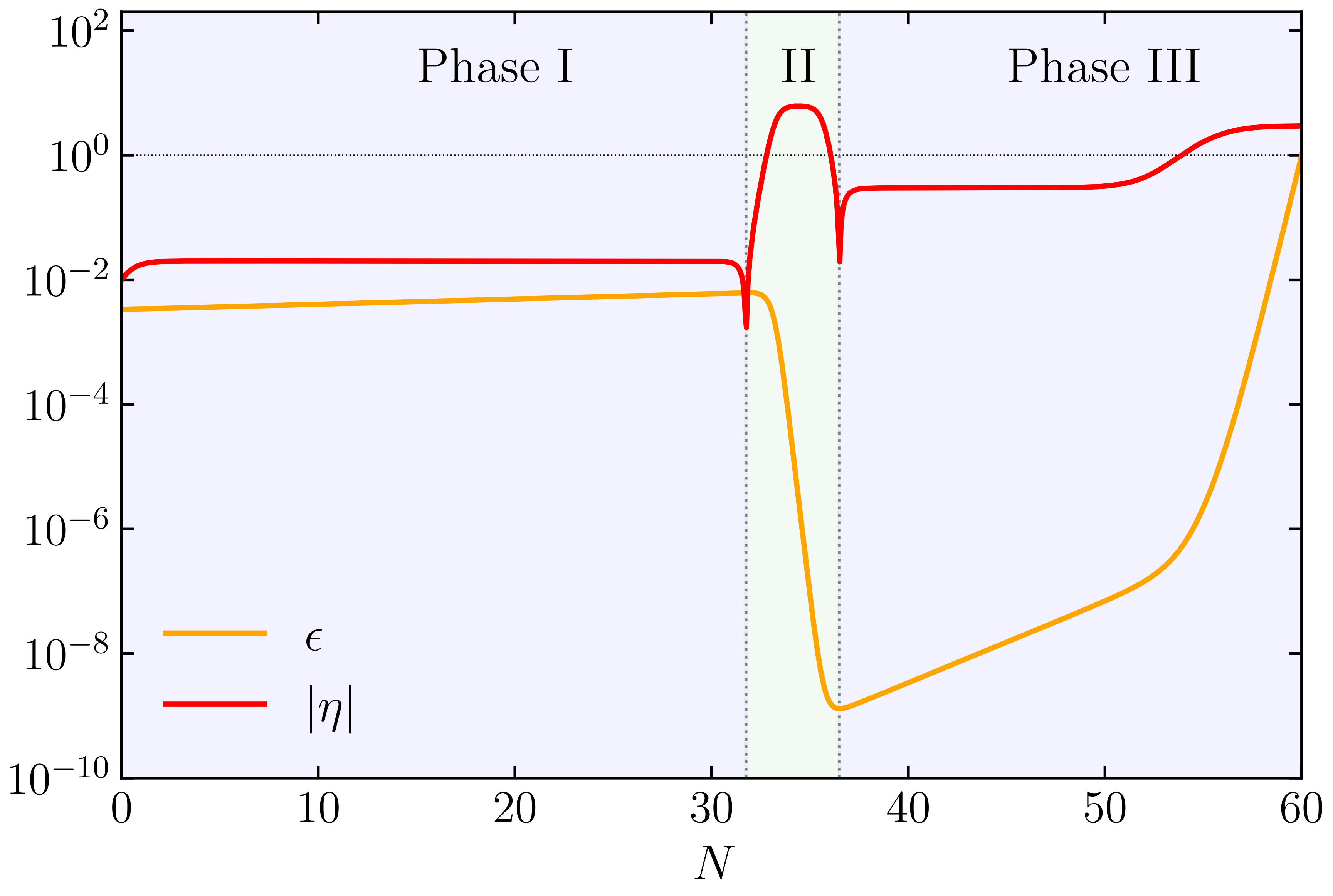
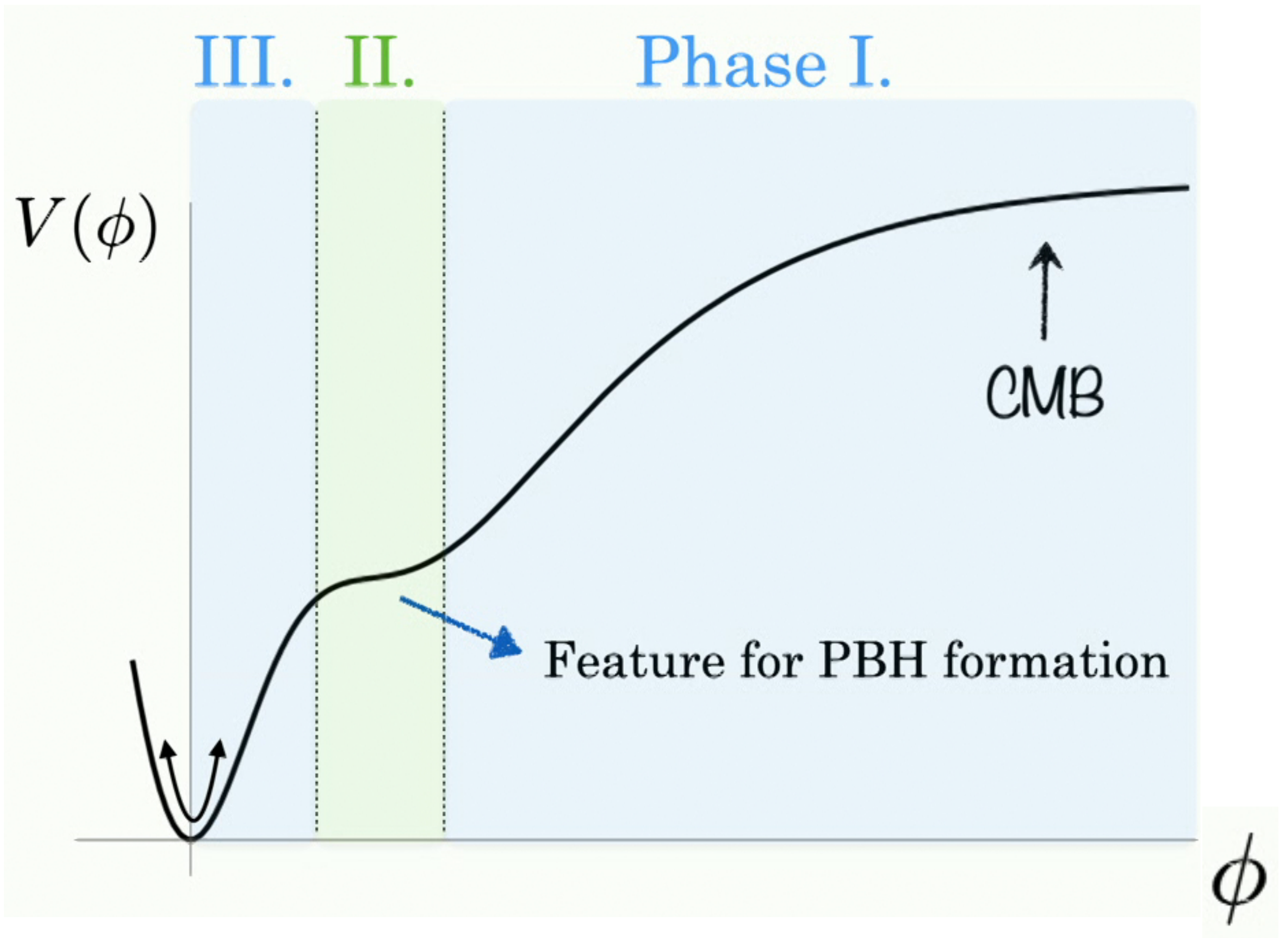


▷ **Ultra slow-roll inflation**: $V' = 0$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \;\Rightarrow\; \ddot{\phi} = -3H\dot{\phi} \;\Rightarrow\; \eta \simeq -6$$

(this implies $\phi \sim a^{-3} \Rightarrow$ decaying mode controls the dynamics)

[Kinney,...,Germani-Prokopec, Dimopoulos, ...]



