Phenomenology of gravitational waves from hyperbolic encounters

M. Teuscher **A. Barrau** **K. Martineau**
based on work by **J. García-Bellido**

Outline

This is an ongoing project!

I. **Motivations**

II. **Searching for optimal scenarios**

III. **Prospects at Ultra High Frequencies**

IV. **Conclusion**

Motivations for studying hyperbolics

More probable events than bounded orbits?

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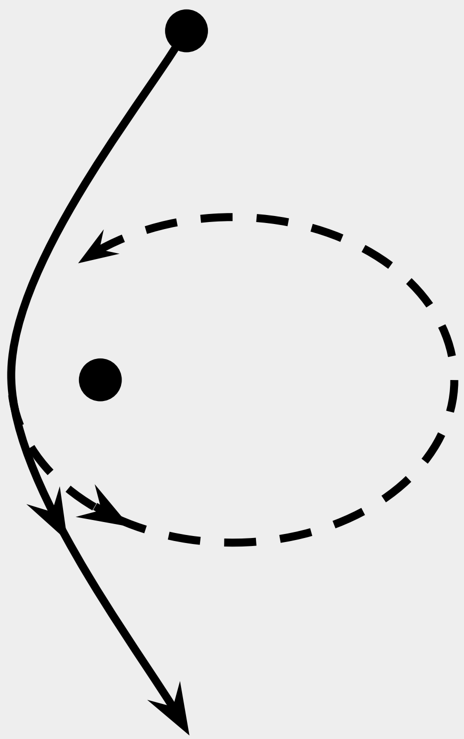
Close encounters
∈ ***dense*** PBH clusters



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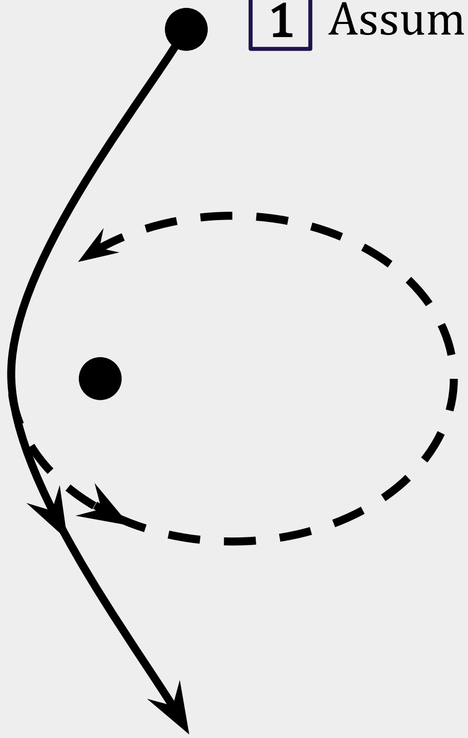

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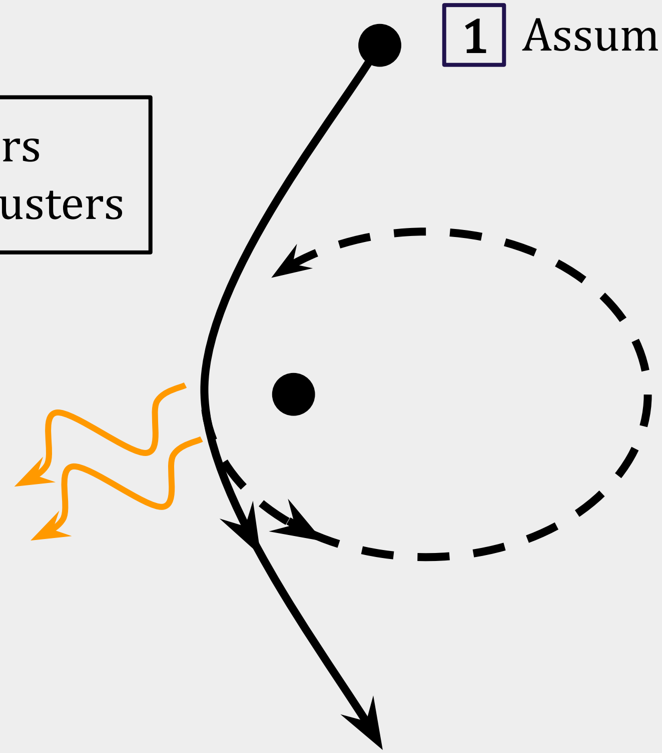

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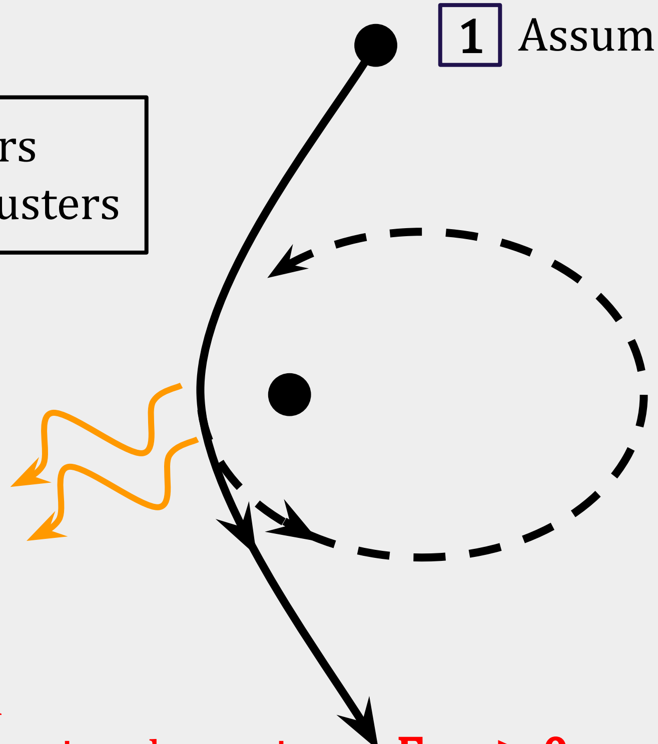

2 **GW bremsstrahlung**

$\mathbf{E_{OUT} > 0}$

3 True with
- Newtonian dynamics
- weak energy dissipation

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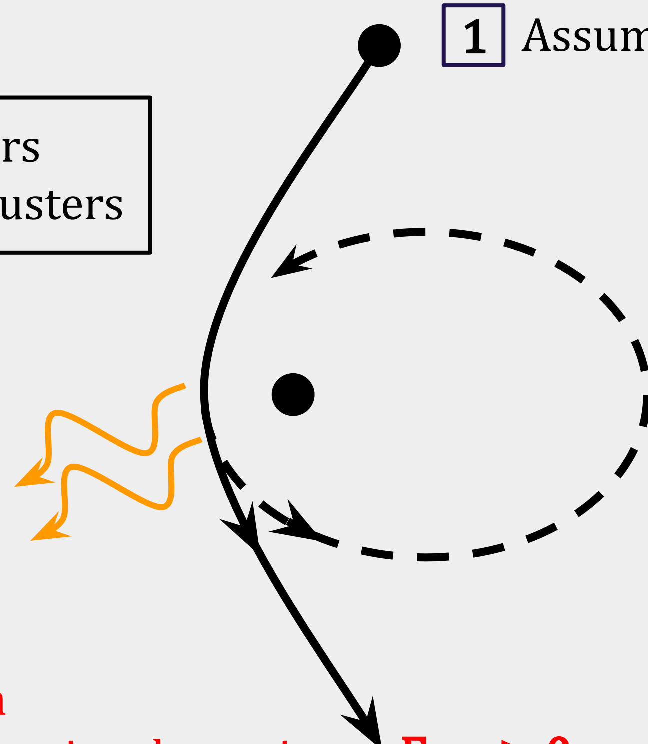

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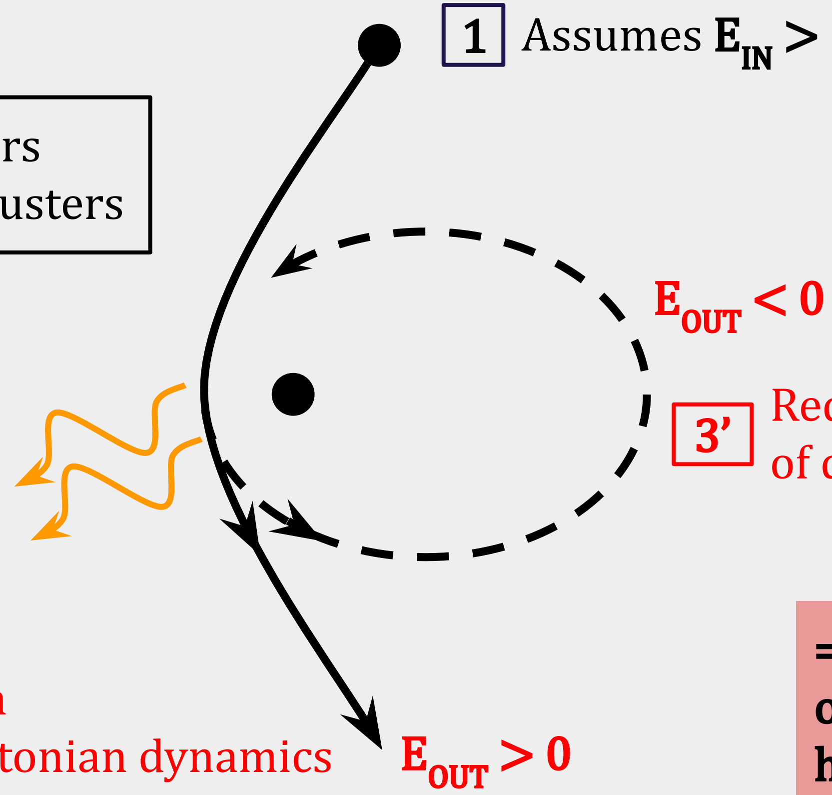

