Non-unitary fermions and extended symmetry

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At the crossroads of physics and mathematics :
the joy of integrable combinatorics
A conference in the honour of Philippe Di Francesco's 60th birthday
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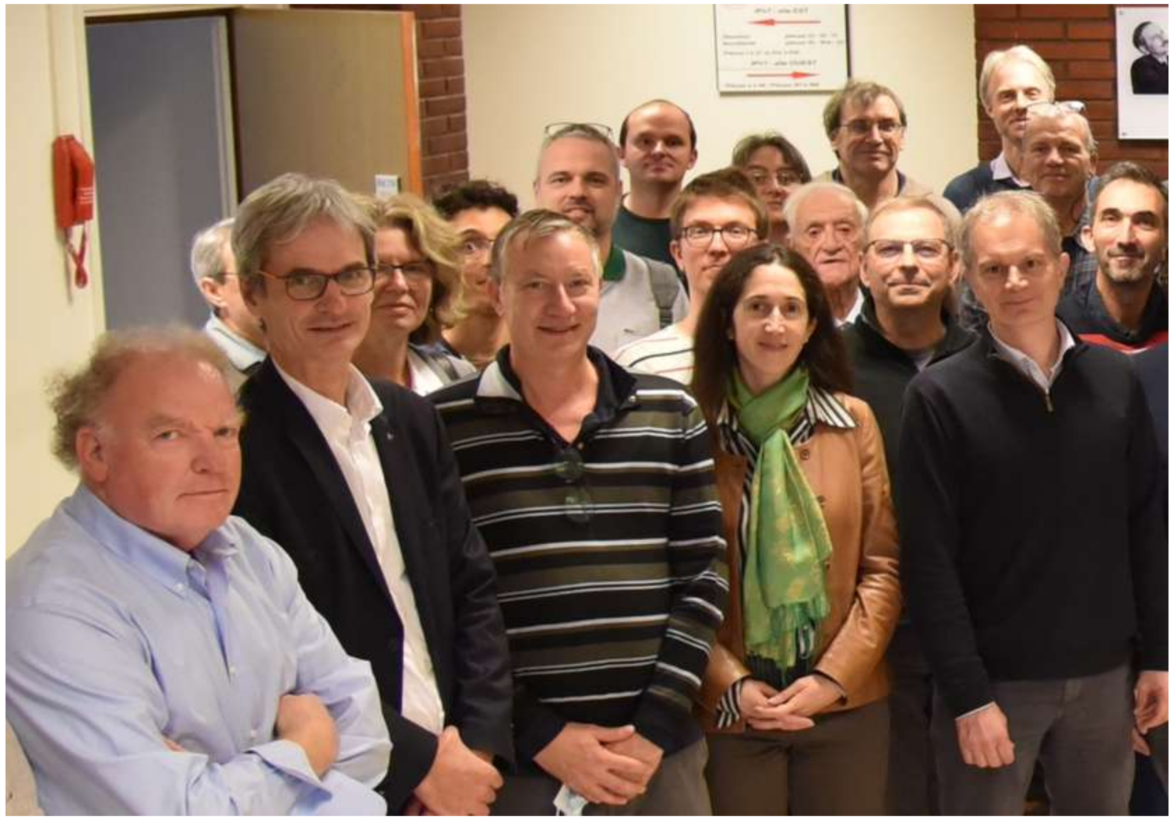


IPhT@60

IGST 2017

The model

- **long-range** deformation of the XX model
- particular point of a **q-deformation** of the Haldane-Shastry spin chain
- naturally expressed in terms of **Temperley-Lieb** algebra and free fermions
- no regular translational invariance ⟶ **quasi-translation invariance**
- the boundary conditions render the fermions **non-unitary**
- the **even and odd length** chains have **radically different** properties

Non-unitary fermions and gl(1|1)

- consider a 1-dimensional lattice with N sites and the **fermionic degrees of freedom**

$$\{f_i, f_j^+\} = (-1)^i\,\delta_{ij}\,, \quad \{f_i, f_j\} = \{f_i^+, f_j^+\} = 0$$

$$f_j = (-\mathrm{i})^j\,c_j,\ f_j^+ = (-\mathrm{i})^j\,c_j^\dagger$$

- they generate a **global gl(1|1) algebra** with anti-commutation relations

$$\left[\mathrm{N}, \mathrm{F}_1\right] = -\mathrm{F}_1\,, \quad \left[\mathrm{N}, \mathrm{F}_1^+\right] = \mathrm{F}_1^+\,, \quad \left\{\mathrm{F}_1, \mathrm{F}_1^+\right\} = \mathrm{E}$$

$$\mathrm{F}_1 = \sum_{i=1}^{N} f_i\,, \quad \mathrm{F}_1^+ = \sum_{i=1}^{N} f_i^+\,, \qquad \mathrm{N} = \sum_{i=1}^{N} (-1)^i\, f_i^+ f_i\,, \quad \mathrm{E} = \sum_{i=1}^{N} (-1)^i = -1$$