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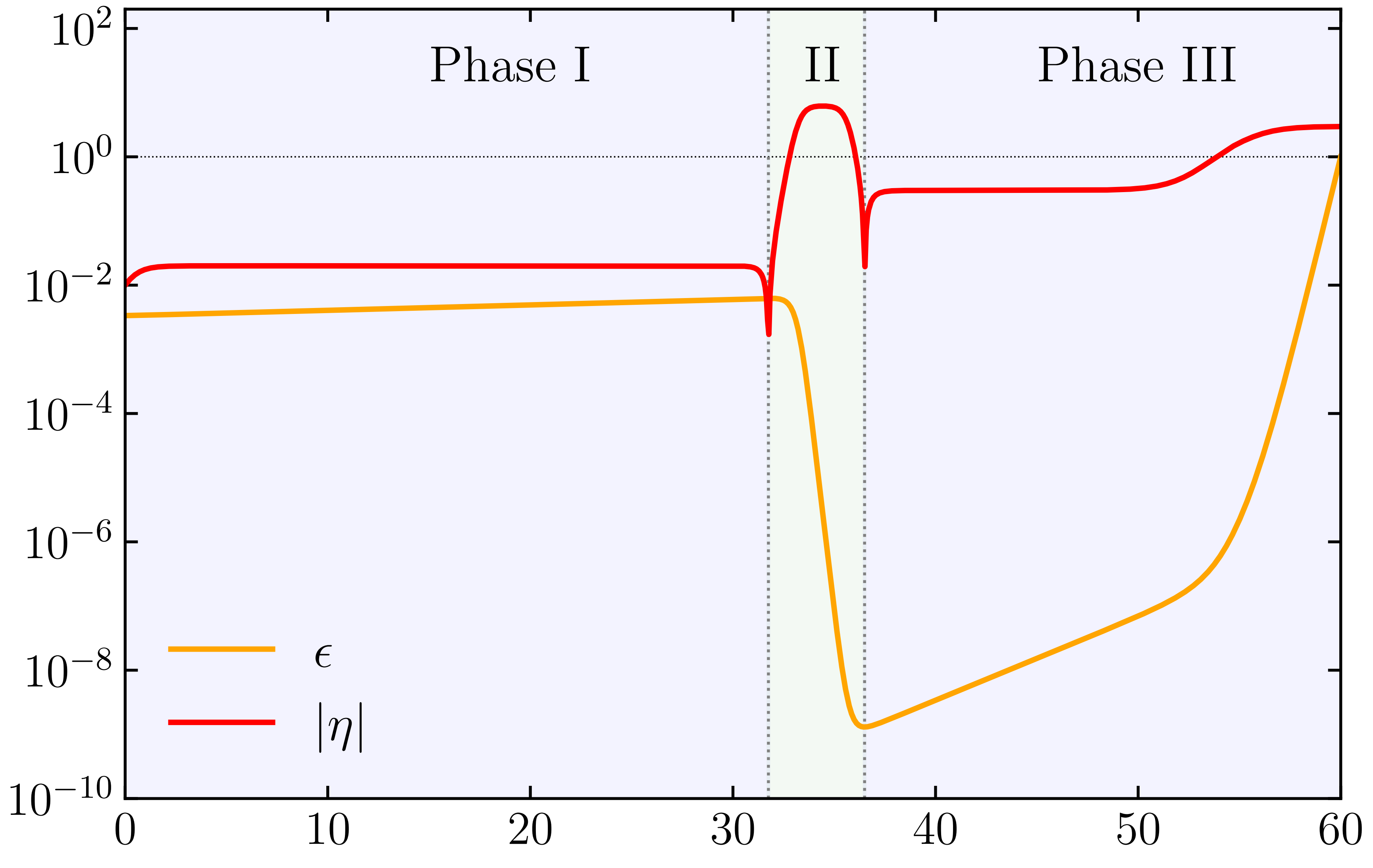
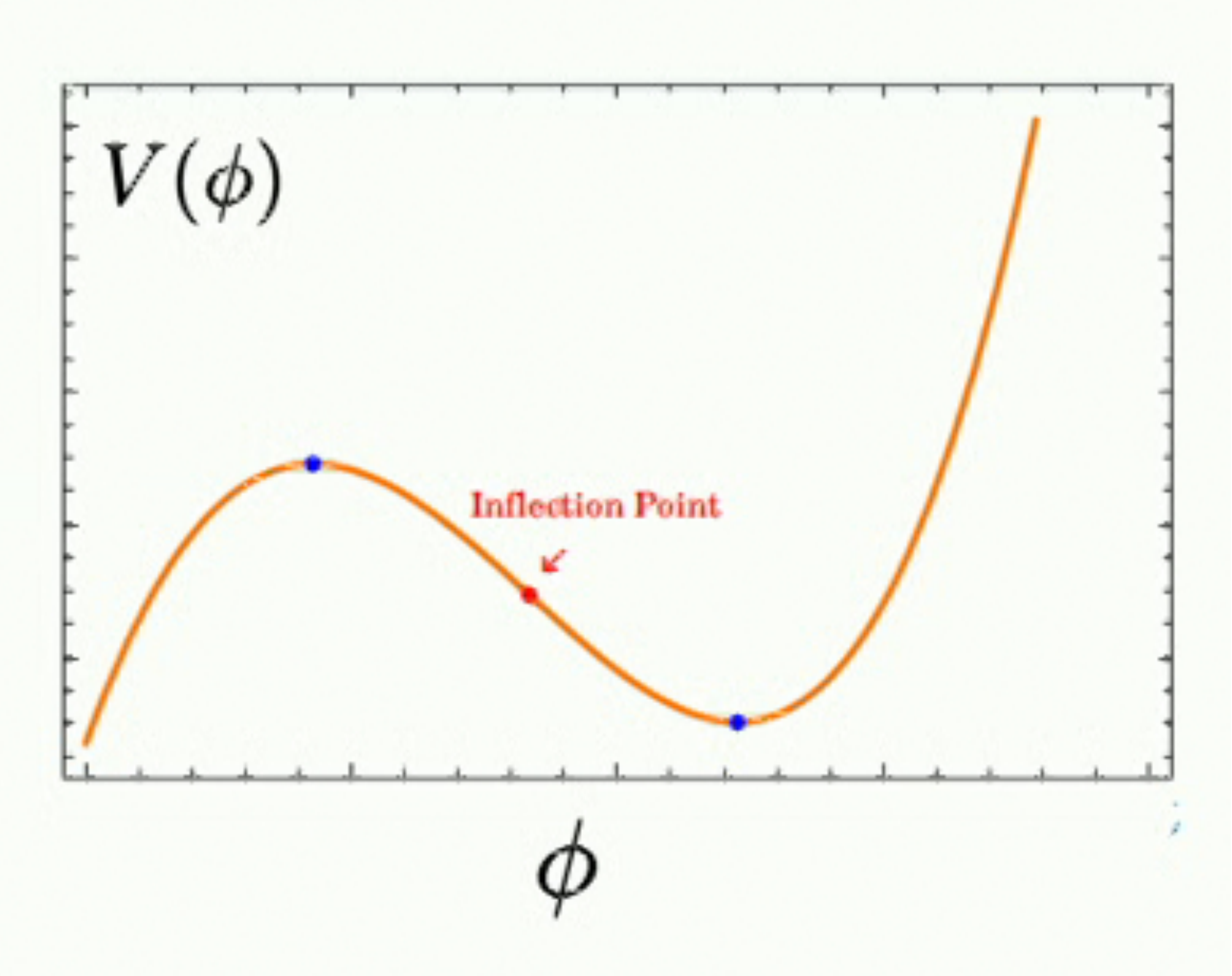


- **Constant roll inflation**: $V' < 0$

  Scalar climbs a hill overshooting local minimum

$$\eta = 2\epsilon - 6 + \frac{2V'}{|\dot{\phi}| H} < -6$$



Inflation and PBH

η must become large and negative

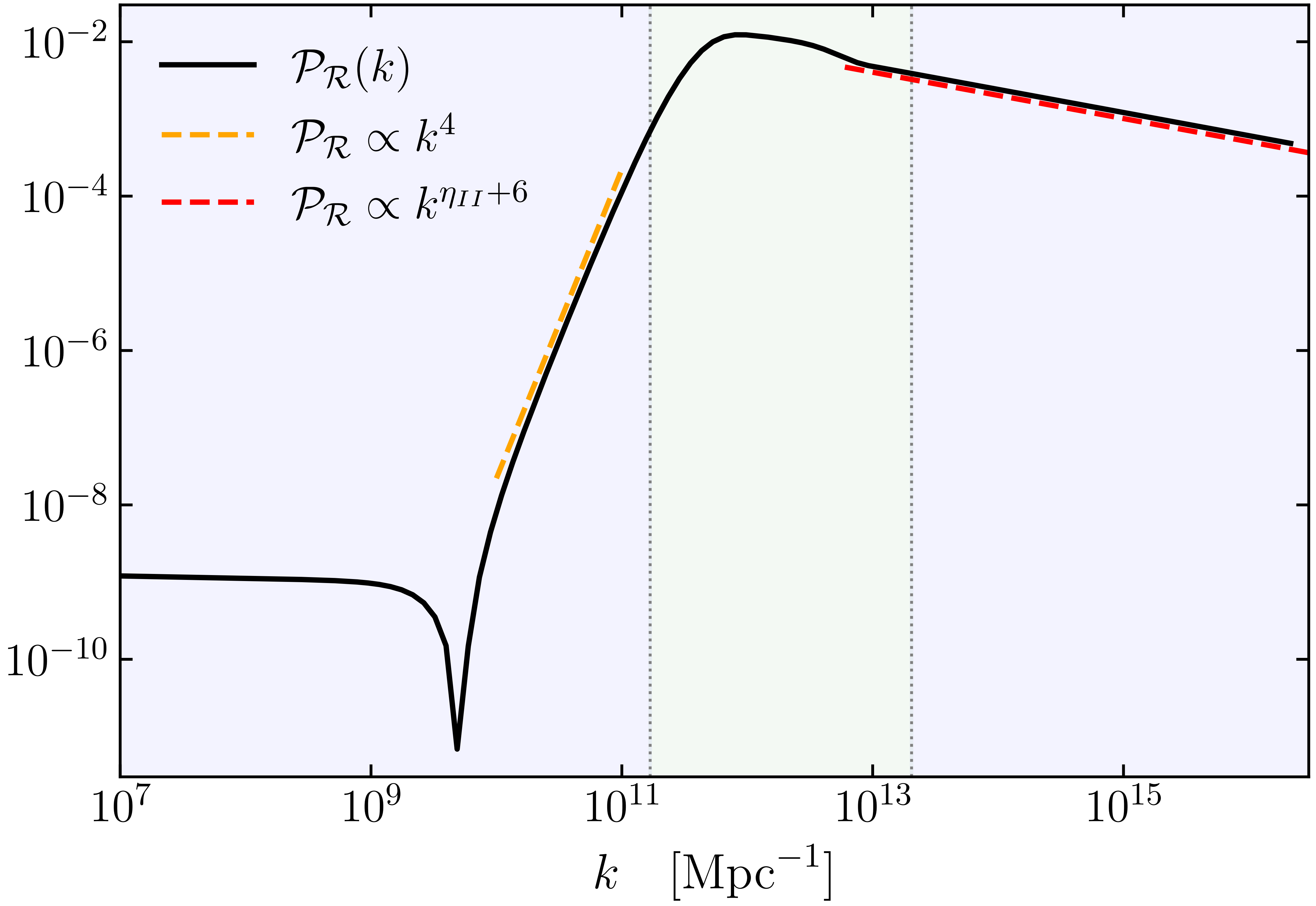
$\Rightarrow$ We get a **rapid enhancement of the spectrum**