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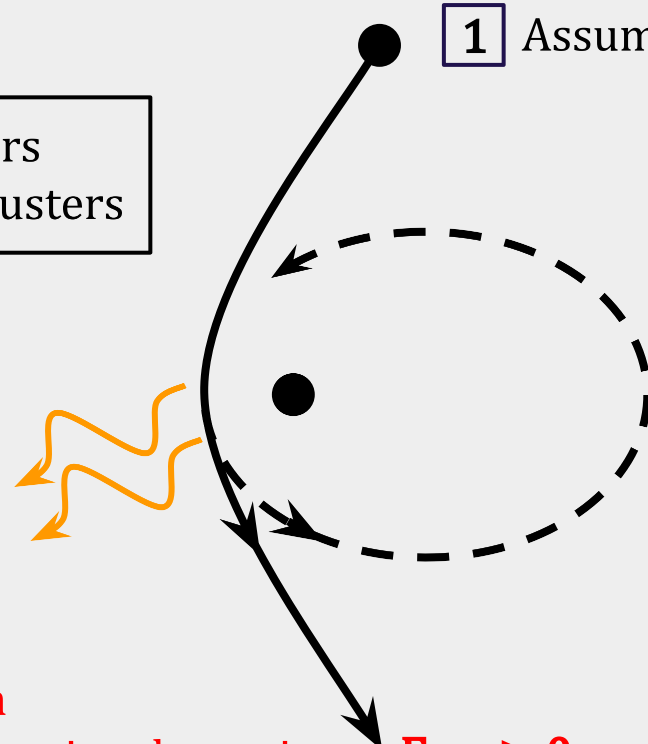

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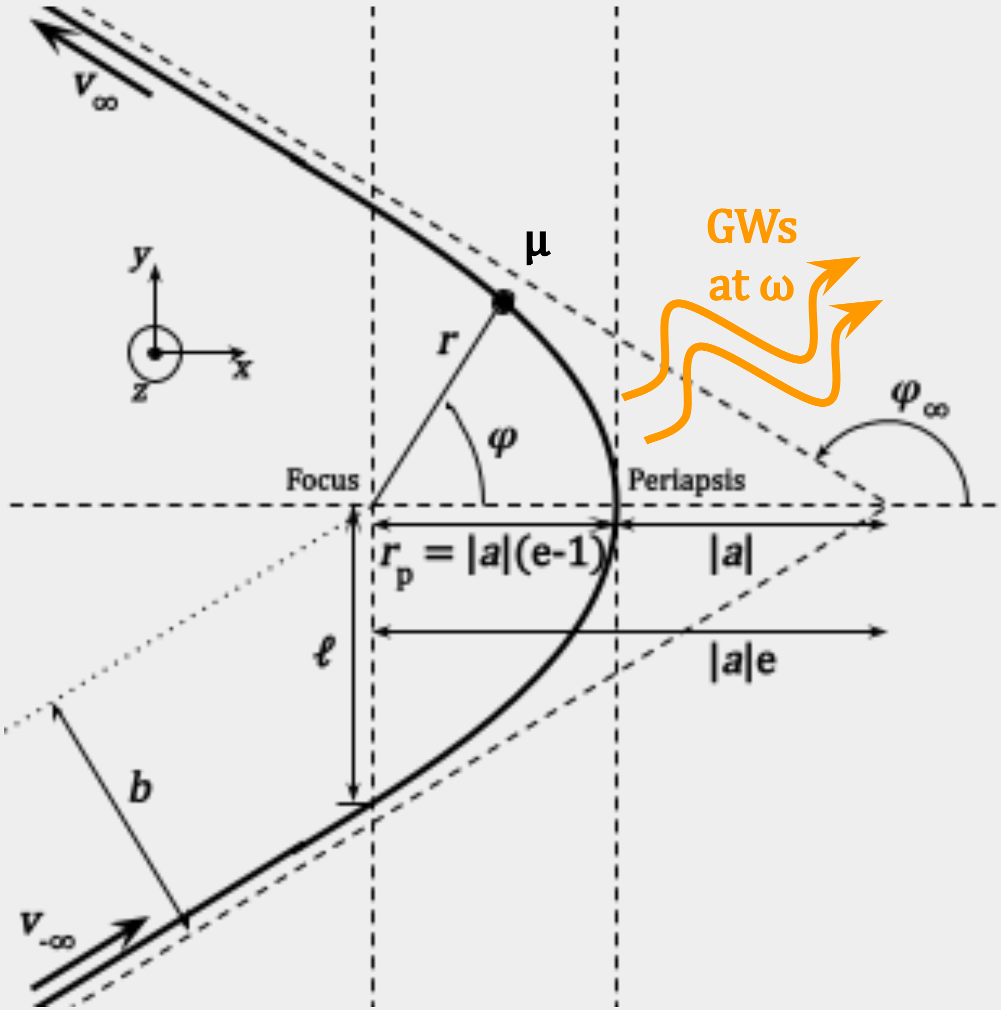
Very **scarce literature**: no textbook !
Studied e.g. in [García-Bellido *et al.* 1711.09702 / 2307.00915, De Vittori *et al.* 1207.5359]

GWs on hyperbolas



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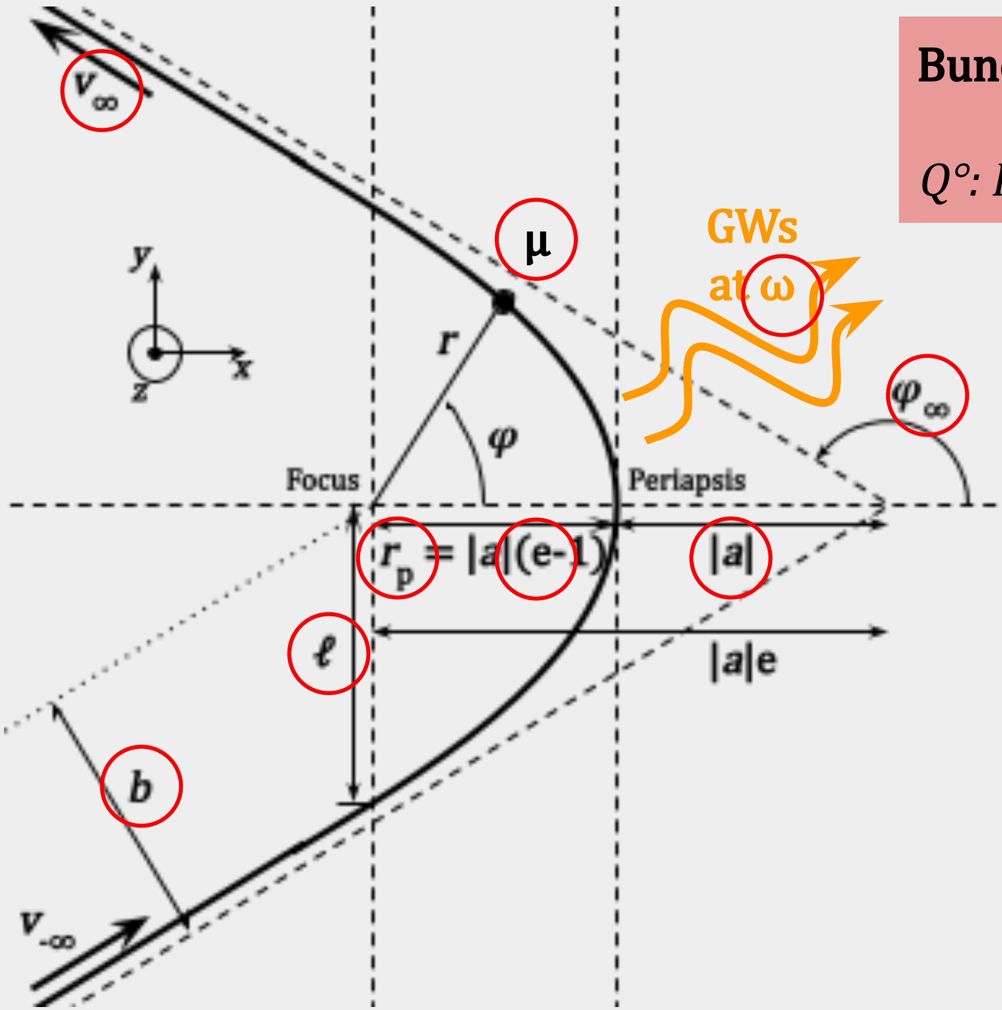

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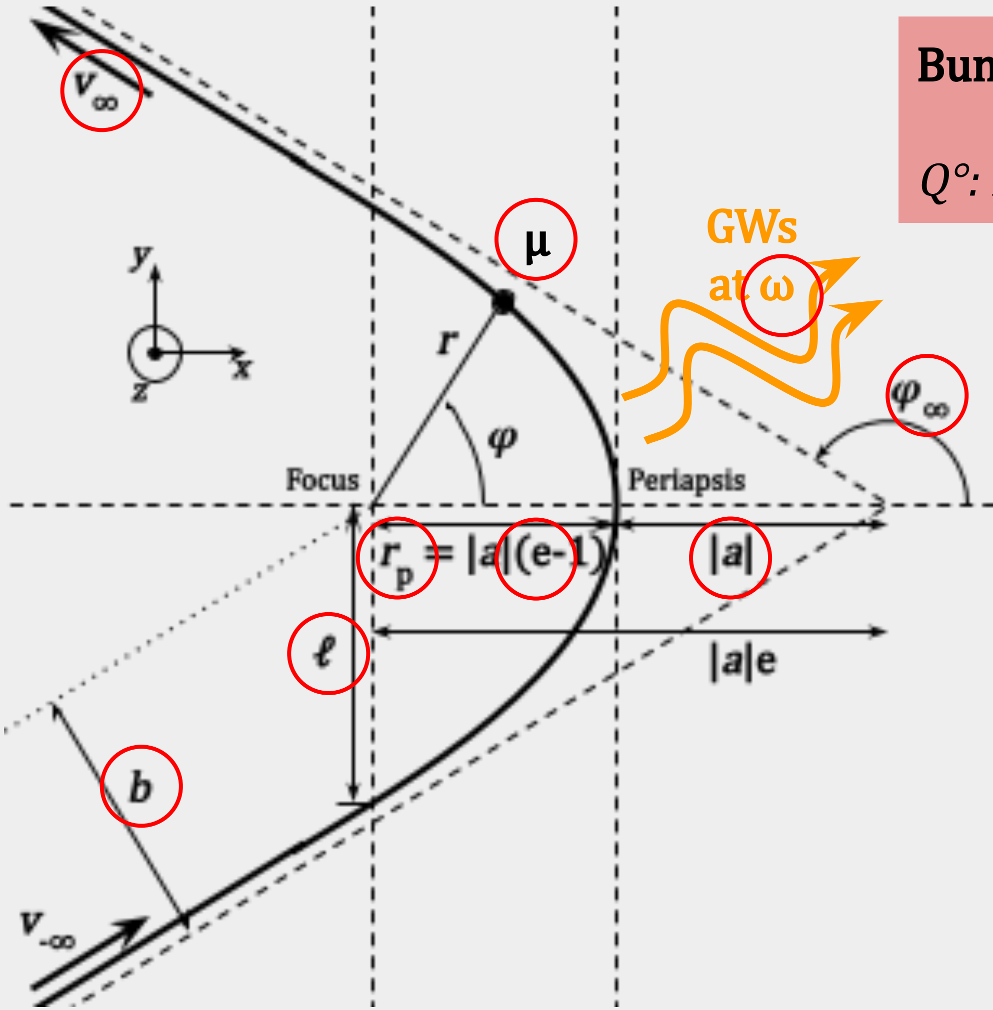

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→ GW detector fixes **frequency**

⇒ 2 remaining D.o.F.

