Ruijsenaars wavefunctions as $SL(2,\mathbb{Z})$ matrix coeffecients

(joint with P. Di Francesco, R. Kedem, S. Khoroshkin, G. Schrader)

Goal: find Ruijsenaars wavefunctions, aka Macdonald eigenfunctions

Tools: $SL(2,\mathbb{Z})$-action & quantum cluster varieties

Starring: quantum open Toda chain

$$G = SL_n \quad (\& \ n=2)$$

① Spherical DAHA

$\mathbb{SH}_{q,t}(SL_n)$ — spherical DAHA (double affine Hecke algebra)

$\mathbb{SH}_{q,t}$ "lives" on a $T^2 \setminus D^2$ ← punctured torus

a) $\forall\, v \in H_1(T^2 \setminus D^2) \simeq \mathbb{Z}^2 \quad \exists\, E_v \in \mathbb{SH}_{q,t}$

b) $SL(2,\mathbb{Z}) \circlearrowright \mathbb{SH}_{q,t} \qquad g \circ E_v = E_{gv}$

c) $\mathbb{SH}_{q,t} \simeq \mathbb{C}(q,t)\langle E_{k,0},\, E_{0,k} \rangle_{1 \le k \le n}$

$SL(2,\mathbb{Z})$ — mapping class group of $T^2 \setminus D^2$

Cherednik's representation:

$$S\mathbb{H}_{q,t} \hookrightarrow \mathbb{C}(q,t)\langle \vec{\Lambda}, \vec{T} \rangle \big/ \; T_i \Lambda_j = q^{\delta_{ij}} \Lambda_j T_i$$

$$E_{k,0} \longmapsto e_k(\vec{\Lambda})\Big|_{\Lambda_1 \cdots \Lambda_n = 1} \quad \longleftarrow \text{elementary symmetric function}$$

$$E_{0,k} \longmapsto M_k(\vec{\Lambda}, \vec{T}) \quad \longleftarrow \text{Macdonald operator}$$

Example: $n = 2$

$$E_{1,0} \longmapsto \Lambda + \Lambda^{-1}$$

$$E_{0,1} \longmapsto \frac{t\Lambda - t^{-1}\Lambda^{-1}}{\Lambda - \Lambda^{-1}} T + \frac{t^{-1}\Lambda - t\Lambda^{-1}}{\Lambda^{-1} - \Lambda} T^{-1}$$

② Strategy

1) Find $(\mathcal{H},\rho)$ — $SL(2,\mathbb{Z})$-equivariant Hilbert space rep-n of $\mathbb{S}\mathbb{H}_{q,t}$

2) $\prod_j \delta(\Lambda_j - X_j)$ — eigenfunctions of $E_{k,0}$:

$$e_k(\vec{\Lambda}) \prod_j \delta(\Lambda_j - X_j) = e_k(\vec{X}) \prod_j \delta(\Lambda_j - X_j)$$

eigenvalue

3) $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow \rho(E_{0,k}) = \rho(S \circ E_{k,0}) = \rho(S)\,\rho(E_{k,0})\,\rho(S)^{-1}$$

$$\Rightarrow \rho(S) \circ \prod_j \delta(\Lambda_j - X_j) \text{ — Ruijsenaars wavefunction}$$

③ Character varieties

S — surface w/ punctures
G — group

decorated local system:
1) $\varphi \in \mathrm{Hom}(\pi_1(S), G)$
2) $\forall p$ — puncture F_p — flag, inv-t under monodromy around p.

$\underline{\mathrm{Loc}}_{G,S}$ — space of decorated local systems

$P_{G,S} := \underline{\mathrm{Loc}}^{\circ}_{G,S}/G$

decorated character variety

open subspace: everything in generic position

$\mathcal{O}_q(\mathcal{P}_{G,S})$ — quantised algebra of f-s

Γ_S — mapping class group of S, acts on $\mathcal{O}_q(\mathcal{P}_{G,S})$

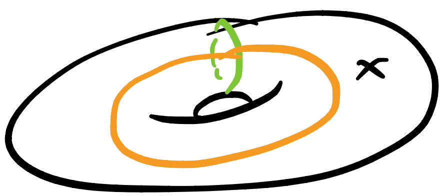
Now, $S = T^2 \setminus D^2$ ← punctured torus