▷ We wake up the decaying mode which participates to the dynamics

▷ Interesting phenomena:

- Dip in the spectrum, due to distruptive interference growing/decaying modes

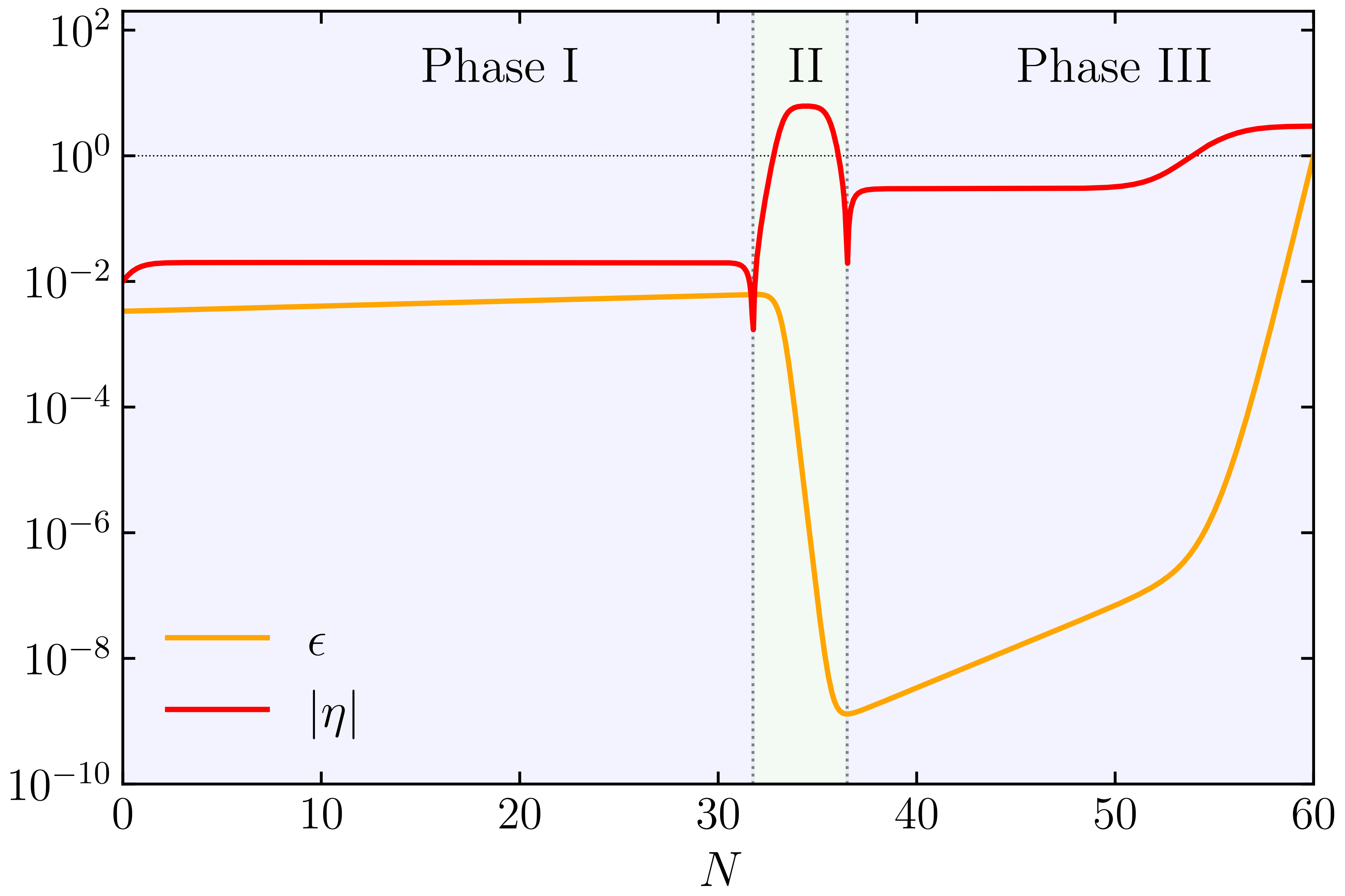
- Limit k^4 in the slope of the growing spectrum



[Byrnes et al]

(for recent review see e.g. [Özsoy, GT])

Inflation and PBH

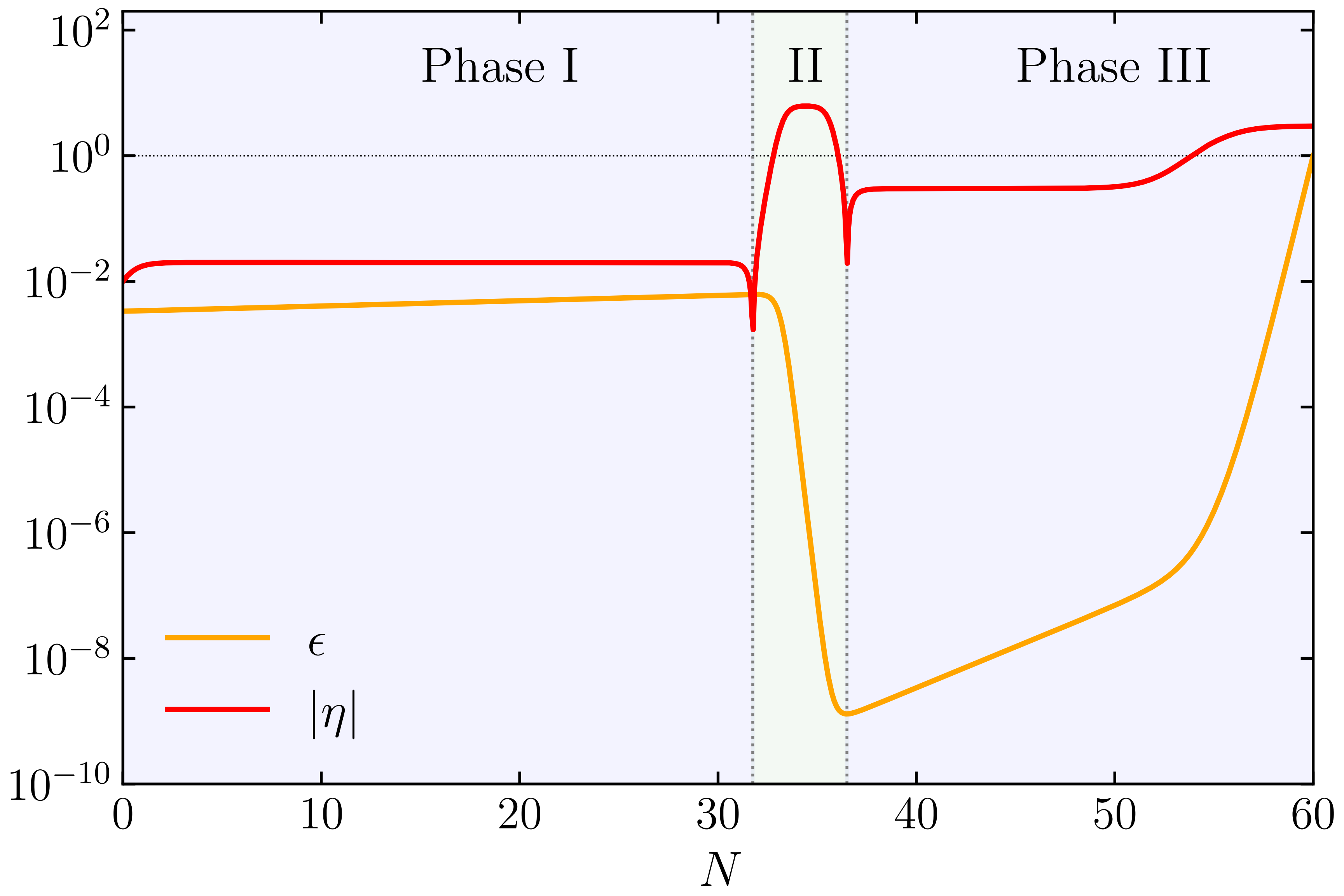


The non-slow-roll phase should be brief
to avoid excessive effects of quantum diffusion

$$\frac{d\phi}{dN} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\,\xi(N)$$

[Vennin et al]