Aside: frequency of aperiodic signal

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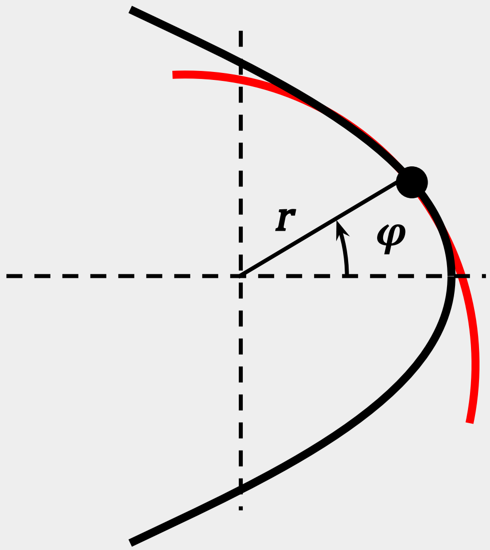
Geometrical definition



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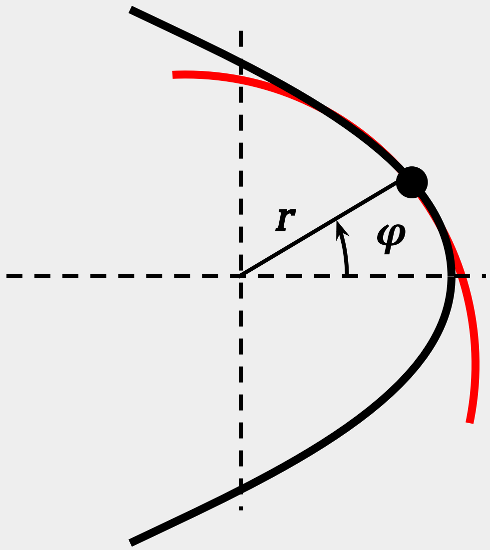

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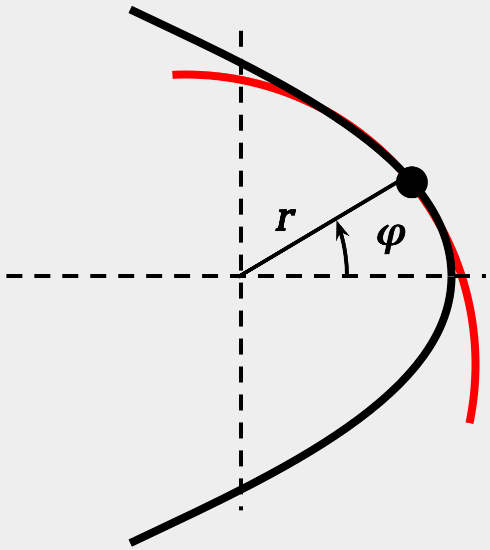

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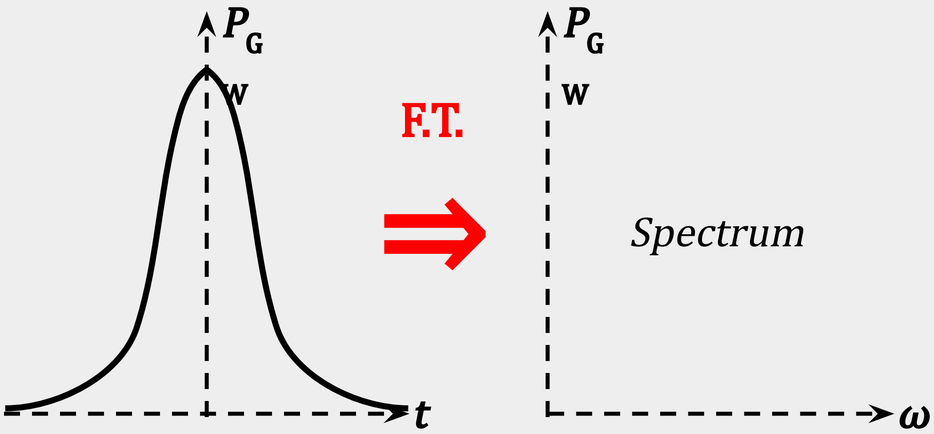

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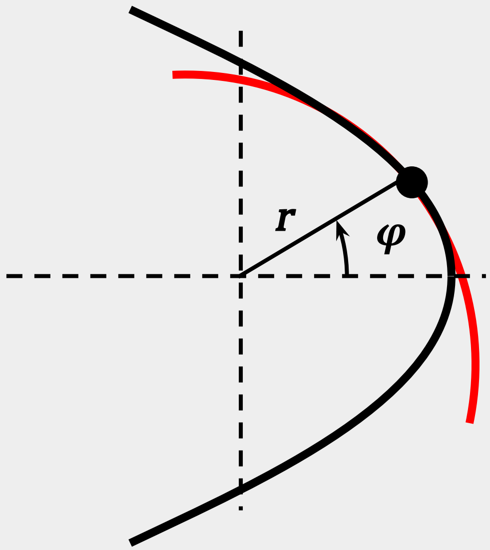

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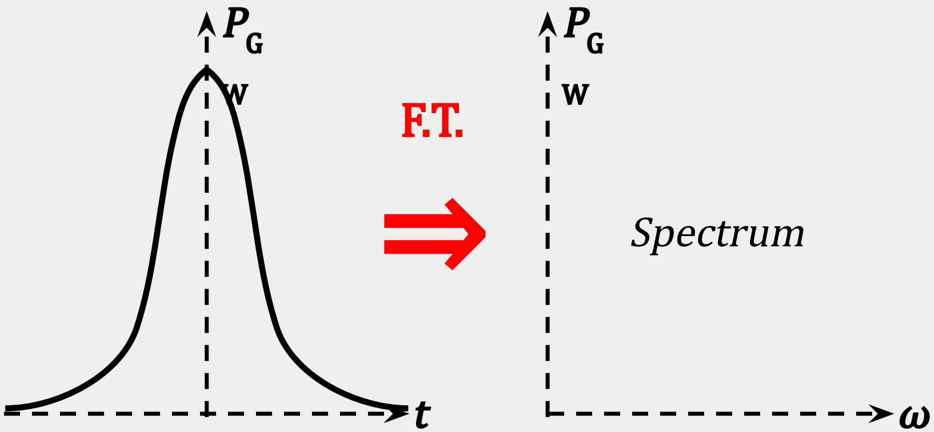

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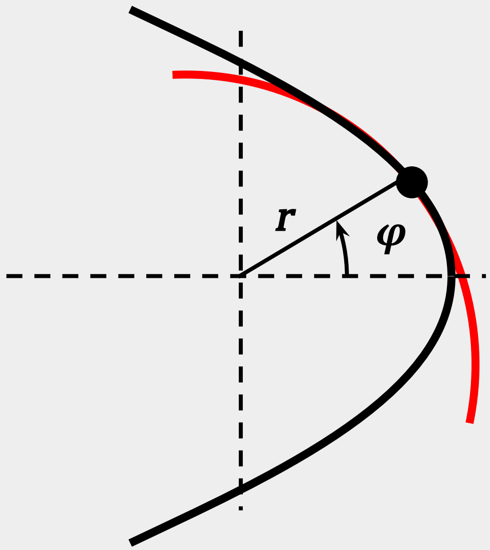

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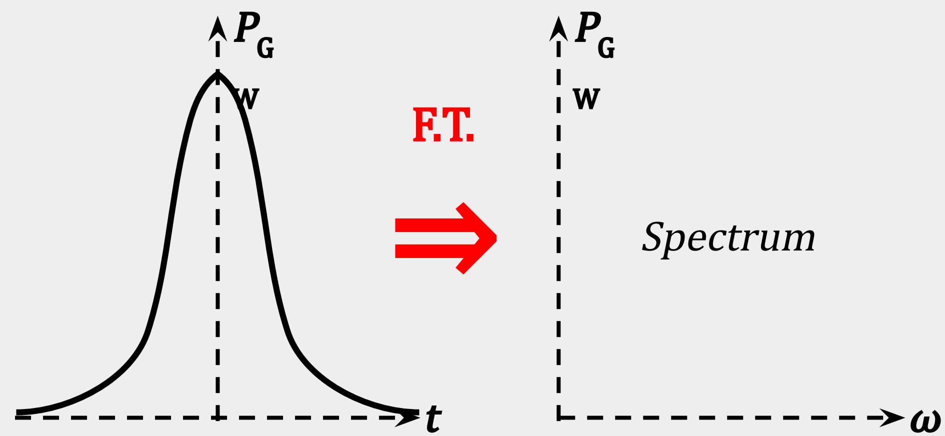