- **gl(1|1)** (anti-)commutes with the **two-site operators**

$$g_i \equiv f_i + f_{i+1}\,, \quad g_i^+ = f_i^+ + f_{i+1}^+\,, \qquad 1 \leqslant i < N$$

Non-unitary fermions and Temperley-Lieb

- the **two-site operators** $g_i \equiv f_i + f_{i+1}\,, \quad g_i^+ = f_i^+ + f_{i+1}^+\,, \qquad 1 \leqslant i < N$

can be used to generate the free-fermion **Temperley-Lieb** algebra

$$e_i^2 = 0\,, \quad e_i\, e_{i\pm1}\, e_i = e_i\,, \quad [e_i, e_j] = 0 \text{ if } |i-j| > 1 \qquad\qquad e_i \equiv g_i^+ g_i$$

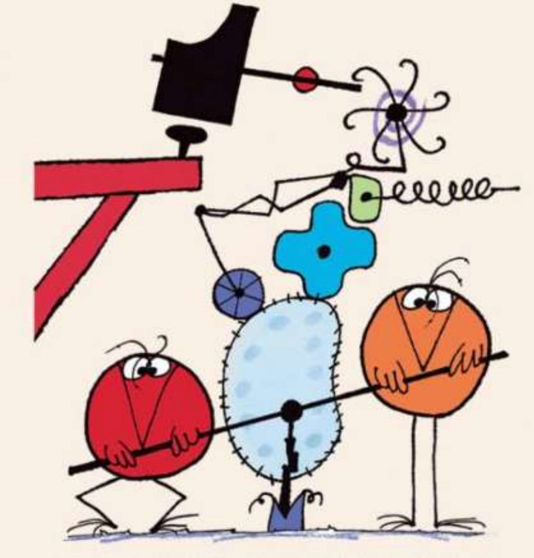
- **TL** generators commute with the global gl(1|1) plus $\mathrm{F}_2 = \sum_{i<j}^{N} f_i\, f_j\,, \quad \mathrm{F}_2^+ = \sum_{i<j}^{N} f_i^+\, f_j^+$

[Gainutdinov, Read, Saleur, 11]

↙

$$U_{\mathrm{q}}\,\mathfrak{sl}(2)|_{\mathrm{q}=\mathrm{i}}$$

The Hamiltonians at N odd



- the interaction can be defined in terms of **commutators of TL generators**

$$e_{[i,j]} \equiv [[\cdots[e_i, e_{i+1}], \cdots], e_j] \qquad\qquad e_{[i,i]} \equiv e_i$$

- two **chiral Hamiltonians** $\mathrm{H}^{\mathrm{L,R}} = \frac{\mathrm{i}}{2} \sum_{1\leqslant i\leqslant j<N} h_{ij}^{\mathrm{L,R}}\, e_{[i,j]}$

with explicit coefficients $h_{ij}^{\mathrm{R}} = (-1)^{j-i}\, h_{N-j,N-i}^{\mathrm{L}} = -h_{ij}^{\mathrm{L}}$ $\longrightarrow$ $\mathrm{H}^{\mathrm{L}} = -\mathrm{H}^{\mathrm{R}}$

- **one non-chiral Hamiltonian** $\mathrm{H} = -\frac{1}{4N} \sum_{1\leqslant i\leqslant j<k\leqslant l<N} \left(h_{ij;kl}^{\mathrm{L}} + h_{ij;kl}^{\mathrm{R}}\right) \left\{e_{[i,j]}, e_{[k,l]}\right\}$

- the **quasi-translation operator** $\mathrm{G} = (1 + t_{N-1}\, e_{N-1}) \cdots (1 + t_1\, e_1)\,, \quad \mathrm{G}^N = 1$ $\qquad t_k \equiv \tan\frac{\pi k}{N}$

$$\left[\mathrm{G}, \mathrm{H}^{\mathrm{L}}\right] = \left[\mathrm{G}, \mathrm{H}\right] = \left[\mathrm{H}^{\mathrm{L}}, \mathrm{H}\right] = 0$$

generalises the XXZ Hamiltonian $H_{\mathrm{XXZ}} = -\sum_{j=1}^{N} e_j$ at $\Delta = \frac{\mathrm{q} + \mathrm{q}^{-1}}{2} = 0$