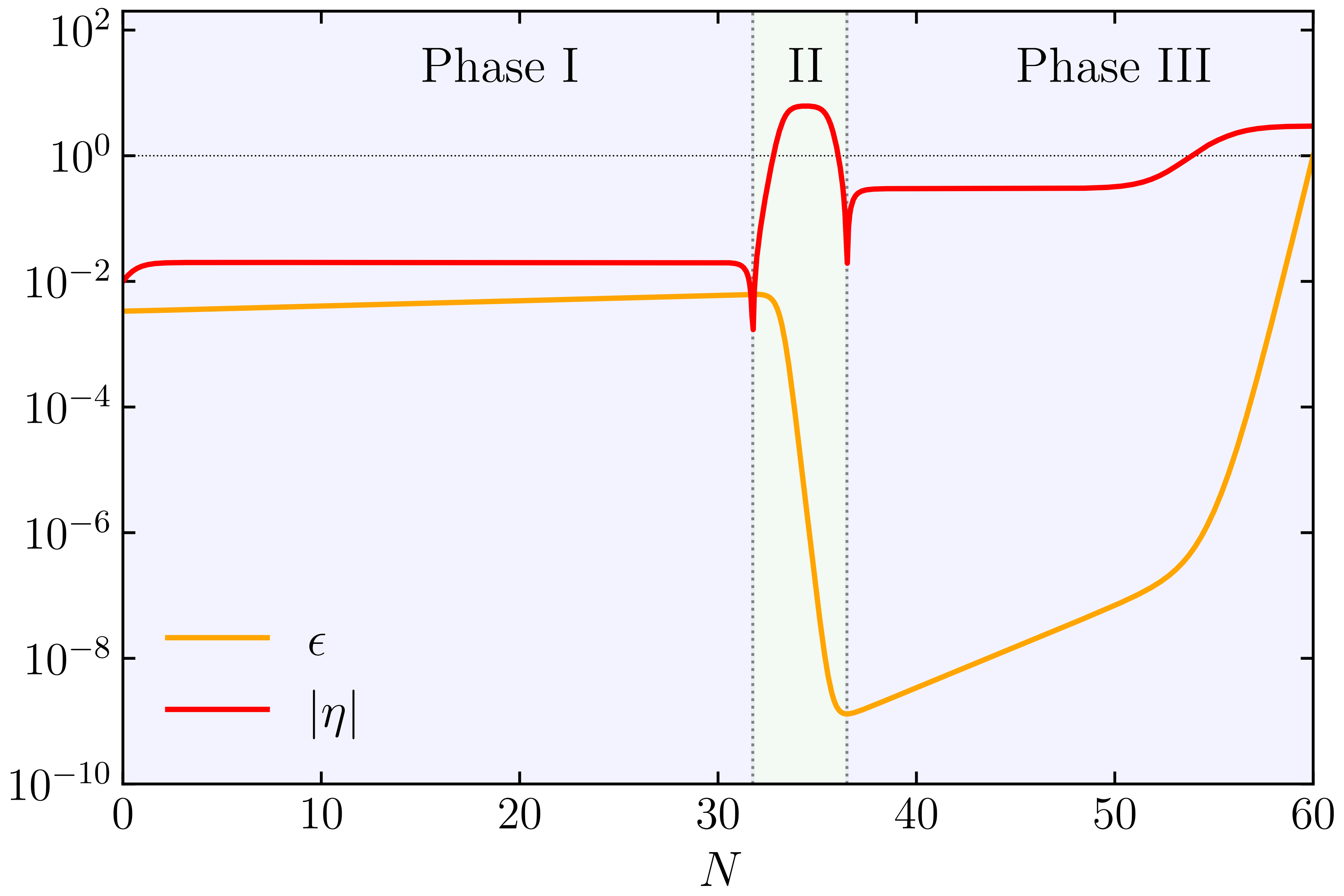
Inflation and PBH



Calculations can be carried on with the help of numerics

- **Analytic control is possible**
  for $\eta = -6$ and for a model of Starobinsky
- Or by designing piecewise models with constant slopes for ϵ and η
  [Karam et al, Franciolini et al, Domenech et al, ...]

Inflation and PBH

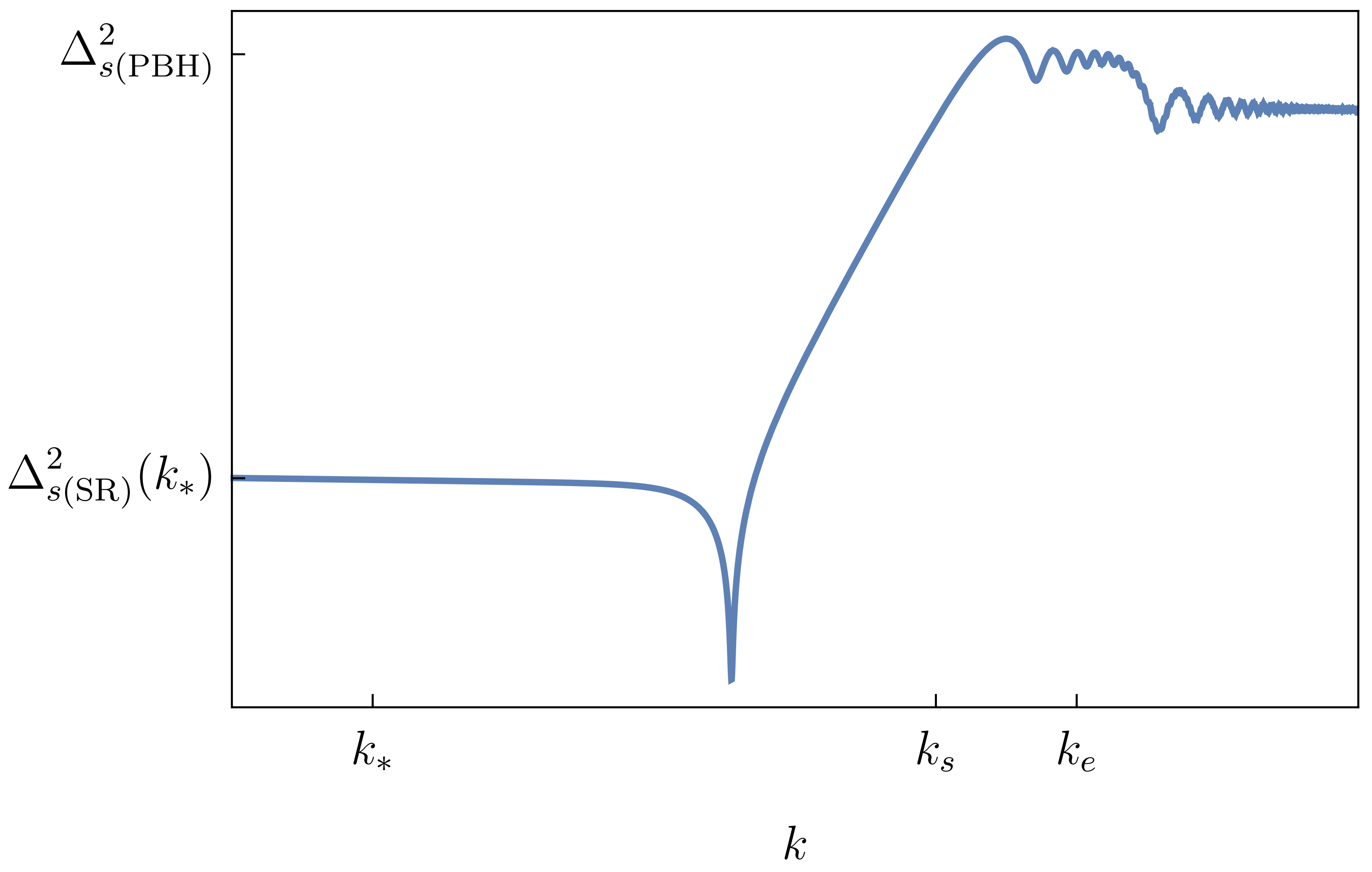


Calculations can be carried on with the help of numerics

- **Analytic control is possible** for $\eta = -6$ and for a model of Starobinsky
- Subtleties associated with decaying mode, and connections between slow-roll and non-slow-roll phases.

▷ **Good thing** Observables sensitive on details of the model.

▷ **Bad things** Degeneracies likely to occurr, and we lack an analytical understanding of what is going on



Idea: take $|\eta|$ large, and use $1/|\eta|$ as expansion parameter

This might lead to a **reliable** analytical framework!

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▷ At the same time, take $\Delta N_{\rm nsr} \ll 1$, and the product $|\eta|\,\Delta N_{\rm nsr} = \text{fixed} \equiv 2\,\Pi_0$

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▷ At the same time, take $\Delta N_{\rm nsr} \ll 1$, and the product $|\eta|\,\Delta N_{\rm nsr} = \text{fixed} \equiv 2\,\Pi_0$

▷ Straightforward to solve for mode functions, and compute correlators in an expansion in $1/|\eta|$ and ϵ. E.g. for the power spectrum (take $\epsilon \ll 1$):

$$\frac{\Delta_\zeta(\kappa)}{\Delta_\zeta(0)} = 1 - 4\kappa\,\Pi_0\,\cos\kappa\,j_1(\kappa) + 4\kappa^2\,\Pi_0^2\,j_1^2(\kappa) \qquad +\mathcal{O}(1/|\eta|)$$

with $\kappa = k/k_\star$ and $j_1(\kappa) = \frac{\sin\kappa}{\kappa^2} - \frac{\cos\kappa}{\kappa}$

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▷ At the same time, take $\Delta N_{\rm nsr} \ll 1$, and the product $|\eta|\,\Delta N_{\rm nsr} \, = \text{fixed} \equiv 2\,\Pi_0$

▷ Practically, what do we do? Whenever meeting $\Delta N_{\rm nsr}$, substitute with $2\Pi_0/|\eta|$. At the end, take limit $|\eta| \to \infty$

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Depends on one parameter only!