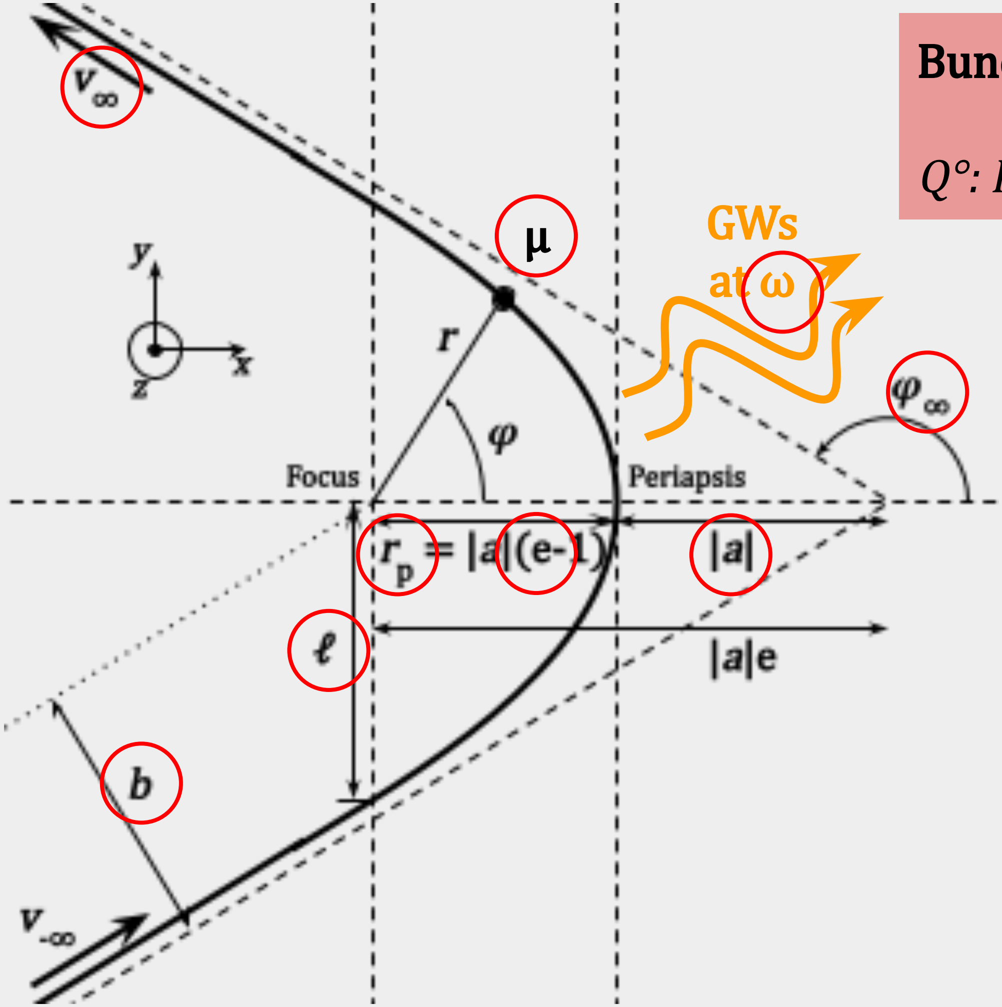
$$\omega_{\text{max}} \equiv \omega\big|_{\text{spectrum peak}}$$

[García-Bellido *et al.* 1711.09702]: **equivalent!**

And even: $$r_{\text{min}}^3 \omega_{\text{max}}^2 = \mathcal{G}M(e+1)$$

⇒ Pseudo Kepler law!

GWs on hyperbolas



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→ **frequency** → fixed by detector

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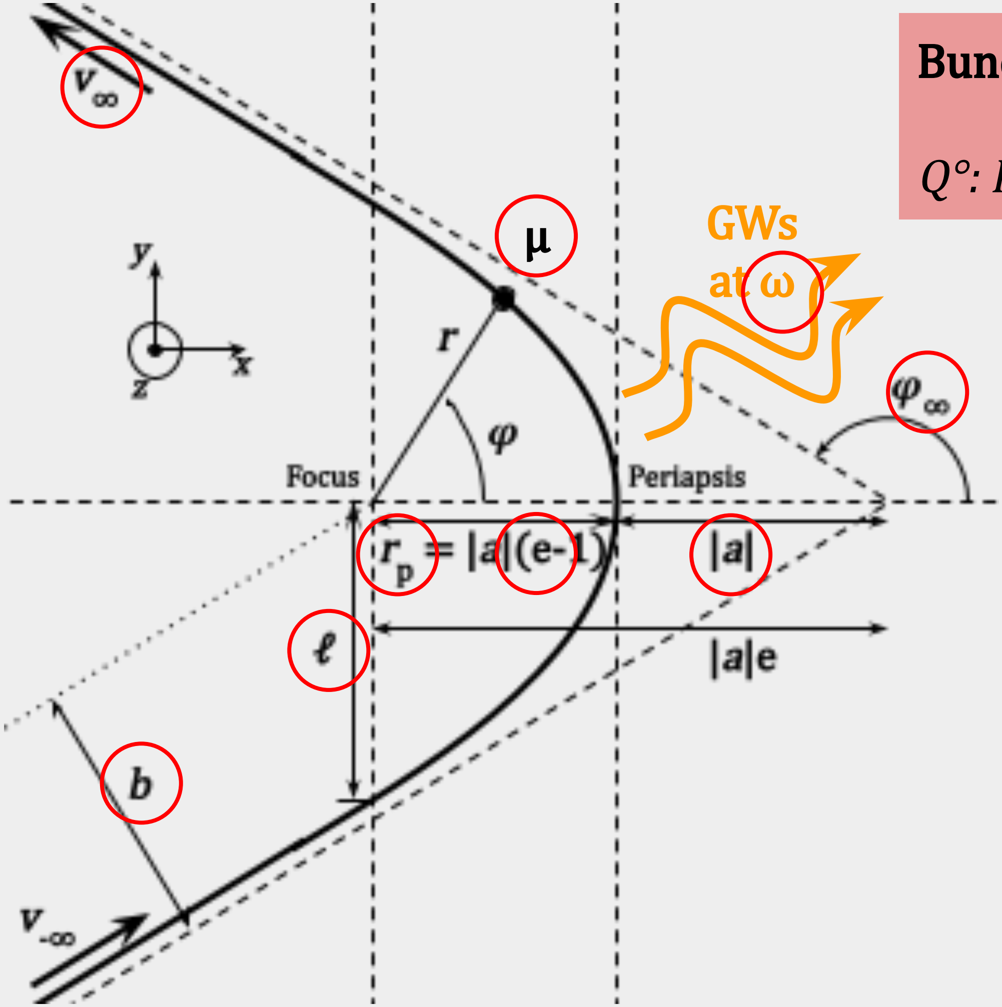

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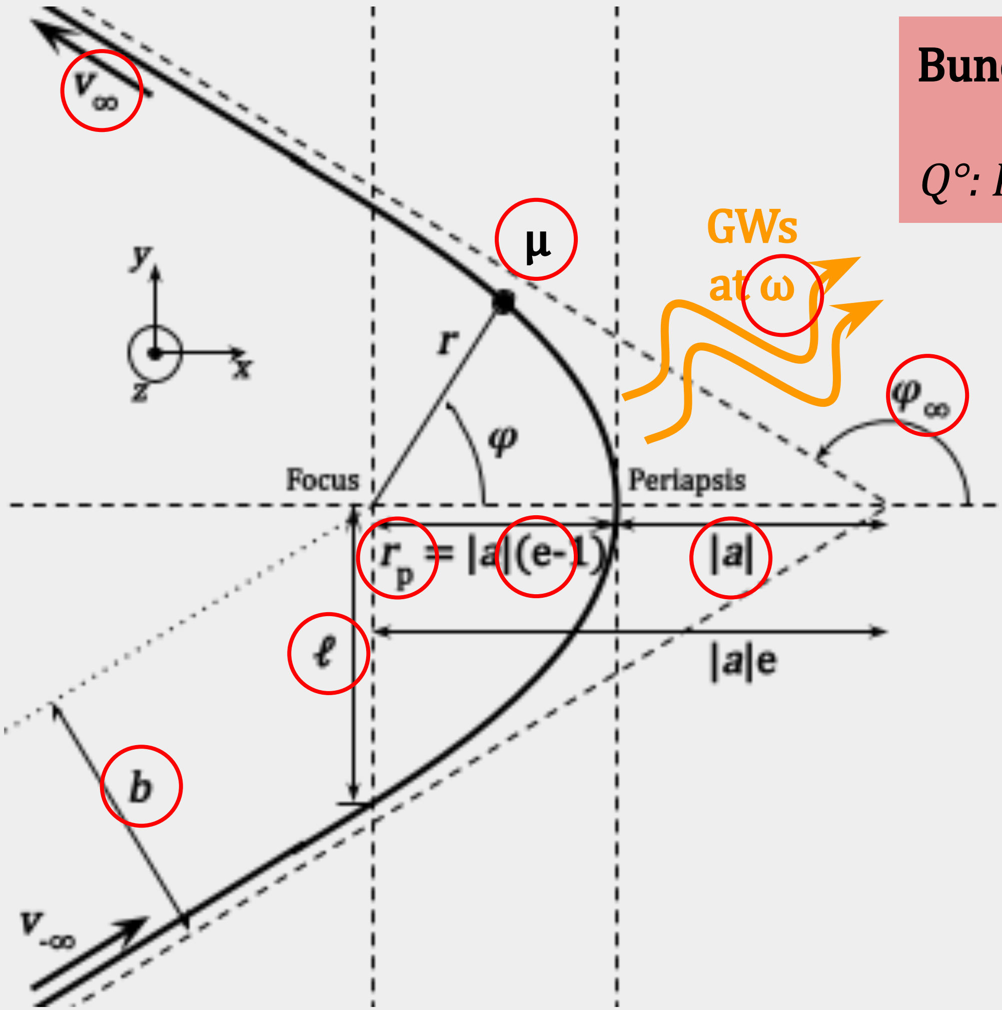