$\mathcal{P}^{RS}_{G,S} := \mathcal{P}_{G,S}$ s.t. monodromy around D^2 is conjugate to $\begin{pmatrix} \lambda & & & 0 \\ & \ddots & & \\ & & \lambda & \\ 0 & & & \lambda^{1-n} \end{pmatrix}$

<u>**Note:**</u> $\mathcal{P}^{RS}_{G,S} = \mathcal{P}_{G,S}$ for $n=2$

<u>**Fact:**</u> $\mathbb{SH}_{q,t} \simeq \mathcal{O}_q(\mathcal{P}^{RS}_{G,S}) \circlearrowleft \Gamma_S = SL_2(\mathbb{Z})$

(Lost in history)



$tr_{\Lambda^k \mathbb{C}^n}(\varphi(\bigcirc)) = E_{k,0}$

$tr_{\Lambda^k \mathbb{C}^n}(\varphi(\bigcirc)) = E_{0,k}$

④ Quantum cluster varieties

$\mathcal{X}$ – cluster Poisson variety

$\mathbb{L}^q := \mathcal{O}_q(\mathcal{X})$ – quantum cluster variety

$\exists$ a Γ-equivariant Hilbert space rep-n of $\mathbb{L}^q$

"cluster modular group"

Upshot: $\mathcal{P}_{G,S}$ has cluster structure

$\Rightarrow \exists$ a Γ_S-equivariant Hilbert space rep-n of $\mathbb{L}^q_{G,S} := \mathcal{O}_q(\mathcal{P}_{G,S})$

Same for $\mathcal{P}^{RS}_{G,S}$ & $\mathbb{L}^q_{RS}$ $(= \mathbb{L}^q_{SL_2, S^2 \setminus 0^2}$ if $n=2)$

⑤ Embedding $SH_{q,t} \hookrightarrow \mathbb{L}^q_{RS}$

1) $SH_{q,t}$ is gen-d by $E_{k,0}$, $E_{0,k}$

$\exists$ faithful rep-n

$E_{k,0} \longmapsto e_k(\vec{\Lambda}) \longleftarrow$ elementary symmetric function

$E_{0,k} \longmapsto M_k(\vec{\Lambda}, \vec{D}) \longleftarrow$ Macdonald operator

2) $\mathbb{L}^q_{RS}$ contains $L_{k,0}$, $L_{0,k}$

$\exists$ faithful rep-n

$L_{k,0} \longmapsto H_k(\vec{x}, \vec{p}) \longleftarrow$ Toda Hamiltonian

$L_{0,k} \longmapsto \hat{L}_{0,k}(\vec{x}, \vec{p}, t)$

<u>Note:</u> $e_k(\vec{\Lambda})$ — eigenvalues of H_k

$\Psi_{\vec{\lambda}}(\vec{x})$ – joint eigenfunctions for H_k

Whittaker function

$$H_k \Psi_{\vec{\lambda}}(\vec{x}) = e_k(\vec{\Lambda}) \Psi_{\vec{\lambda}}(\vec{x}) \qquad \vec{\Lambda}_j = e^{\lambda_j}$$

<u>Thm:</u> (Kashaev, Schrader – S.)

Sklyanin measure

$$W : L^2(\mathbb{R}^n, d\vec{x}) \xrightarrow{\sim} L^2_{sym}(\mathbb{R}^n, m(\vec{\lambda})d\vec{\lambda})$$

$$f(\vec{x}) \longmapsto \int f(\vec{x}) \overline{\Psi_{\vec{\lambda}}(\vec{x})} \, d\vec{x}$$

is a unitary equivalence

<u>Lemma:</u> $\hat{L}_{0,k}(\vec{X}, \vec{P}, t) \Psi_{\vec{\lambda}}(\vec{x}) = M_k(\vec{\Lambda}, \vec{D}, t) \Psi_{\vec{\lambda}}(\vec{x})$

Thm: ($n=2$: folklore; $n>2$: not yet written, but Di Francesko, Kedem, Schrader, S.)