▷ $$\lim_{\kappa\to\infty}\frac{\Delta_\zeta(\kappa)}{\Delta_\zeta(0)} = (1+\Pi_0)^2$$

Idea: take $|\eta|$ large, and use $1/|\eta|$ as expansion parameter

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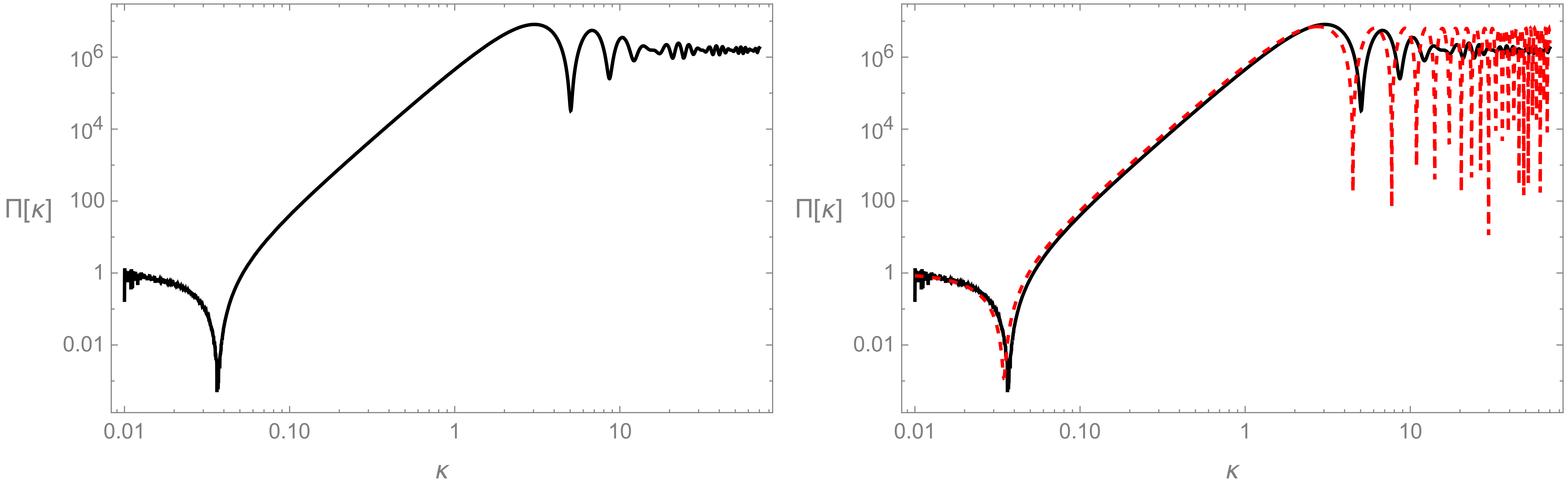
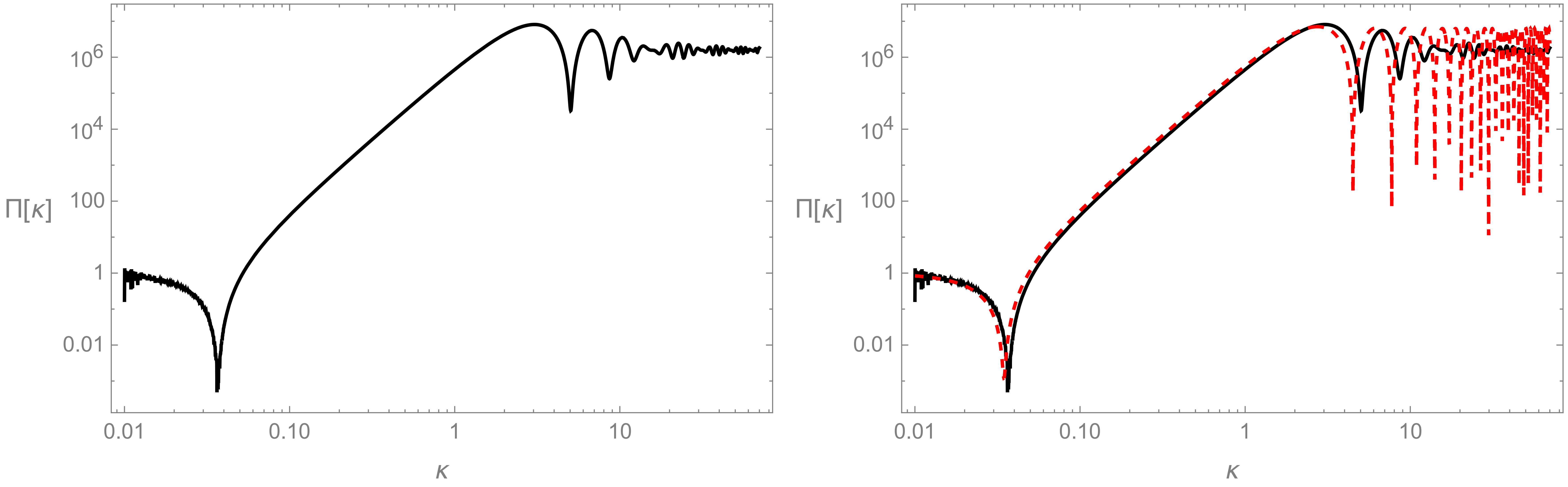


it catches pretty well the large-scale behaviour, up to the peak

($\mathcal{O}(1/|\eta|)$ corrections can be included, and improve the small-scale behaviour)

Idea: take $|\eta|$ large, and use $1/|\eta|$ as expansion parameter



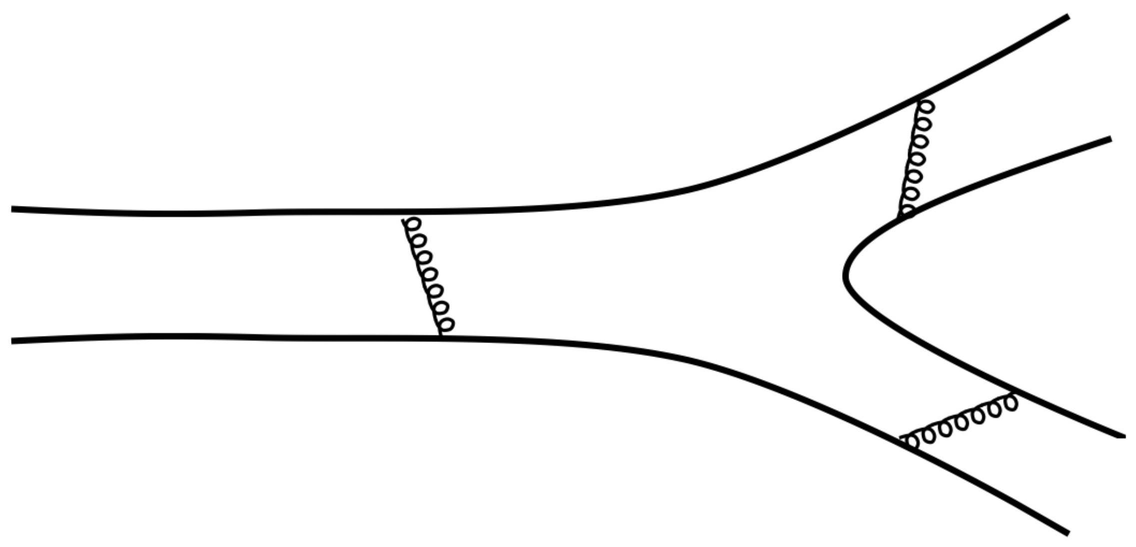
Also spectral index can be computed analytically, at leading order in $1/|\eta|$:

$$n_\zeta - 1 = \frac{2\,\kappa\,\Pi_0\left[(1-2\kappa^2)\sin(2\kappa) - 2\kappa\cos(2\kappa)\right]}{\kappa^2 + 4\kappa\Pi_0\cos\kappa\,(\kappa\cos\kappa - \sin\kappa) + 4\Pi_0^2\,(\kappa\cos\kappa - \sin\kappa)^2} - \frac{\Pi_0^2\left[4 - (4-8\kappa^2)\cos(2\kappa) + 4\kappa(\kappa^2-2)\sin(2\kappa)\right]}{\kappa^2 + 4\kappa\Pi_0\cos\kappa\,(\kappa\cos\kappa - \sin\kappa) + 4\Pi_0^2\,(\kappa\cos\kappa - \sin\kappa)^2}$$

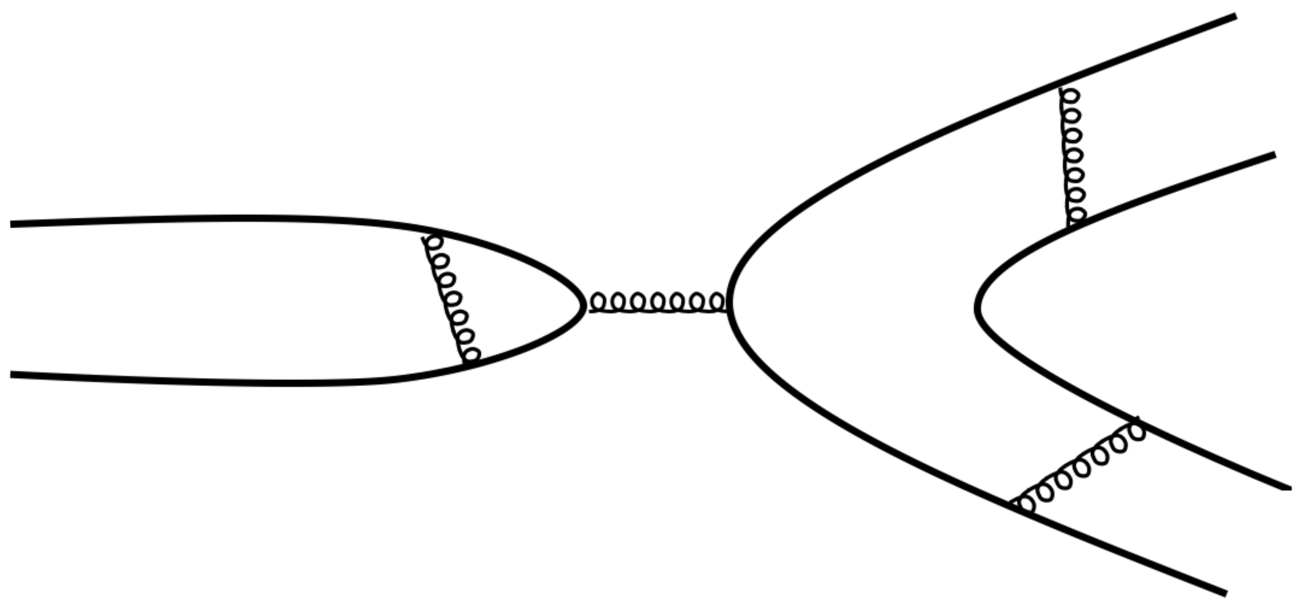
Analogy: Large-N limit of $SU(N)$ QCD

▷ Model studied by 't Hooft: computations simplify taking number N of colors large, and expand in $1/N$. Call g the QCD coupling constant, consider limits

$$g \to 0 \quad , \quad N \to \infty \quad , \quad g^2\, N \equiv g_0^2 = \text{fixed}$$



$$\sim \frac{1}{\sqrt{N}}$$



$$\sim \frac{1}{N^{3/2}}$$

▷ Analogy with PBH inflationary models

$$\Delta N_{\rm nsr} \to 0 \quad , \quad |\eta| \to \infty \quad , \quad |\eta|\, \Delta N_{\rm nsr} = \text{fixed}$$