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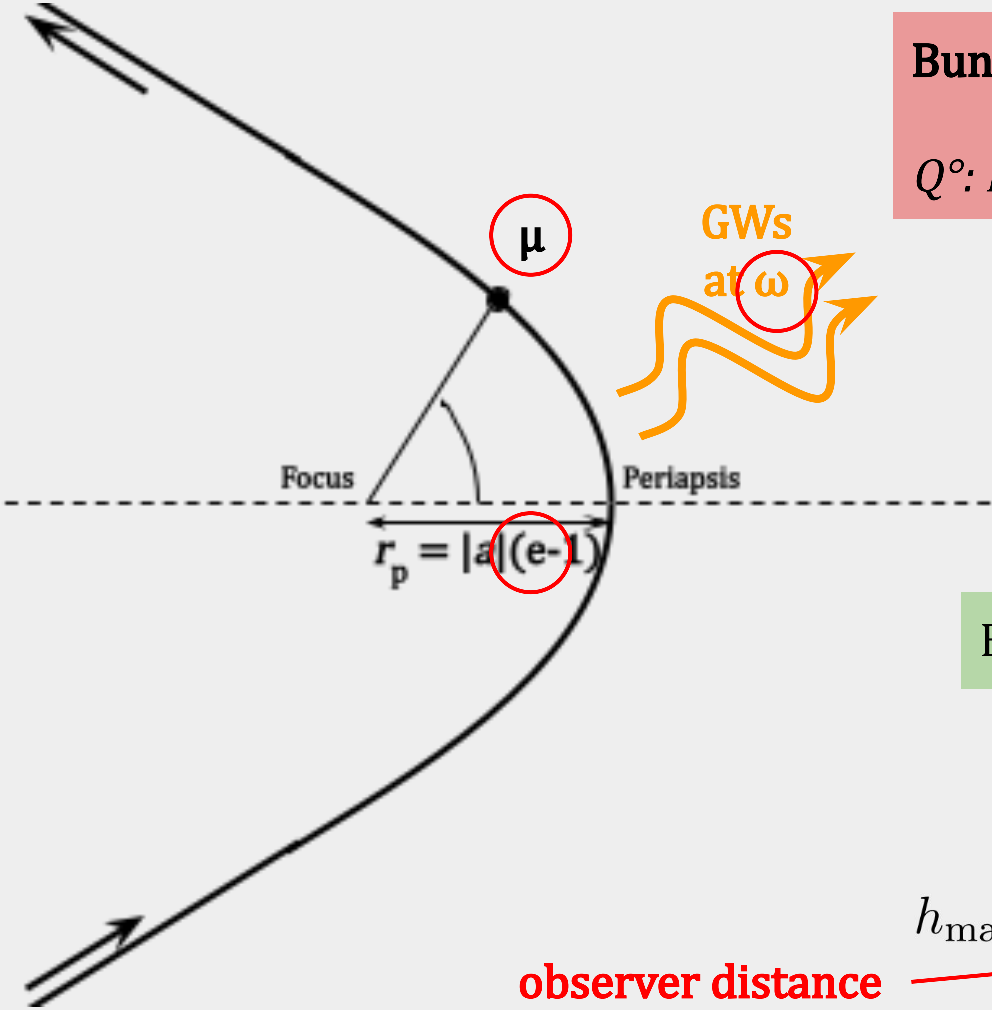

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Based on [García-Bellido *et al.* 1711.09702]:

$$h \equiv \sqrt{2(h_+^2 + h_\times^2)} \qquad \textit{(equal mass BHs)}$$

$$h_{\max} = \frac{1}{R}\omega_{\max}^{2/3}(\mathcal{G}M)^{5/3}\frac{\sqrt{18(e+1)+5e^2}}{(e+1)^{1/3}}$$

observer distance → R

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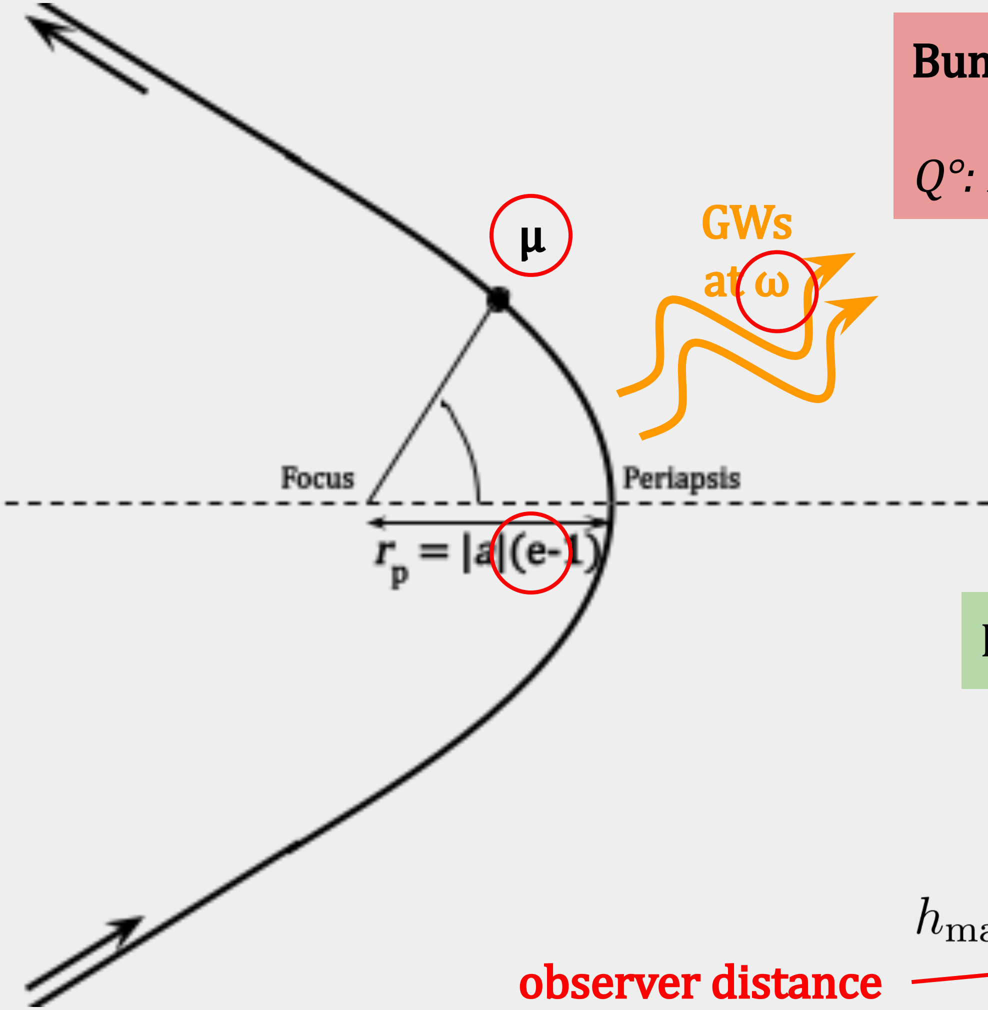

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Goals :

- Probe 2D parameter space with M and e
- Best possible h_{GW} for a given detector ? on which trajectory ?
- How far can we detect such events ?

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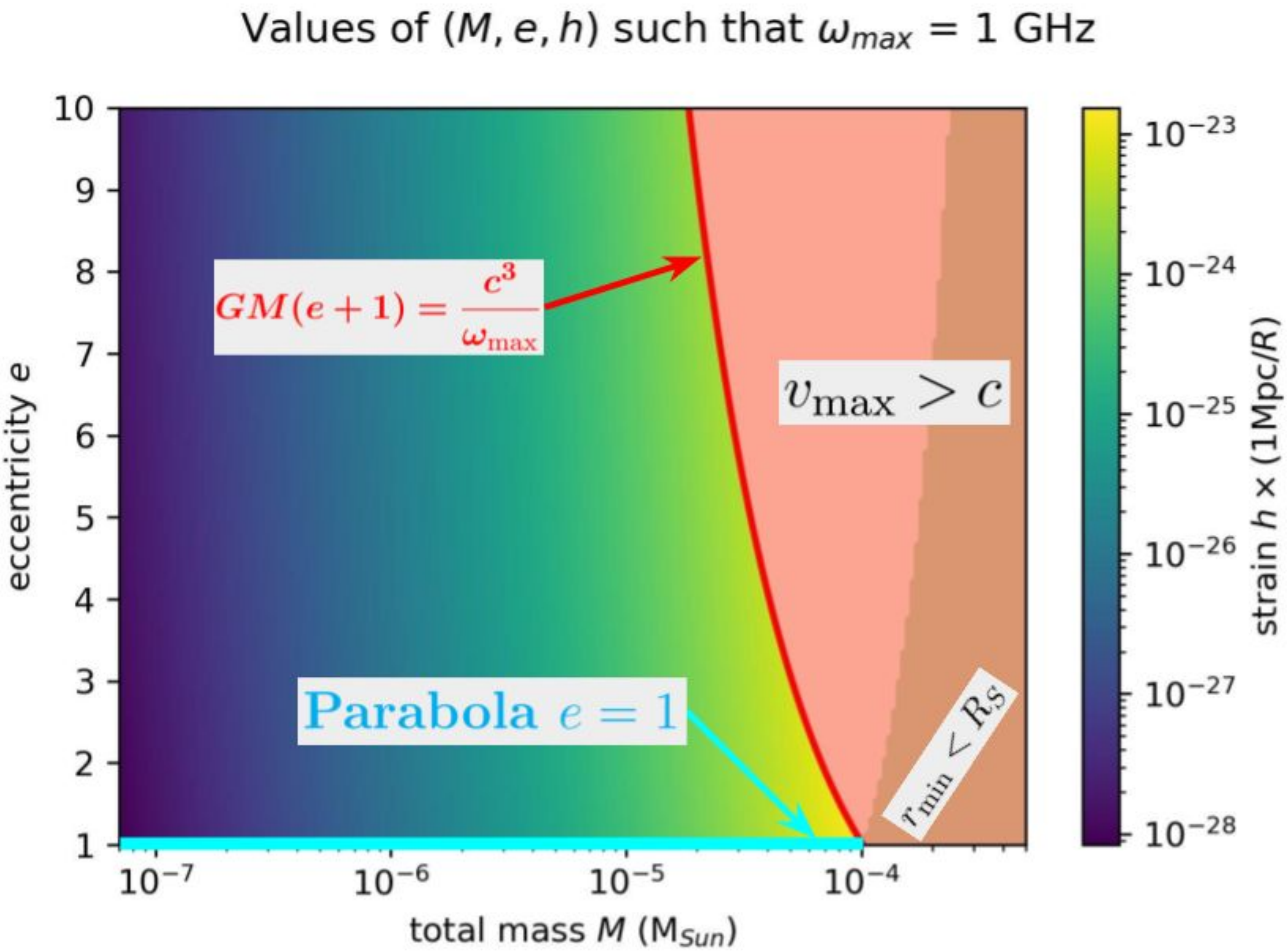
Parameter space for hyperbolas