Discrete symmetries

The model has real spectrum in spite of being non-unitary, due to **PT symmetry**

- **Parity P**: $\mathrm{P}(f_i) \;=\; f_{N+1-i}$

$$\mathrm{P}(\mathrm{H}^{\mathrm{L}}) = -\mathrm{H}^{\mathrm{L}}\,, \quad \mathrm{P}(\mathrm{H}) = \mathrm{H}\,, \quad \mathrm{P}(\mathrm{G}) = \mathrm{G}^{-1}$$

- **Time reversal T** (anti-linear): $\mathrm{T}(e_i) = e_i$

$$\mathrm{T}(\mathrm{H}^{\mathrm{L}}) = -\mathrm{H}^{\mathrm{L}}\,, \quad \mathrm{T}(\mathrm{H}) = \mathrm{H}\,, \quad \mathrm{T}(\mathrm{G}) = \mathrm{G}$$

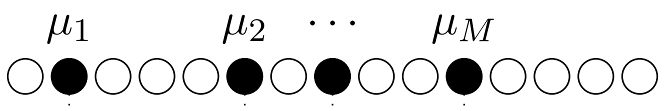
$$\mathrm{p} = -\mathrm{i}\log\mathrm{G}$$

- **Charge conjugation C** (anti-linear):

$$\mathrm{C}(\mathrm{H}^{\mathrm{L}}) = -\mathrm{H}^{\mathrm{L}}\,, \quad \mathrm{C}(\mathrm{H}) = \mathrm{H}\,, \quad \mathrm{C}(\mathrm{G}) = \mathrm{G}$$

The spectrum for N odd

The spectrum is given in terms of collections of integers $\{\mu_m\}$ reminiscent of the **Haldane-Shastry motifs**, with huge degeneracies **[Haldane, 88; Shastry, 88]**

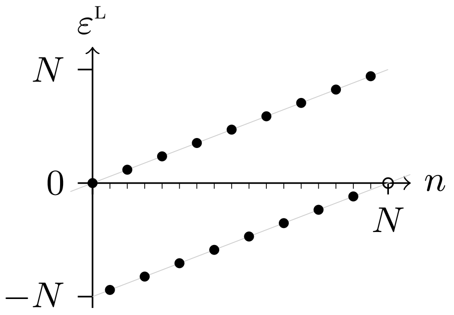


M magnon motif $\quad \mu_{m+1} > \mu_m + 1\,, \qquad 1 \leqslant m < M$

- quasi-momentum $\quad \mathrm{p} = -\mathrm{i}\log\mathrm{G} \qquad p = \frac{2\pi}{N}\sum_m \mu_m \bmod 2\pi$

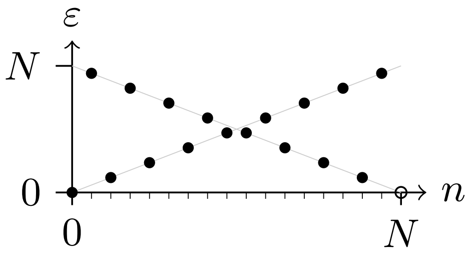
- chiral Hamiltonian

$$E^{\mathrm{L}}_{\{\mu_m\}} = \sum_{m=1}^{M} \varepsilon^{\mathrm{L}}_{\mu_m} \qquad \varepsilon^{\mathrm{L}}_n = \begin{cases} n\,, & n \text{ even}\,, \\ n - N\,, & n \text{ odd}\,, \end{cases}$$



- non-chiral Hamiltonian

$$E_{\{\mu_m\}} = \sum_{m=1}^{M} \varepsilon_{\mu_m} \qquad \varepsilon_n = |\varepsilon^{\mathrm{L}}_n|$$



How to solve the model

The model descends from a q-deformation of the **Haldane-Shastry** model with quantum affine symmetry (which explains the spectrum and the degeneracies)

As such, the highest weight wave functions are written in terms of particular **Macdonald polynomials**

[Bernard, Gaudin, Haldane, Pasquier, 93; Cherednik 92; Uglov 95; Lamers 18; Lamers, Pasquier, D.S., 22]

We want to solve it in terms of (non-unitary) fermions ⟶ **use quasi-translations**

- start with the first site and **translate the fermions** via

$$\Phi_i \equiv \mathrm{G}^{1-i}\, f_1\, \mathrm{G}^{i-1}\,, \qquad \Phi_i^+ \equiv \mathrm{G}^{1-i}\, f_1^+\, \mathrm{G}^{i-1}$$