Large $|\eta|$ limit of inflation

- New framework based on a novel consistent perturbative expansion. Hopefully useful for carrying on computations in the context of PBH physics

- Works in a particular limit of parameter space. The hope is to learn something useful for real-world physics. (In analogy with large-N QCD.)

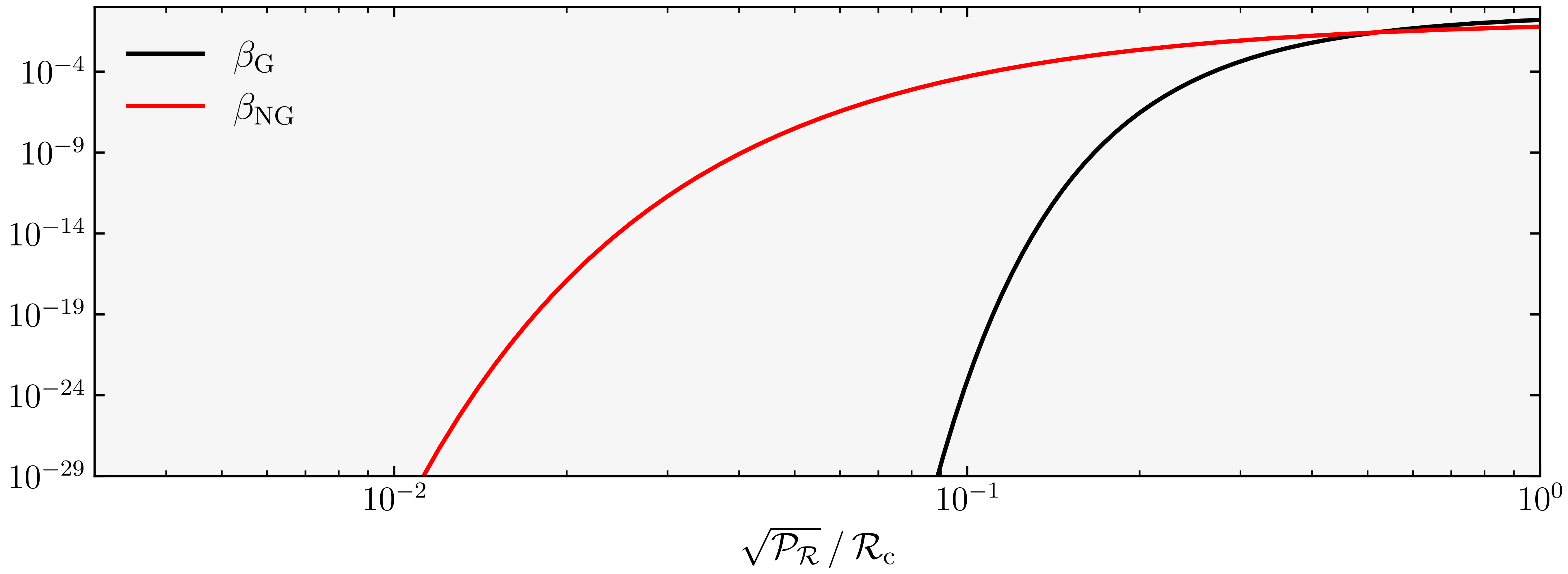
Higher-order correlation functions

▷ Non-Gaussian effects around the peak of the spectrum plays an important role for PBH formation. Analytic control of non-Gaussianity would be welcome!

$$\langle \zeta_{\mathbf{k_1}} \zeta_{\mathbf{k_2}} \zeta_{s\mathbf{k_3}} \rangle \;=\; (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)\, B_\zeta(k_1, k_2, k_3)$$

We can reduce the required amplitude of P_ζ for producing PBH at small scales:

[Byrnes et al, Atal-Germani, Passaglia et al,. . . ,Taoso-Urbano]



Higher-order correlation functions and the large-$|\eta|$ approach

▷ A single dominant term in the third order Hamiltonian of single-field inflation [Maldacena, Kristiano-Yokoyama]

$$\mathcal{H}_{\rm int} \;=\; -\frac{1}{2}\int d^3x\, a^2(\tau)\epsilon(\tau)\,\eta'(\tau)\,\zeta^2(\tau,\vec{x})\,\zeta'(\tau,\vec{x})$$

$$\eta'(\tau) \;=\; \Delta\eta\left[-\delta(\tau-\tau_1)+\delta(\tau-\tau_2)\right]$$

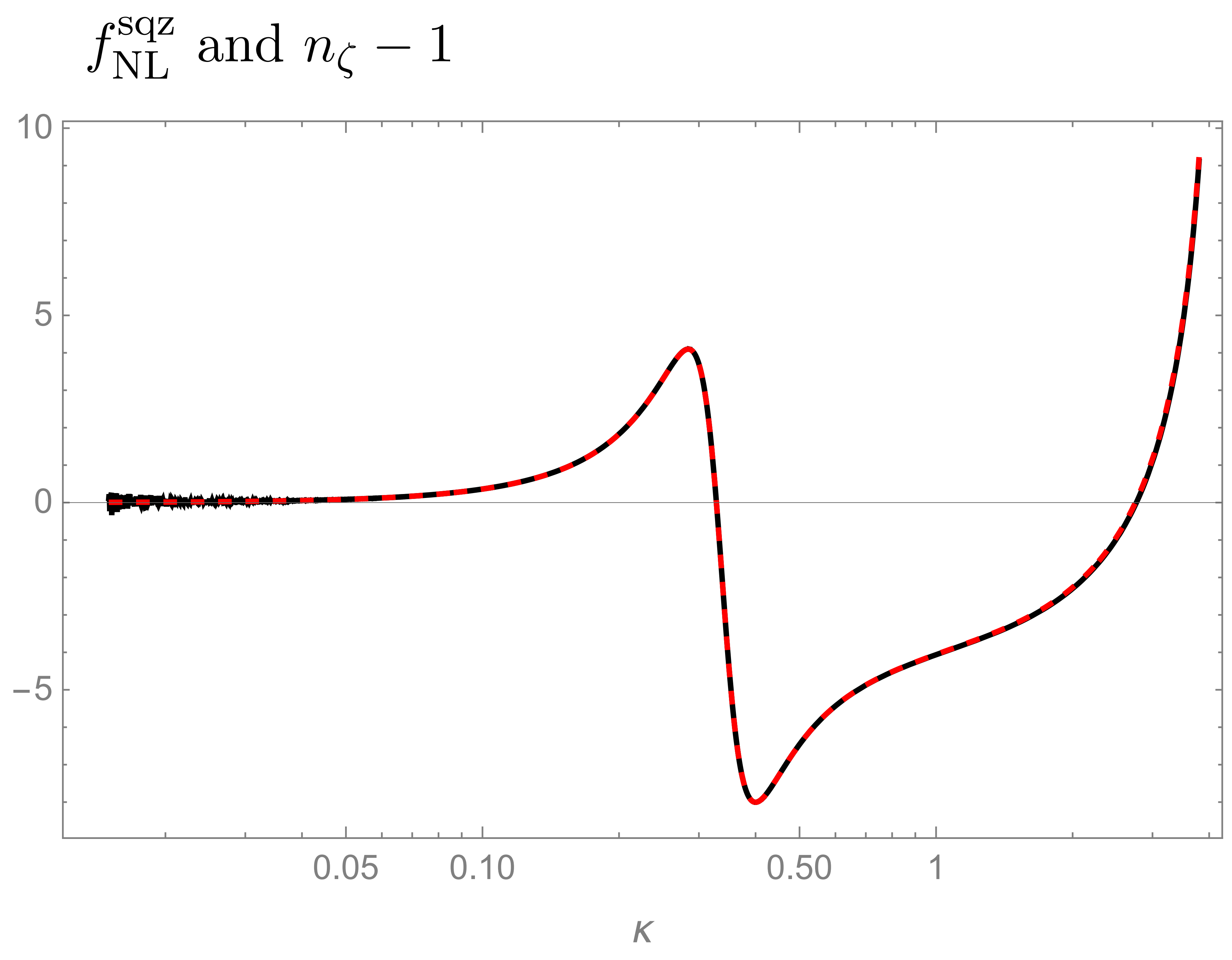
▷ Plug mode functions and compute large-η limit of bispectrum. At leading order in $1/|\eta|$ one gets an analytic expression