Optimal and suboptimal GW strains

Event is observable at *f=ω/2π* only if:

$$M \leqslant \frac{c^3}{2\pi\mathcal{G}f(e+1)} \leqslant \boxed{2\times 10^{-5}\ \mathrm{M}_\odot\left(\frac{1\ \mathrm{GHz}}{f}\right)}$$

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Releasing $\boldsymbol{M}$ AND $\boldsymbol{e}$ $\Rightarrow$ $\boldsymbol{\omega_{\mathrm{max}}}$ only parameter

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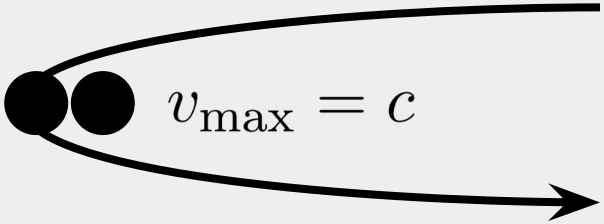

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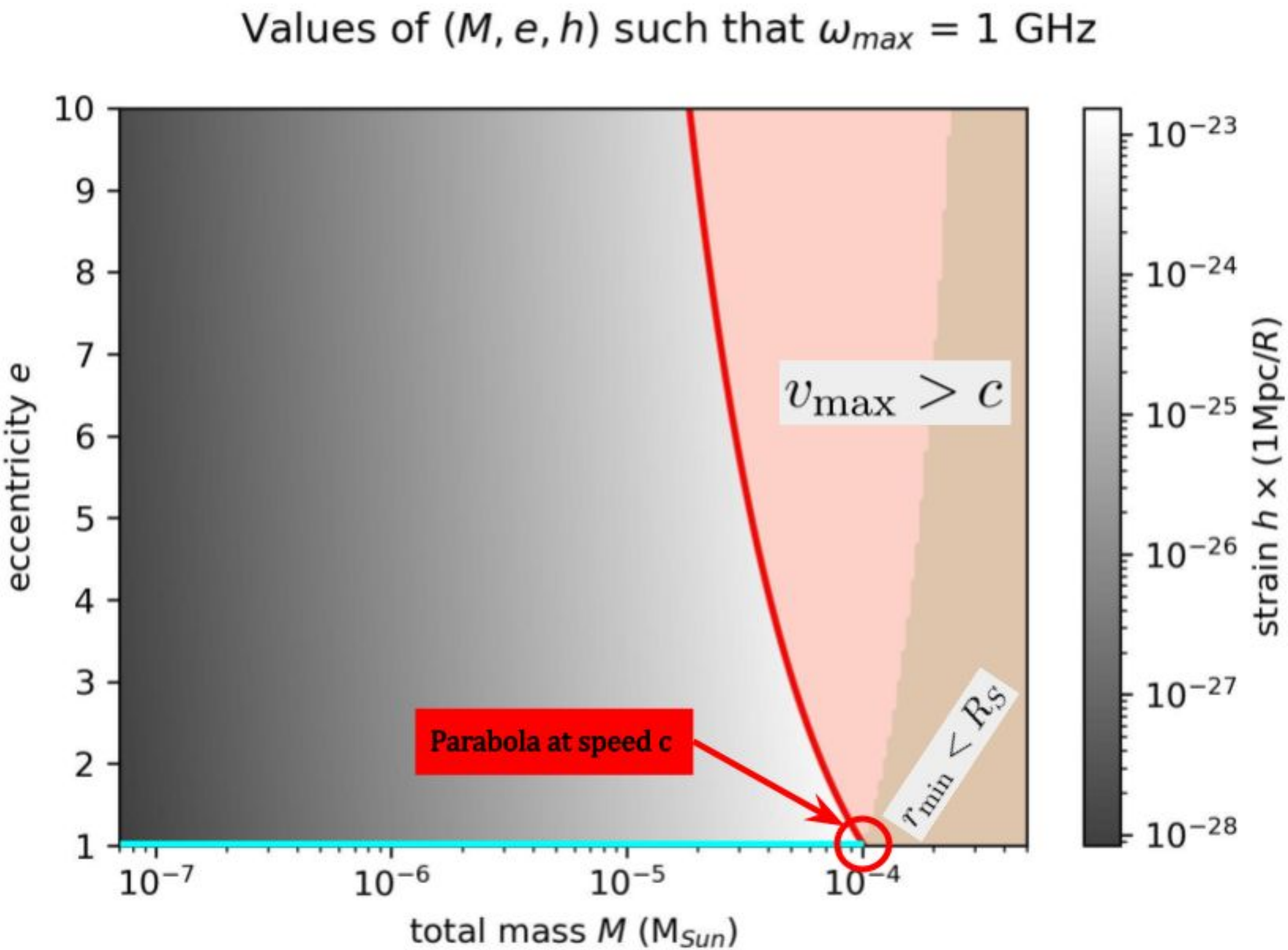

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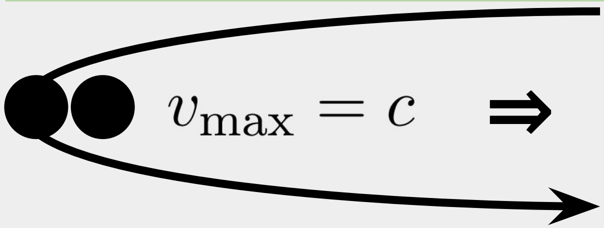

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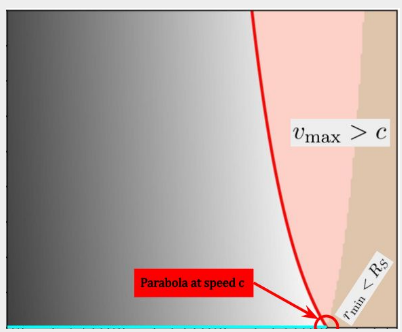

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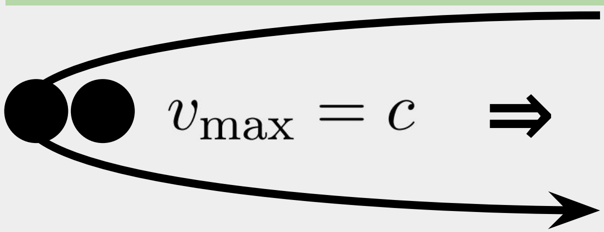

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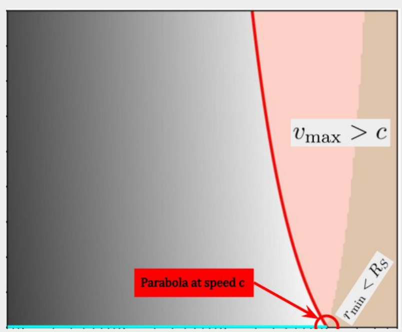

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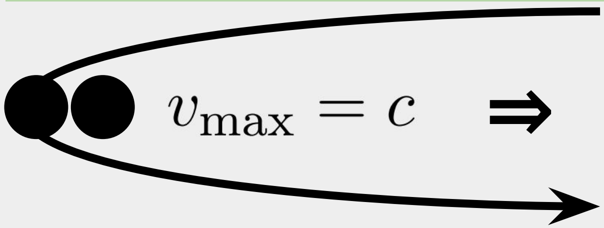

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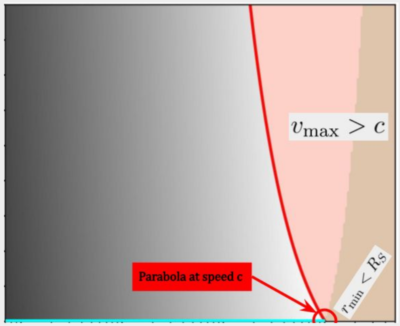

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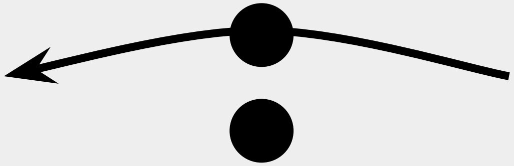
Drift of best scenario to higher e !



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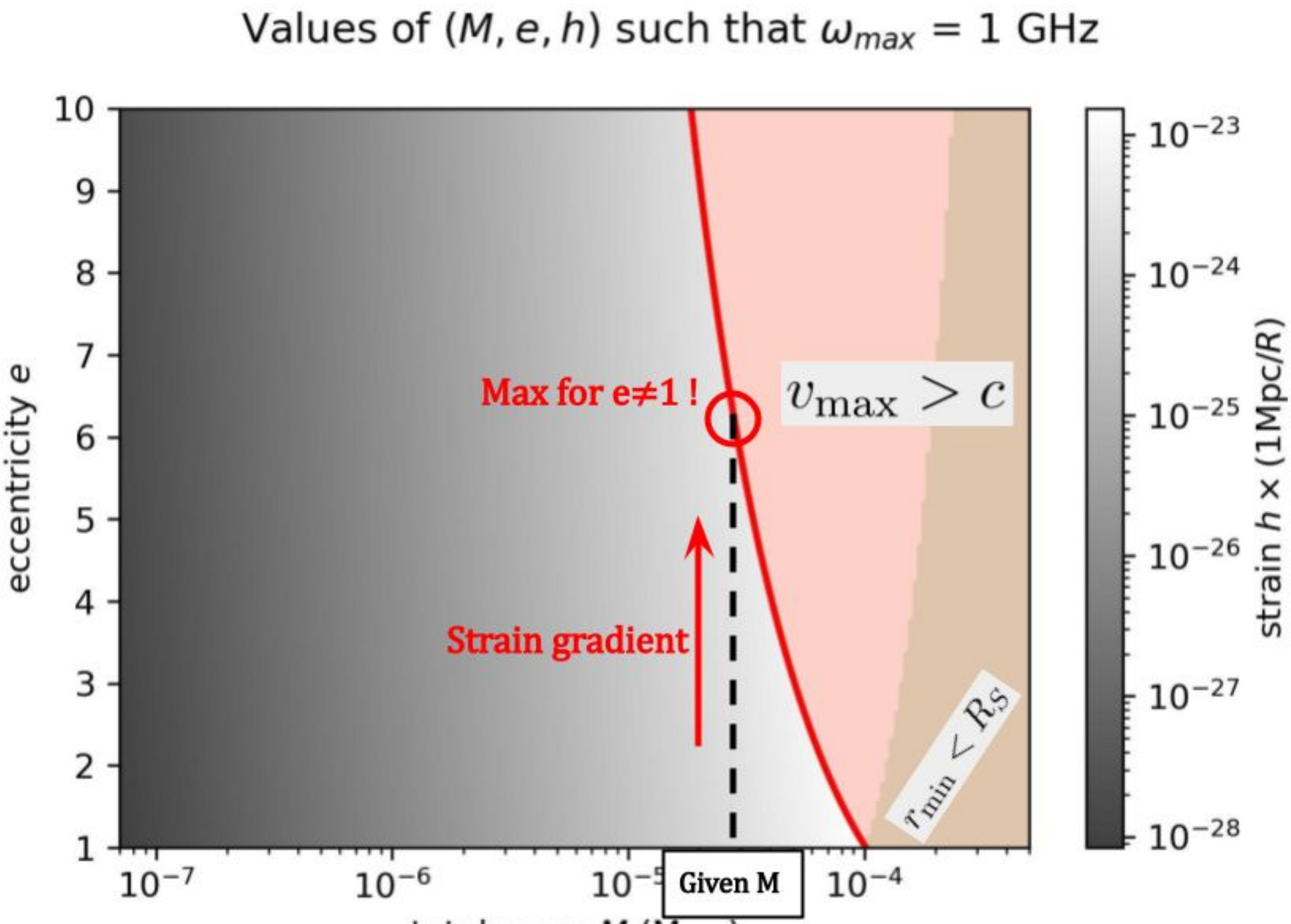