- the transformation is **periodic** due to $\mathrm{G}^N = 1$

$$\Phi_{i+N} = \Phi_i\,, \qquad \Phi_{i+N}^+ = \Phi_i^+$$

- the price to pay is that the commutation relations are **non-local** (but **translationally invariant**)

$$\{\Phi_i, \Phi_j^+\} = -(1 + t_{j-i})\,, \quad \{\Phi_i, \Phi_j\} = \{\Phi_i^+, \Phi_j^+\} = 0$$

How to solve the model

- next we use the **Fourier modes** of the quasi-translated fermions

$$\tilde{\Psi}_n \equiv \frac{a_n}{N} \sum_{j=1}^{N} \mathrm{e}^{-2\mathrm{i}\pi nj/N}\, \Phi_j \,, \quad \tilde{\Psi}_n^+ \equiv \frac{a_n}{N} \sum_{j=1}^{N} \mathrm{e}^{2\mathrm{i}\pi nj/N}\, \Phi_j^+$$

$a_0 \equiv \mathrm{i}$ and $a_n \equiv \mathrm{i}^{n+1/2}$ else

- to get canonical commutation relations we **rescaled** the Fourier modes

$$\{\tilde{\Psi}_n, \tilde{\Psi}_m^+\} = \delta_{nm} \,, \quad \{\tilde{\Psi}_n, \tilde{\Psi}_m\} = \{\tilde{\Psi}_n^+, \tilde{\Psi}_m^+\} = 0$$

- the zero modes are **generators of the gl(1|1) algebra**

$$\frac{1}{a_0}\, \tilde{\Psi}_0 = \sum_{i=1}^{N} f_i = \mathrm{F}_1 \,, \quad \frac{1}{a_0}\, \tilde{\Psi}_0^+ = \sum_{i=1}^{N} f_i^+ = \mathrm{F}_1^+$$

- the other modes are **linear combinations of the two-site operators**

$$\frac{1}{a_n}\, \tilde{\Psi}_n = \sum_{i=1}^{N-1} M_{ni}\, g_i \,, \quad \frac{1}{a_n}\, \tilde{\Psi}_n^+ = \sum_{i=1}^{N-1} \bar{M}_{ni}\, g_i^+$$

$$g_i \equiv f_i + f_{i+1} \,, \quad g_i^+ = f_i^+ + f_{i+1}^+$$

How to solve the model

- in these variables **the chiral Hamiltonian** becomes purely quadratic

$$\mathrm{H}^{\mathrm{L}} = \sum_{n=1}^{N-1} \varepsilon^{\mathrm{L}}_{n}\, \tilde{\Psi}^{+}_{n}\, \tilde{\Psi}_{n}$$

- it can be diagonalised on the Fourier Fock space spanned by

$$|n_1, \ldots, n_M\rangle \equiv \tilde{\Psi}^{+}_{n_1} \ldots \tilde{\Psi}^{+}_{n_M} |\varnothing\rangle$$

$$0 \leqslant n_1 < \cdots < n_M < N$$

- compatibility with the motif rule thanks to

$$\varepsilon^{\mathrm{L}}_{n} + \varepsilon^{\mathrm{L}}_{n+1} = \varepsilon^{\mathrm{L}}_{2n+1 \bmod N}$$

- the **non-chiral Hamiltonian** is quartic

$$\mathrm{H} = \sum_{n=1}^{N-1} \varepsilon_n\, \tilde{\Psi}^{+}_{n}\, \tilde{\Psi}_{n} + \sum_{\substack{1 \leqslant m < n < N \\ 1 \leqslant r < s < N}} \tilde{V}_{mn;rs}\, \tilde{\Psi}^{+}_{m}\, \tilde{\Psi}^{+}_{n}\, \tilde{\Psi}_{r}\, \tilde{\Psi}_{s}$$

$$\tilde{V}_{mn;m+k,n-k} = (-1)^{k+1} 4\, \delta_{m\,\mathrm{odd}}$$

$$0 \leqslant 2k < n-m$$

eigenvalues

statistical repulsion

selection rule: $\varepsilon^{\mathrm{L}}_{m} + \varepsilon^{\mathrm{L}}_{n} = \varepsilon^{\mathrm{L}}_{r} + \varepsilon^{\mathrm{L}}_{s}$

To do list

- Study the system for **even length:** spectrum identically zero; Jordan blocks
- Identify the **extended symmetry:** gl(1|1) Yangian?
- Interpret the **staggering** & the **linear dispersion relations** in the odd case
- **CFT limit:** gl(1|1) Kac-Moody algebra?
- **Free field** realisation and **vertex operators algebra**
- **Wave functions** in the fermionic representation & **Macdonald polynomials**
- **Other roots of unity**: q^3=1 and gl(2|1) symmetry

Happy birthday Philippe!