$$B_\zeta(k_1,k_2,k_3) \;=\; \text{too long to fit in the slide} \rightarrow$$

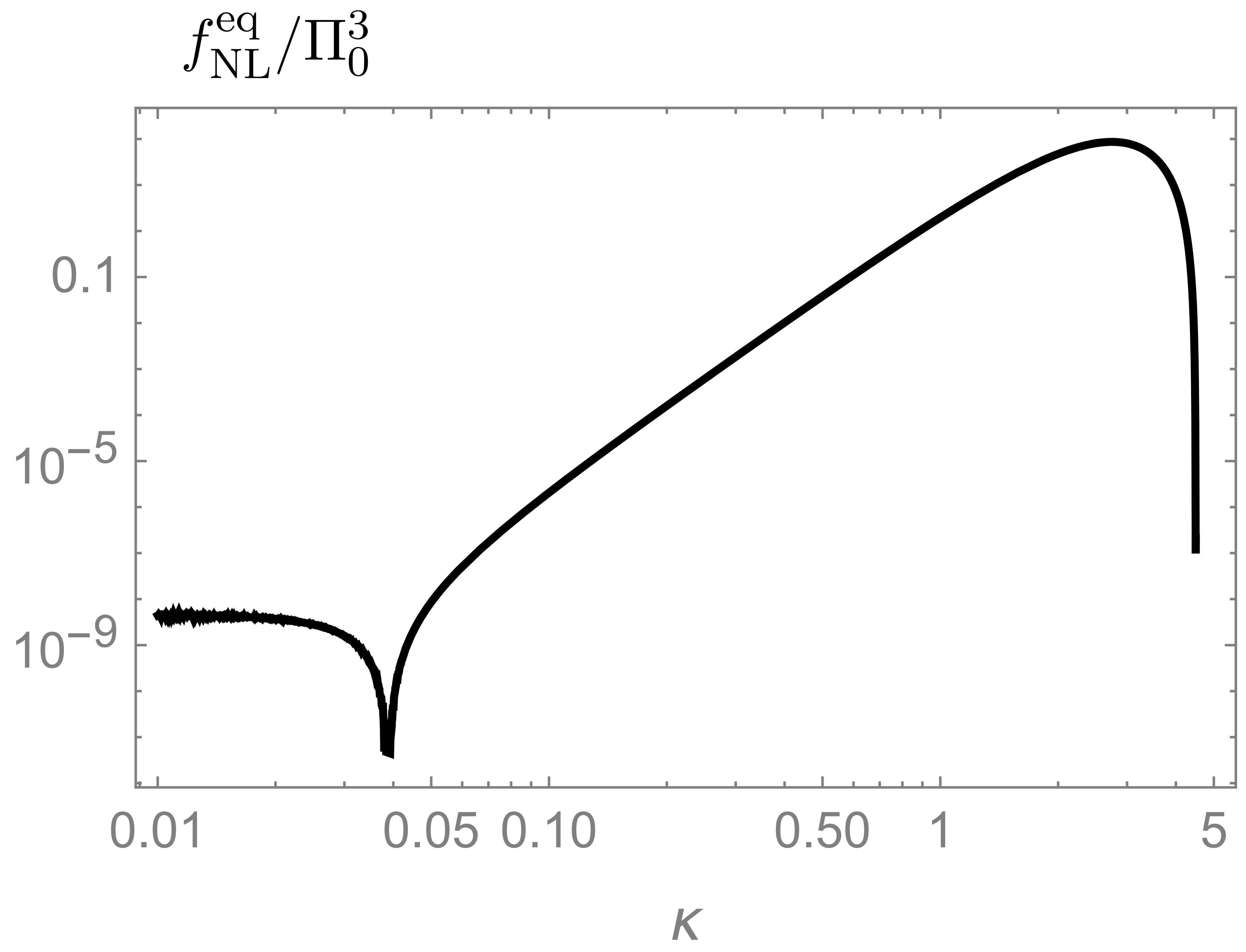
`Depends on one parameter only!`

Higher-order correlation functions and the large-$|\eta|$ approach

Squeezed limit satisfies Maldacena consistency relation

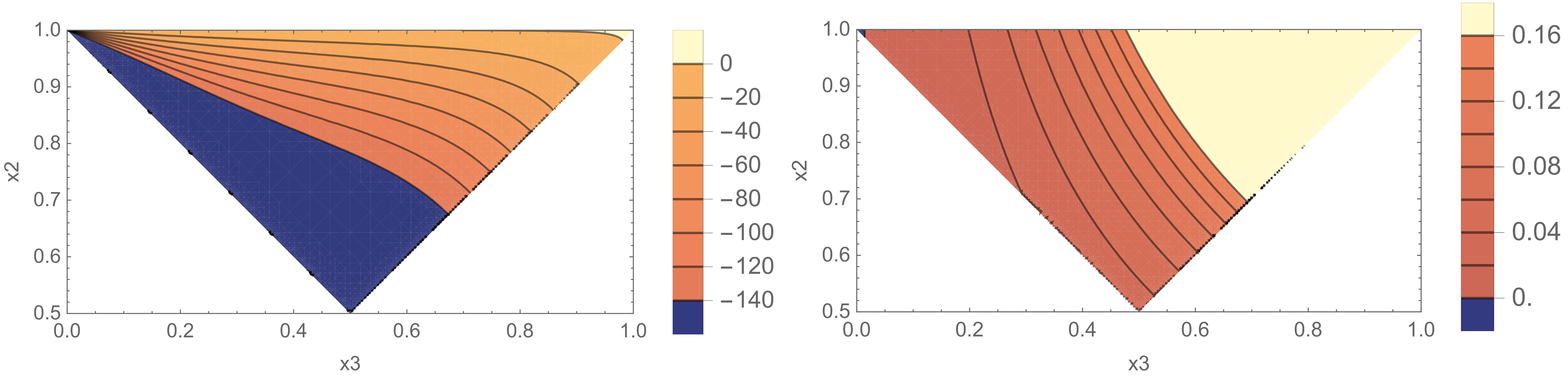


Equilateral limit has a growth towards small scales



Higher-order correlation functions and the large-$|\eta|$ approach

The bispectrum is strongly dependent on the scale,
and at each scale it has rich shape dependence



At the dip

Towards small scales

$f_{\rm NL}$

Subtle issues: Loop corrections and PBH

work in progress

Subtle issues: Loop corrections and PBH

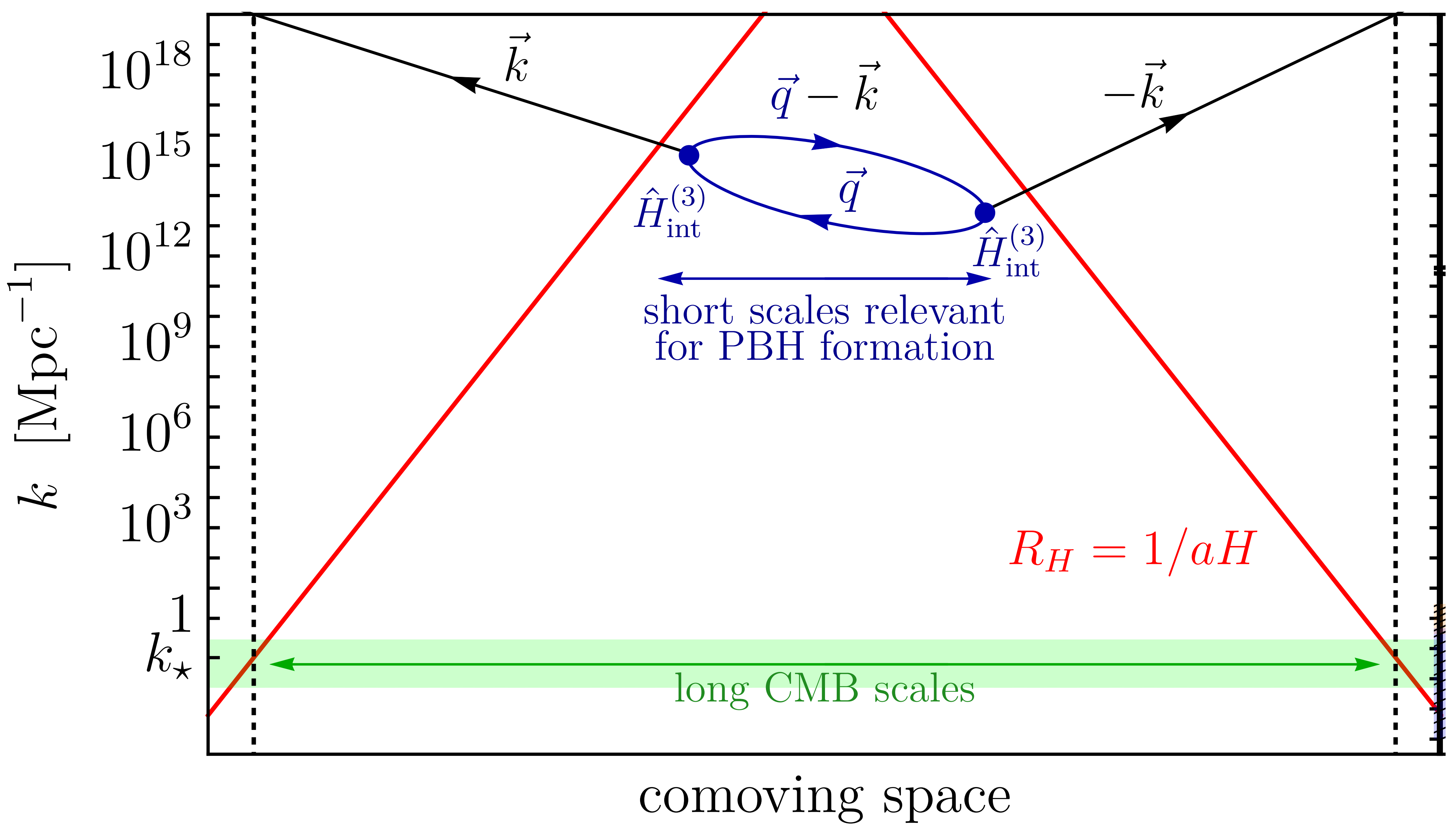
$$\langle \mathrm{in} \left| \bar{T} e^{-i\int \mathcal{H}_{\mathrm{int}}(\tau')d\tau'}\, \mathcal{O}(\tau)\, T e^{i\int \mathcal{H}_{\mathrm{int}}(\tau')d\tau'} \right| \mathrm{in} \rangle$$

$$\zeta''_{\mathbf{p}} + \frac{(a^2\epsilon)'}{a^2\epsilon}\zeta'_{\mathbf{p}} + \frac{(a^2\epsilon\eta')'}{4a^2\epsilon}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\zeta_{\mathbf{k}}\zeta_{\mathbf{p-k}} = 0$$

- ▷ In single-field slow-roll inflation, loop corrections are small [. . . , Weinberg,. . .]