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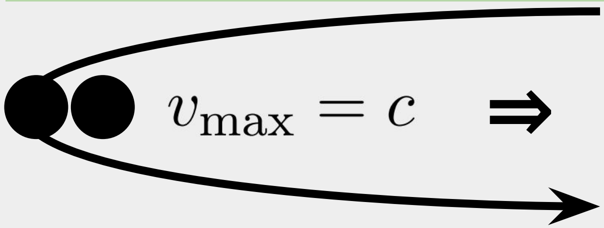

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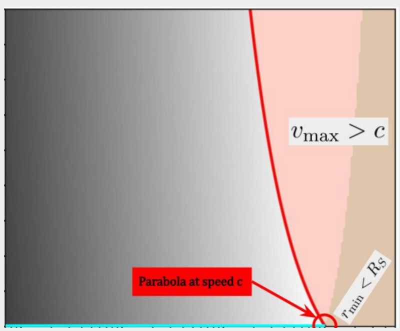

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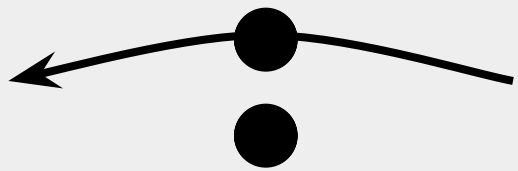

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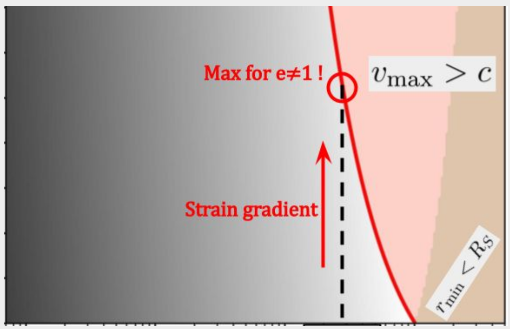

$$h_{\max}(R,f,M) \underset{M\ll10^{-5}M_\odot\,GHz/f}{\approx} \frac{\sqrt{5}}{R}\frac{\mathcal{G}M}{c^2}$$

$$= 1,1\times10^{-22}\left(\frac{1\ \mathrm{kpc}}{R}\right)\left(\frac{M}{10^{-6}\ \mathrm{M}_\odot}\right)$$

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Haloscopes signal-to-noise ratio (SNR)

[c.f. Killian's talk]: Event duration must be carefully assessed for proper sensitivity reckoning

Haloscopes: $\boldsymbol{f}$ ~ a few **GHz**, quality factor $\boldsymbol{Q \sim f/\Delta f \sim 10^5}$

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(newly derived for hyperbolas)

$$t_{\Delta f} \approx \frac{1}{\pi f}\sqrt{1+\frac{1}{e}\sqrt{\frac{\Delta f}{f}}}$$

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horribly small!

Taking best scenario $h_{\max} = \frac{1}{R}\frac{\sqrt{41}}{8\pi}\frac{c}{f}$:

$$\mathrm{SNR} \geqslant 1 \implies R \leqslant 60\ \mathrm{A.U.}\left(\frac{\text{experimental vars.}}{\text{typical values}}\right)$$

Outline

Summary

Take-away message:

Great strain can be generated on **unbounded** and **highly eccentric** orbits
(comparable to inspirals!)

Counter-intuitive behavior at **fixed frequency**:
best scenario is a parabola yet strain grows with *e* !

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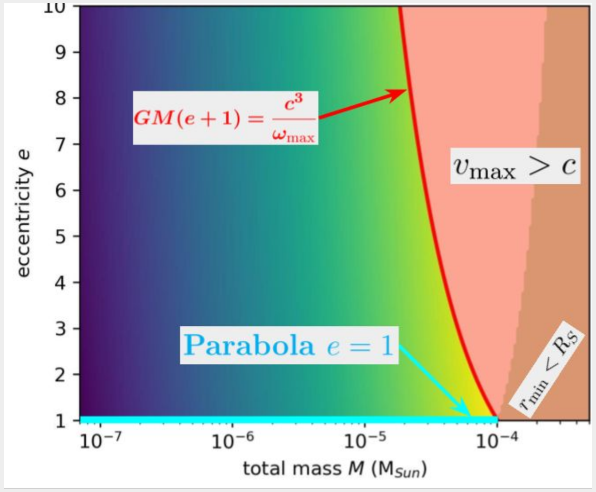
Clarified situation on hyperbolic trajectories:



Reusable results for others!

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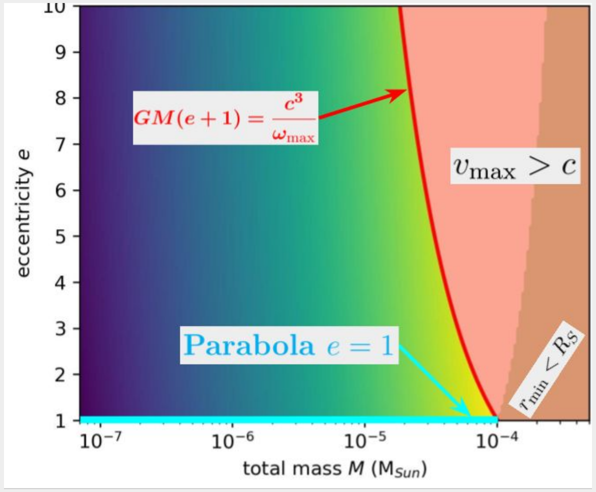


Reusable results for others!

Major drawback: **signal-to-noise** ratios **for high-frequency** detectors are **terrible**
Signal lasts way too short at these frequencies

Left to do: reckoning the number of events if we relax ***f***

Fiducial values at LVK frequencies

$$M \leqslant \frac{c^3}{2\pi\mathcal{G} f(e+1)} \leqslant \boxed{2\times 10^3\ \mathrm{M}_\odot \left(\frac{10\ \mathrm{Hz}}{f}\right)}$$

$$h_{\max}(R,f) = \frac{1}{R}\frac{\sqrt{41}}{8\pi}\frac{c}{f}$$

$$= 2,5\times 10^{-19}\left(\frac{1\ \mathrm{Gpc}}{R}\right)\left(\frac{10\ \mathrm{Hz}}{f}\right)$$

$$h_{\max}(R,f,M) \underset{M\ll 10^3 M_\odot\, 10Hz/f}{\approx} \frac{\sqrt{5}}{R}\frac{\mathcal{G}M}{c^2}$$

$$= 1,1\times 10^{-20}\left(\frac{1\ \mathrm{Gpc}}{R}\right)\left(\frac{M}{10^2\ \mathrm{M}_\odot}\right)$$

$$t_{\Delta f} \sim 10^{-4}\ \mathrm{s}\left(\frac{1\ \mathrm{Hz}}{f}\right)\left(\frac{10^5}{f/\Delta f}\right)^{1/2}\sqrt{1+\frac{1}{e}}$$

Removing divergences in formulae

VI.2 Fixing divergencies

The problem with parabolic orbits is that $e \to 0$, but $v_\infty \to 0$ as well, while $b \to \infty$... All these parameters are not independent and mix in the equations, so it's not obvious to see how to obtain the parabolic limit from the hyperbolic formulae. Here I make two claims on how to do it, which turn out to be equivalent:

