Stochastic sources for primordial perturbations

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Outline

Based on 2208.14978[1] and 2304.05978[2]:

1. Review of stochastic differential equations
2. Dissipative inflation
   - 2.1 Fluctuation dissipation theorem
   - 2.2 Stochastic equations of motion
   - 2.3 Inflaton perturbations and phenomenology
3. Another example: stochastic inflation
   - 3.1 Basic formalism
   - 3.2 General remarks

[1] In collaboration in G. Ballesteros, M.A.G. García, M. Pierre and J. Rey
[2] In collaboration in G. Ballesteros and M. Pierre

☞ Appendix C1 in [Ballesteros, Garcia, APR, Pierre, Rey 2022]

Langevin equation: continuum limit of the *stochastic difference equation*

$$\Delta\boldsymbol{\phi} = \boldsymbol{f}(\boldsymbol{\Phi}, t)\,\Delta t + \boldsymbol{g}(\boldsymbol{\Phi}, t)\,\Delta W_t$$

The *Wiener increment* ΔW_t is a **gaussian** random variable with

$$\langle \Delta W_t \rangle = 0, \qquad \langle \Delta W_t^2 \rangle = \Delta t$$

Discrete case: **numerical solution + Montecarlo average**

Alternatively, **continuum limit** provides a PDE for the PDF $P(\boldsymbol{\Phi}, t)$:
Fokker-Planck equation

$$\dot{\boldsymbol{\Phi}} = \boldsymbol{f}(\boldsymbol{\Phi}, t) + \boldsymbol{g}(\boldsymbol{\Phi}, t)\,\xi(t), \quad \langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t')$$

$$\frac{\partial P(\boldsymbol{\Phi}, t)}{\partial t} = \underbrace{-\sum_i \frac{\partial}{\partial \Phi_i} \left[P(\boldsymbol{\Phi}, t)\, f_i(\boldsymbol{\Phi}, t)\right]}_{\text{Drift (deterministic)}} + \underbrace{\sum_{i,j} \frac{\partial^2}{\partial \Phi_i\, \partial \Phi_j} \left[P(\boldsymbol{\Phi}, t)\, D_{ij}(\boldsymbol{\Phi}, t)\right]}_{\text{Difussion (stochastic)}}, \quad D = \frac{1}{2}\boldsymbol{g}\boldsymbol{g}^T$$

The Fokker-Planck equation can be

- Solved as a PDE
- Used to construct system of ODEs for n-th moment of $\boldsymbol{\Phi}(t)$

See [Calzetta, Hu 2008] for a rigurous description. Schematically,

- A **dissipative** process induces **fluctuations** around equilibrium
- The **variance of the fluctuations** is proportional to the **strength of dissipation** and the **temperature** of the system

Toy model: particle coupled to harmonic oscillators

[Calzetta, Hu 2008] [Ballesteros, Garcia, APR, Pierre, Rey 2022]

$$H = \underbrace{\frac{p^2}{2M} + V(q)}_{\text{heavy particle}} + \underbrace{\sum_{i=1}^{N} \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2 \right)}_{\text{oscillators (environment)}} \underbrace{- q \sum_{i=1}^{N} g_i x_i + q^2 \sum_{i=1}^{N} \frac{g_i^2}{2 m_i \omega_i^2}}_{\text{interaction Hamiltonian}} .$$

EOM of heavy particle: $M\ddot{q} + V'(q) + \int_0^t ds\, \gamma(t-s)\dot{q}(s) = \xi(t)$,

$$\gamma = \sum_i \frac{g_i^2}{m_i\omega_i^2}\cos(\omega_i t)\,,\ \text{(\textbf{dissipation}, “microphysics”)};$$

$$\xi(t) = \sum_i g_i\left[\left(x_i(0) - \frac{g_i}{m_i\omega_i^2}q(0)\right)\cos(\omega_i t) + \frac{p_i(0)}{m_i\omega_i}\sin(\omega_i t)\right]\ \text{(\textbf{fluctuation}, “unknown d.o.f.”)}.$$

Stochastic information on $x_i(0)$, $p_i(0)$ can be obtained from equipartition theorem (**assuming thermal equilibrium**):

$$m_i\omega_i^2\langle x_i(0)x_j(0)\rangle = \frac{1}{m_i}\langle p_i(0)p_j(0)\rangle = \delta_{ij}\langle E_i\rangle = T\delta_{ij}\,,\quad \langle x_i(0)p_j(0)\rangle = 0$$

In consequence, $\boxed{\langle\xi(t)\xi(t')\rangle \propto \gamma T}$ (details in [Ballesteros, Garcia, APR, Pierre, Rey 2022]).