- ▷ In PBH forming scenarios, the same mechanism that enhances the spectrum can also amplify loop corrections at large scales.

[Kristiano-Yokoyama, Riotto, Firouzjahi, Franciolini et al, Fumagalli,. . .]

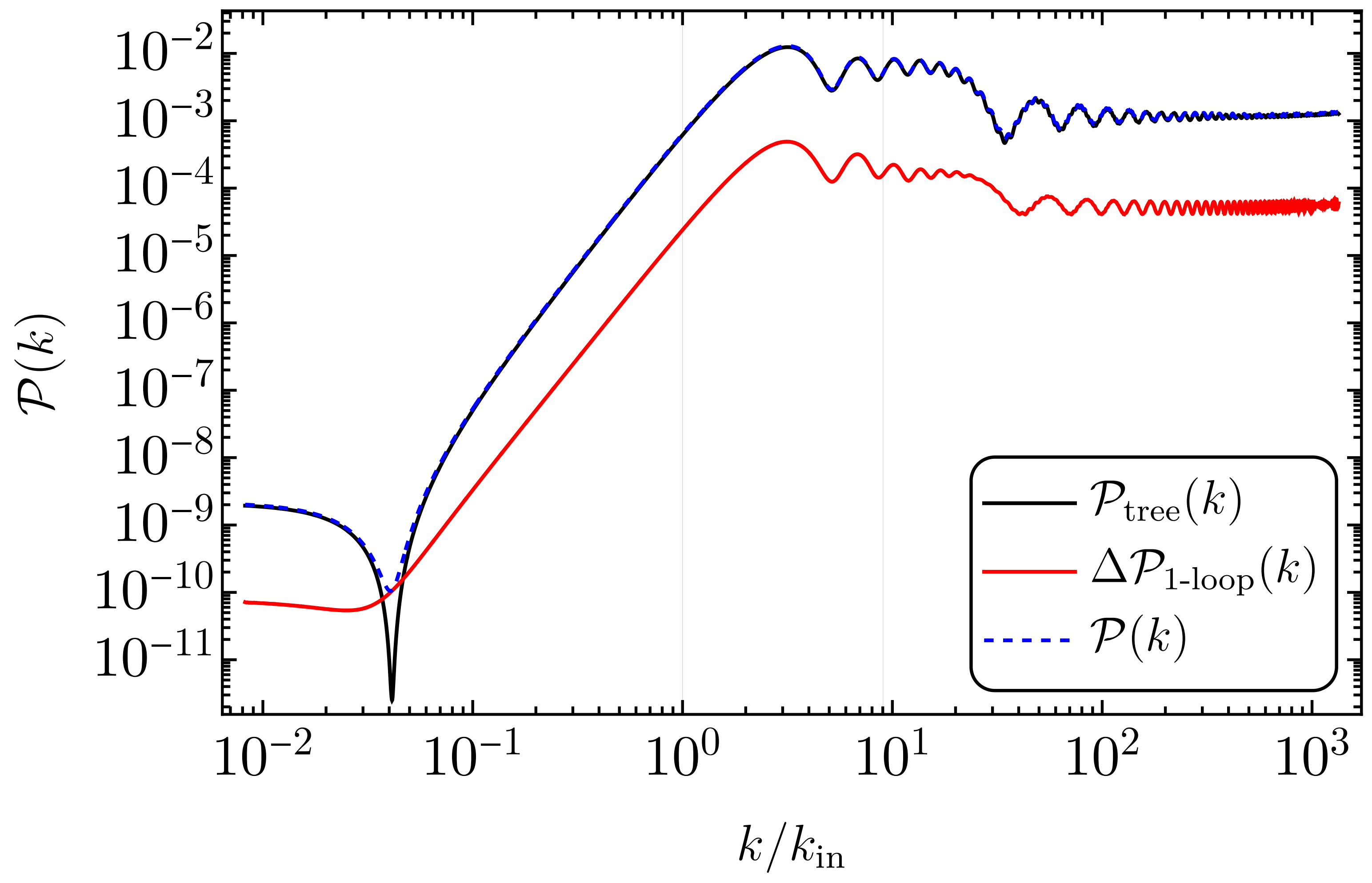


[Franciolini et al]

Subtle issues: Loop corrections and PBH

$$\langle \mathrm{in} \left| \bar{T} e^{-i\int \mathcal{H}_{\mathrm{int}}(\tau')d\tau'}\, \mathcal{O}(\tau)\, T e^{i\int \mathcal{H}_{\mathrm{int}}(\tau')d\tau'} \right| \mathrm{in} \rangle$$

- $\eta = -6$ and sudden transition between SR and USR: loops are dangerously large, and UV quadratic divergences should be renormalized [Kristiano-Yokoyama]
- Smooth transition between SR and USR: loops can be placed under control [Riotto, Firouzjahi, Franciolini et al,...]; model dependent issue



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Subtle issues: Loop corrections and PBH

$$\langle \text{in} \left| \bar{T} e^{-i\int \mathcal{H}_{\text{int}}(\tau')d\tau'} \, \mathcal{O}(\tau) \, T e^{i\int \mathcal{H}_{\text{int}}(\tau')d\tau'} \right| \text{in} \rangle$$

large-$|\eta|$ approach simplifies considerably formulas

in the case of a sudden transition

$$\Delta^{\text{loop}}(\kappa) \, = \, \Delta^{\text{tree}}(\kappa) \left[1 + L_{\text{UV}}(\kappa) + L_{\text{IR}}(\kappa) \right]$$

$$L_{\text{UV}}(\kappa) \, = \, -\Delta_0 \, \frac{\Pi_0 \, \Lambda_{\text{UV}}^2}{1+\Pi_0} \left(\frac{5}{6} + \frac{3 j_1(\kappa) - \kappa}{3\kappa} \right)$$ $\Rightarrow$ *at large scales it can be renormalized*

$$L_{\text{IR}}(\kappa) \, = \, -\frac{\Delta_0 \, \Pi_0}{6} \, \kappa^2 \, \ln\left(\mu / \Lambda_{\text{IR}} \right)$$ $\Rightarrow$ *due to secular effects of superhorizon modes*

(Δ_0 is the spectrum at large scales)

Subtle issues: Loop corrections and PBH

$$\langle \text{in} \left| \bar{T} e^{-i\int \mathcal{H}_{\text{int}}(\tau')d\tau'}\, \mathcal{O}(\tau)\, T e^{i\int \mathcal{H}_{\text{int}}(\tau')d\tau'} \right| \text{in} \rangle$$

large-$|\eta|$ approach

$$\Delta^{(\text{loop})}(\kappa) = \Delta_0 - \frac{4\,\Delta_0\,\Pi_0}{3}\left[1 + \frac{\Delta_0}{8}\ln\left(\mu/\Lambda_{\text{IR}}\right)\right]\kappa^2 + \mathcal{O}(\kappa^4)$$

$$\Downarrow$$

very small contribution $\Rightarrow$ κ^2-suppressed

Subtle issues: Loop corrections and PBH

$$\langle \mathrm{in} \left| \bar{T} e^{-i\int \mathcal{H}_{\rm int}(\tau')d\tau'}\, \mathcal{O}(\tau)\, T e^{i\int \mathcal{H}_{\rm int}(\tau')d\tau'} \right| \mathrm{in} \rangle$$

...but recently [Fumagalli] found that we were all missing boundary terms in the interaction Hamiltonian, that once included further reduce the size of loops to κ^3-suppressed corrections.

...also [Tada et al] reach a similar conclusions exploiting the effects of boundary terms

....on the other hand, [Firouzjahi] repeated the computations independently and found that the inclusion of boundary terms do not help after all to suppress loop corrections...

Conclusions

- Single-field models of inflation able to strongly enhance fluctuations at small scales can lead to interesting dark matter candidates (PBH, vector DM)
  - ▷ To properly understand their consequences, an analytical understanding of their features would be helpful.
- Since the slow-roll parameter $|\eta|$ is larger than one for a fraction of the inflationary phase, I considered the case $|\eta|$ large, and promoted $1/|\eta|$ to an expansion parameter.
- Formulas simplify, and obtain analytical expressions for the two and three point functions in agreement with previous studies and with expectactions.
- It will be interesting to further apply these methods and analytical formulas to study PBH formation, including the effects of non-Gaussianities, and to the analysis of loop corrections in these scenarios.