1. Claim 1: the starting point of computations is the conic equation $r(\phi) = \frac{p}{1+e\cos(\phi-\phi_0)}$ where p is the semi-latus rectum. Here there is no problem when $e = 1$, but in calculations we start replacing e.g. $p = -a(e^2-1) = -\dfrac{b^2}{a} = \dfrac{\kappa}{v_\infty^2}(e^2-1)$ and now divergences start appearing: $e^2 - 1 \to 0$ but $v_\infty \to 0$ as well, also $a, b \to \infty$... Hence the claim: **to avoid divergences, the limit shall be taken such that p remains fixed.** Assuming the masses are given at the start (they have nothing to do with geometrical considerations). This limit means that

$$\frac{v_\infty^2}{e^2-1} \tag{40}$$

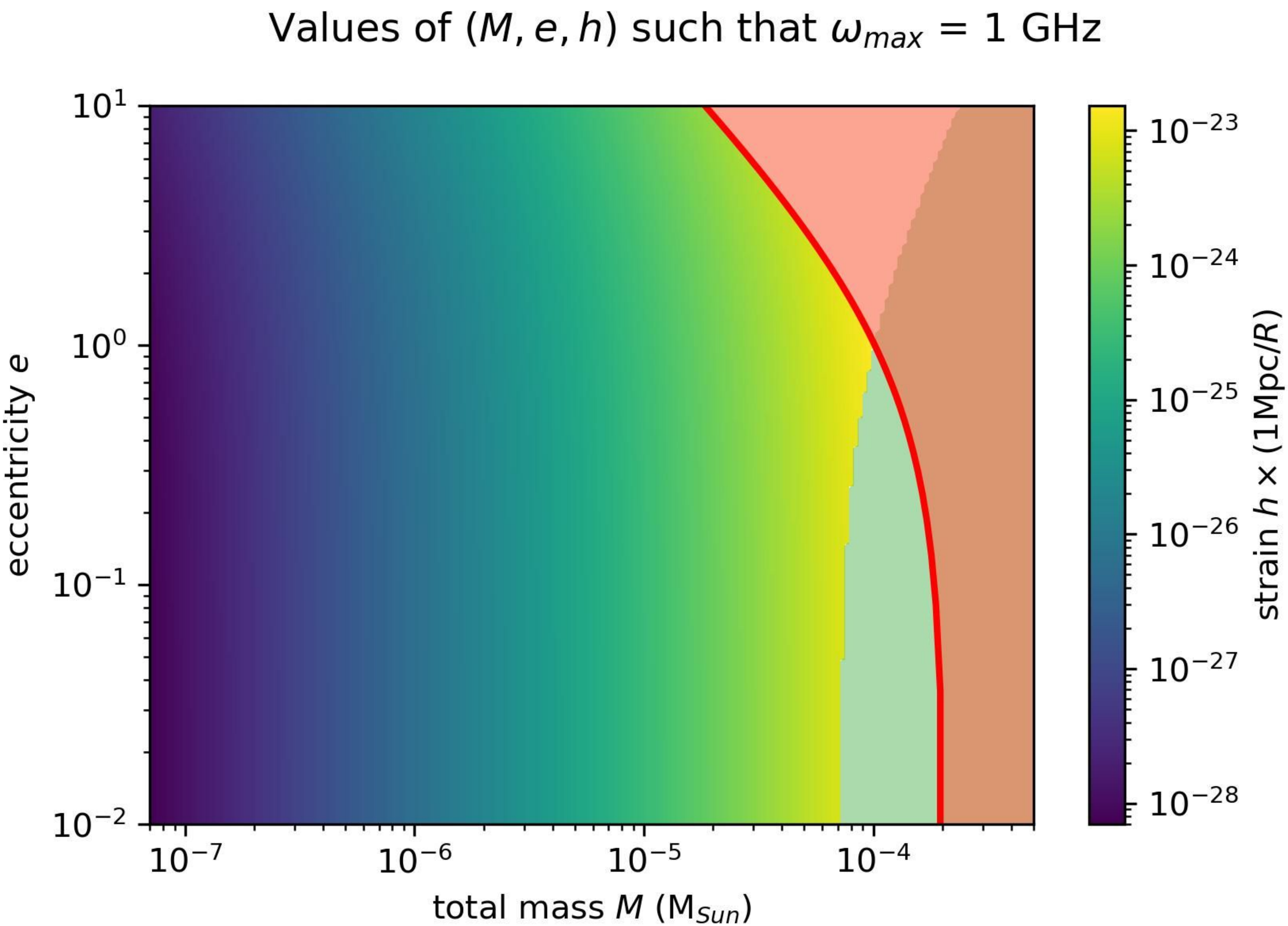
must remain a constant quantity, more precisely

$$\frac{v_\infty^2}{e-1} \underset{e\to 1}{\sim} \frac{\kappa(e+1)}{p} = \frac{2\kappa}{p} \tag{41}$$

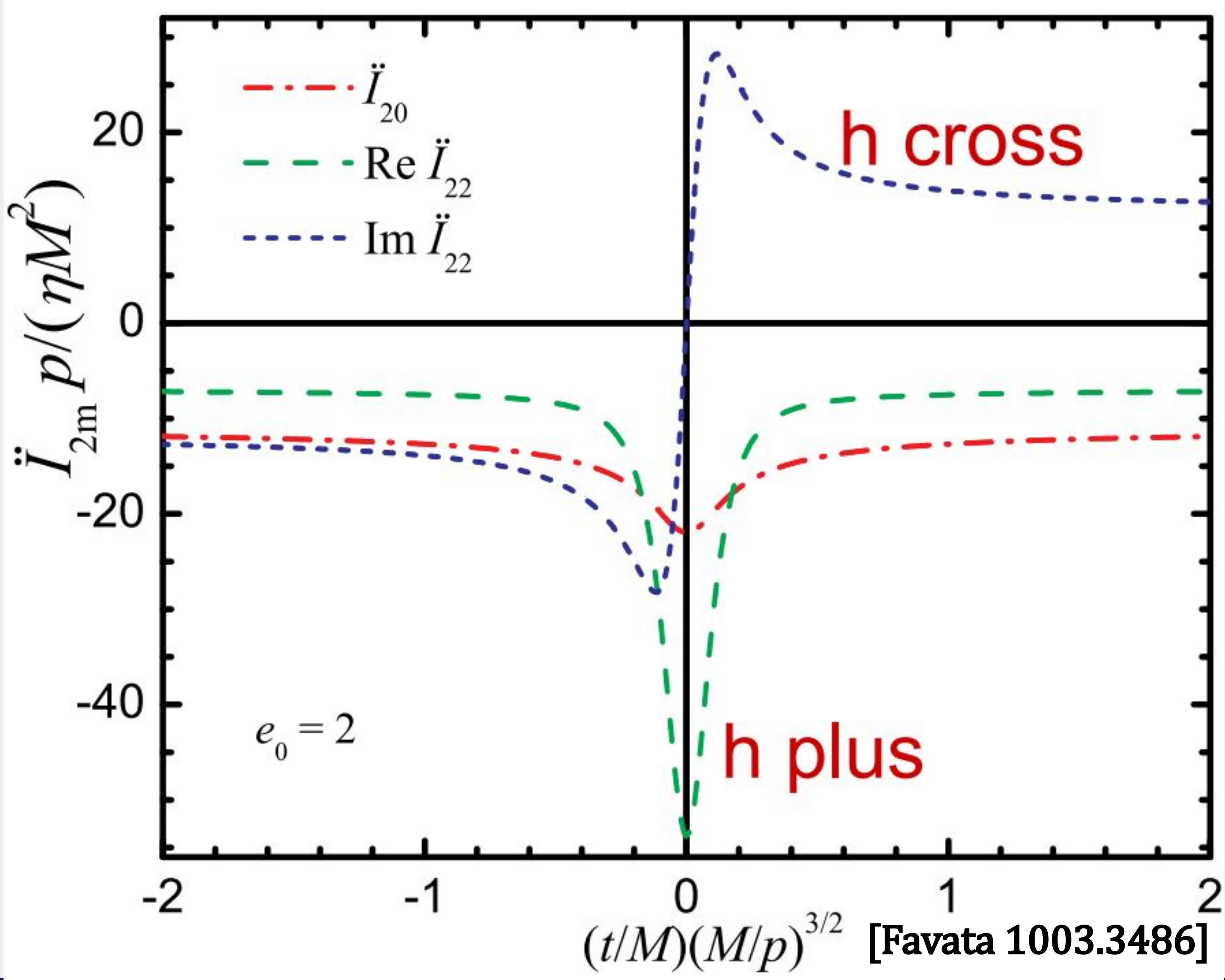
$$h_{c,\max} = \frac{4G\mu v_\infty^2}{Rc^4(e-1)}\sqrt{18(e+1)+5e^2}$$

$$P = \frac{dE}{dt} = -\frac{G}{45c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle = \frac{32G\mu^2 v_0^6}{45c^5\, b^2}\frac{9(e+1)^2}{(e-1)^4}$$

More on the physical constraints



+ and ✕ polariza° & linear memory effect



[Favata 1003.3486]

General suboptimal strain

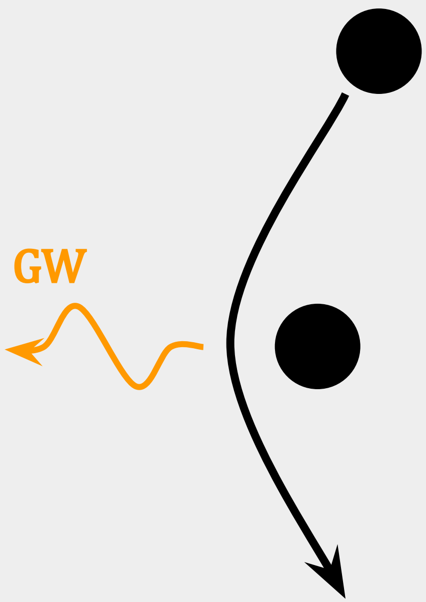
$$h_{c,p}^{\mathrm{MAX}}(R, \omega_p, \kappa) = \frac{\sqrt{5}}{R} \frac{\kappa}{c^2} \sqrt{1 + \frac{8}{5} \frac{\kappa \omega_p}{c^3} + \left(\frac{\kappa \omega_p}{c^3} \right)^2}$$

$$h_{c,p}^{\mathrm{MAX}}(R, \omega_p, \kappa) \approx \frac{\sqrt{5}}{R} \frac{\kappa}{c^2} \tag{32}$$

$$= 1,1 \times 10^{-22} \left(\frac{1\ \mathrm{Mpc}}{R} \right) \left(\frac{M}{10^{-3}\ \mathrm{M}_\odot} \right) . \tag{33}$$

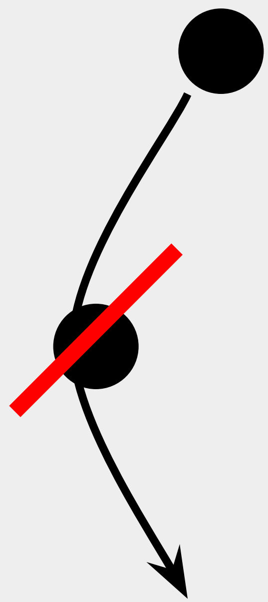
[OLD] Maximization under constraints

Constraint 1:
fixed ω_{max}



$v_{max} / r_{min} =$ const.

Constraint 2:
no merger



$r_{min} > R_S = 2GM/c^2$

Constraint 3:
do not exceed c

$v_{max} < c$

⇒ 2D parameter space with boundaries ; spanned with *M* and *e*