Inflation driven by inflaton coupled to thermalised radiation:

$$T^{\mu\nu} = T^{\mu\nu}_{(\phi)} + T^{\mu\nu}_{(r)}, \qquad \text{Cf. } \textit{warm inflation}$$

Continuity equation: [Bastero-Gil, Berera, Moss, Ramos 2014]

$$\nabla_\nu T^{\mu\nu}_{(\phi)} = Q^\mu = -\nabla_\nu T^{\mu\nu}_{(r)}, \quad Q_\mu = \underbrace{-\Gamma\, u^\nu \nabla_\nu \phi \nabla_\mu \phi}_{\text{dissipation}} + \underbrace{\sqrt{\frac{2\Gamma T}{a^3}} \nabla_\mu \phi \overbrace{\xi(x,t)}^{\text{stoch.}}}_{\text{fluctuation}},$$

$$\langle \xi(\boldsymbol{x},t)\xi(\boldsymbol{x}',t')\rangle = \delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')$$

- Q^0: **energy** transfer; $\quad Q^i$, $i = 1, 2, 3$: **momentum** flux
- Dissipation coefficient depends on microphysics (Lagrangian)
- Fluctuation term follows **fluctuation-dissipation theorem**

$$T^{\mu\nu} = T^{\mu\nu}_{(\phi)} + T^{\mu\nu}_{(r)}$$

$$\nabla_\nu T^{\mu\nu}_{(\phi)} = Q^\mu = -\nabla_\nu T^{\mu\nu}_{(r)}, \quad Q_\mu = \underbrace{-\Gamma(\phi, T)\, u^\nu \nabla_\nu \phi \nabla_\mu \phi}_{\text{dissipation}} + \underbrace{\sqrt{\frac{2\Gamma T}{a^3}} \nabla_\mu \phi\, \xi(x,t)}_{\text{fluctuation}}$$

- Unperturbed FLRW metric
- Perfect fluid: $\rho = \rho_\phi + \rho_r = (\dot\phi^2/2 + V) + \rho_r; p = p_\phi + p_r = (\dot\phi^2/2 - V) + 4\rho_r/3$
- **Fluctuation:** $\xi(x,t)$ perturbatively small $\implies$ **no background fluctuation**
- **Dissipation:** $-\Gamma\, u^\nu \nabla_\nu \phi \nabla_\mu \phi = -\Gamma \dot\phi^2 \rightarrow$ "**friction** term"

$$\ddot\phi + 3H(1+Q)\dot\phi + V_{,\phi} = 0, \qquad Q = \frac{\Gamma}{3H}$$
$$\dot\rho_r + 4H\,\rho_r = \Gamma\dot\phi^2$$

$$T^{\mu\nu} = T^{\mu\nu}_{(\phi)} + T^{\mu\nu}_{(r)}$$

$$\nabla_\nu T^{\mu\nu}_{(\phi)} = Q^\mu = -\nabla_\nu T^{\mu\nu}_{(r)}, \quad Q_\mu = -\Gamma\, u^\nu \nabla_\nu \phi \nabla_\mu \phi + \sqrt{\frac{2\Gamma\, T}{a^3}}\, \nabla_\mu \phi\, \xi(x,t)$$

- Perturbed metric (newtonian gauge)
- Perturbed $T_{\mu\nu}$: $\delta\phi$, $\delta\dot{\phi}$, $\delta\rho_r$, δq_r
- **Fluctuation** $\xi(x,t)$ is space dependent, therefore **sources perturbations** with a variance given by the FDT $\propto \Gamma\, T$.
- **Dissipation** term $-\Gamma\, u^\nu \nabla_\nu \phi \nabla_\mu \phi$ is perturbed:
  - Time component: perturbed version of $-\Gamma(\phi, T)\dot{\phi}^2$ (depends on $\Gamma_{,\phi}$, $\Gamma_{,T}$)
  - Space components: perturbed momentum transfer

Combining **perturbed Einstein** and **continuity** equations in Fourier space yields as system of SDEs

$$\dot{\mathbf{\Phi}}_{\boldsymbol{k}} + A\,\mathbf{\Phi}_{\boldsymbol{k}} = \boldsymbol{B}\,\xi_{\boldsymbol{k}}, \qquad \mathbf{\Phi}_{\boldsymbol{k}} = (\delta\phi_{\boldsymbol{k}}, \delta\dot{\phi}_{\boldsymbol{k}}, \delta\rho_{r,\boldsymbol{k}}, \psi_{\boldsymbol{k}})$$
$$\mathcal{R}_{\boldsymbol{k}} = \boldsymbol{C}^T\,\mathbf{\Phi}_{\boldsymbol{k}}\,,$$

where A, $\boldsymbol{B}$ and $\boldsymbol{C}$ are a **time dependent** matrices which **only depend on background quantities**. To compute the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$, one can:

- Solve $\mathcal{O}(1000)$ times the Lagevin equation,
- Use Fokker-Planck to construct ODE for the covariance matrix $U \equiv \langle \mathbf{\Phi}^{(i)}[\mathbf{\Phi}^{(i)}]^{\dagger}\rangle \longrightarrow \dot{U} = -AU - U\,A^T + \boldsymbol{B}\boldsymbol{B}^T \longrightarrow \langle|\mathcal{R}_k^{(i)}|^2\rangle = \boldsymbol{C}^T\,U\,\boldsymbol{C}$.