$\exists$ $SL(2,\mathbb{Z})$-equivariant injective homomorphism $\iota : \mathbb{SH}_{q,t} \hookrightarrow \mathbb{L}^q_{RS}$ defined by

$$E_{k,0} \longmapsto L_{k,0}, \qquad E_{0,k} \longmapsto L_{0,k}$$

Idea of proof:

$$\begin{tikzcd}
\mathbb{SH}_{q,t} \arrow[d, dashed] & \circlearrowright & L^2_{sym}(\mathbb{R}^n, m(\bar{x})d\bar{x}) \\
\mathbb{L}^q_{RS} & \circlearrowright & L^2(\mathbb{R}^n, d\bar{x}) \arrow[u, "\wr", "W"']
\end{tikzcd}$$

Both faithful

⑥ Ruijsenaars wave functions $(n=2)$

$$
\begin{tikzcd}
\left\{\begin{array}{l} E_{1,0} = \Lambda + \Lambda^{-1} \\ E_{0,1} = M_1 \end{array}\right\} \circlearrowright L^2_{sym}(\mathbb{R}, d\lambda) & L^2(\mathbb{R}, dx) \circlearrowleft \left\{\begin{array}{l} \hat{L}_{1,0} = H_1 \\ \hat{L}_{0,1} \end{array}\right\} \arrow[l, "W", "\sim"'] \\
\left\{\begin{array}{l} E_{0,-1} \\ E_{1,0} = e^{\mu} + e^{-\mu} \end{array}\right\} \circlearrowright L^2_{sym}(\mathbb{R}, d\mu) \arrow[u, "\wr\, W \circ S \circ W^{-1}"'] & L^2(\mathbb{R}, dx) \circlearrowleft \left\{\begin{array}{l} \hat{L}_{0,-1} \\ \hat{L}_{1,0} = H_1 \end{array}\right\} \arrow[l, "W", "\sim"'] \arrow[u, "\wr\, S(t)"']
\end{tikzcd}
$$

$$\Rightarrow \Phi^{\tau}_{\mu}(\lambda) := \int \overline{\Psi_\lambda(x)}\, S\, \Psi_z(x)\, \delta(\mu - z)\, dz\, dx$$

$$= \langle \Psi_\lambda, S \Psi_\mu \rangle \qquad\qquad e^{\tau} = q^{-\frac{1}{2}} t$$

is the Ruijsenaars wave function:

$$M_1(\Lambda, D)\, \Phi^{\tau}_{\mu}(\lambda) = (e^{\mu} + e^{-\mu})\, \Phi^{\tau}_{\mu}(\lambda)$$

Corollary: 1) $\Phi_\mu^{\tilde\tau}(\lambda) = \Phi_{-\mu}^{\tilde\tau}(\lambda) = \Phi_\mu^{\tilde\tau}(-\lambda)$

2) $\Phi_\mu^{\tilde\tau}(\lambda) = \Phi_\lambda^{-\tilde\tau}(\mu)$

Proof: 2) $\langle \Psi_\lambda, S\Psi_\mu \rangle = \langle S\Psi_\lambda, \Psi_\mu \rangle = \overline{\langle \Psi_\lambda, S\Psi_\mu \rangle}$

$= \overline{\langle \Psi_\mu, S\Psi_\lambda \rangle} = \Phi_\mu^{\tilde\tau}(\lambda) = \Phi_\lambda^{-\tilde\tau}(\mu)$

$\langle \Psi_\lambda, S\Psi_\mu \rangle$ recovers formulas in

[Kallnäs, Ruijsenaars]

[Belousov, Derkachov, Kharchev, Khoroshkin]

Macdonald polynomials $=$ Φ for special μ via residue count

⑦ Conclusion:

1) $SL(2,\mathbb{Z})$ – equiv. Hilb. space rep-n of $\mathbb{SH}_{q,t}$ $\rightsquigarrow$ Ruijsenaars wavefunctions
$\rightsquigarrow$ Macdonald polynomials

2) Quantum cluster varieties provide such rep-s

3) Same for other "RLL-algebras", e.g. $U_q(\mathfrak{g})$, $\mathcal{O}_q(G)$, Askey-Wilson

Slogan: Macdonald theory via Toda chain & clusters


Joyeux anniversaire, camarade stakhanoviste!