Details in [Ballesteros, Garcia, APR, Pierre, Rey 2021] ☞ Guillermo Ballesteros' talk.

$$\delta\ddot{\phi}_{\boldsymbol{k}} + (3H+\Gamma)\delta\dot{\phi}_{\boldsymbol{k}} + \left(\frac{k^2}{a^2} + V_{\phi\phi} + \dot{\phi}\,\Gamma_{,\phi}\right)\delta\phi_{\boldsymbol{k}} + \underbrace{\Gamma_{,T}\,\frac{\dot{\phi}T}{4\rho_r}\delta\rho_{r,\boldsymbol{k}} - 4\dot{\psi}_{\boldsymbol{k}}\,\dot{\phi} + (2V_\phi + \Gamma\,\dot{\phi})\psi_{\boldsymbol{k}}}_{\text{coupling to other perturbations}} = \sqrt{\frac{2\Gamma\,T}{a^3}}\,\xi_{\boldsymbol{k}}(t)$$

Notice influence of transfer term $Q_\mu = -\Gamma\,u^\nu\,\nabla_\nu\phi\,\nabla_\mu\phi + \sqrt{\frac{2\Gamma\,T}{a^3}}\,\nabla_\mu\phi\,\xi(x,t)$

- Dissipation transfer term: unperturbed (Γ), perturbed ($\Gamma_{,\phi}$; $\Gamma_{,T}$)
- Fluctuation transfer term: **classical source** for otherwise **quantum** perturbations

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\left(|\mathcal{R}_k^{(h)}|^2 + \langle|\mathcal{R}_k^{(i)}|^2\rangle\right)$$

$|\mathcal{R}_k^{(h)}|^2$ dominates for $Q \ll 1$. $\langle|\mathcal{R}_k^{(i)}|^2\rangle$ dominates for $Q \gg 1$, **enhances** overall $\mathcal{P}_{\mathcal{R}}(k)$.

Transient dissipation → local enhancement of thermal (classical, FDT induced) component of $\mathcal{P}_{\mathcal{R}}(k)$, **peak in the spectrum** ☞ Guillermo Ballesteros' talk

Main idea: splitting field on coarse-graining scales $k_\sigma = \sigma a H$, $\sigma \ll 1$:

$$\bar{\phi}(\boldsymbol{x}, N) = \int \frac{d^3k}{(2\pi)^{3/2}} W(k_\sigma - k) \left(\phi_k \hat{a}_{\boldsymbol{k}} + {\phi_k}^* \hat{a}^\dagger_{\boldsymbol{k}} \right) e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

$$\phi_q(\boldsymbol{x}, N) = \int \frac{d^3k}{(2\pi)^{3/2}} W(k - k_\sigma) \left(\phi_k \hat{a}_{\boldsymbol{k}} + {\phi_k}^* \hat{a}^\dagger_{\boldsymbol{k}} \right) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Define $\xi_\phi^{(Q)} = -\int \frac{d^3k}{(2\pi)^{3/2}} W'(k - k_\sigma) \left(\phi_k \hat{a}_{\boldsymbol{k}} + \text{h.c.} \right) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. **System of SDEs**:

$$\boxed{\begin{aligned} \bar{\phi}' &= \bar{\pi} + \xi_\phi^{(Q)} , \\ \bar{\pi}' &= -(3-\epsilon)\bar{\pi} - \frac{V_{,\phi}|_{\bar{\phi}}}{H^2} + \xi_\pi^{(Q)} , \end{aligned}}$$

where $\langle \xi_\phi^{(Q)}(N) \xi_\phi^{(Q)}(N') \rangle \propto \mathcal{P}_{\delta\phi}(k_\sigma, N)\delta(N - N')$, etc.

- Equivalence quantum fields - classical noise. The commutator $[\xi_\phi^{(Q)}, \xi_\pi^{(Q)}] \to 0$ as $\sigma \to 0$ (cf. "squeezing" [Kiefer, Polarski 1998])
- The power spectrum **coincides** with the one computed in linear perturbation theory [Vennin, Starobinsky 2015], [Ballesteros, Rey, Taoso, Urbano 2020]
- Beyond power spectrum: non-gaussian PDF of $\mathcal{R}$ $\to$ stoch. δN formalism

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- $\xi_\phi^{(Q)}$, $\xi_\pi^{(Q)}$ are **no classical source**; they account for quantum fluctuations. Cf. dissipative case:
  quantum fluctuations $\leftrightarrow$ **noiseless** eq. *vs.* **thermal** fluctuations $\leftrightarrow$ **noisy** eq.
- Stochastic warm inflation [Ballesteros, APR, Pierre 2023], also early work in [Ramos, da Silva 2013]. **No departure** from pert. theory power spectrum.

Conclusions

1. **Langevin equations** arise when studying inflationary dynamics beyond single-field inflation in linear perturbation theory
2. Noise describes **thermal** fluctuations associated to dissipation (**fluctuation-dissipation theorem**)
   - Noise source in EOMs accounts for unknown degrees of freedom of the background
   - Its variance depends on T and strength of dissipation
   - Thermal noise is an extra source for $\mathcal{P}_{\mathcal{R}}$ (on top of quantum fluctuations)
3. Noise describes **backreaction of (quantum) fluctuations** on homogeneous background
   - Formally different from thermal case
   - Power spectrum in this formalism coincides with the one in perturbation theory