Incidence geometry and tiled surfaces

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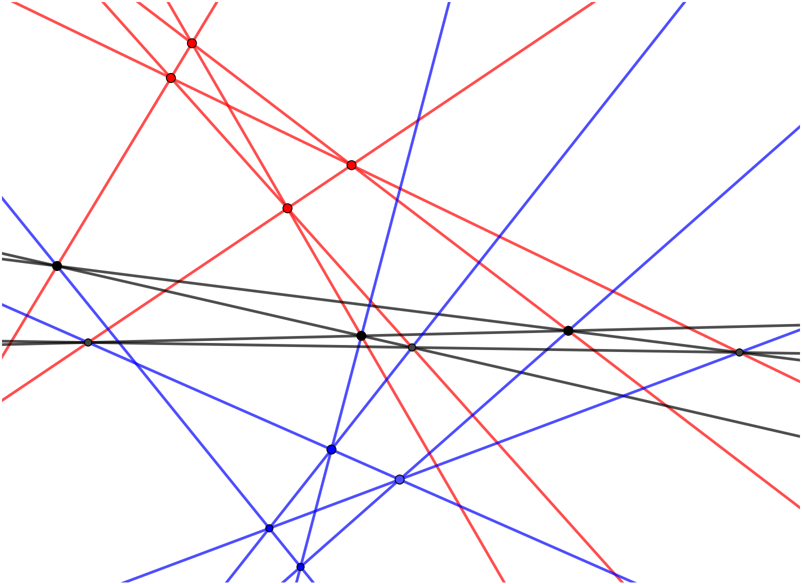
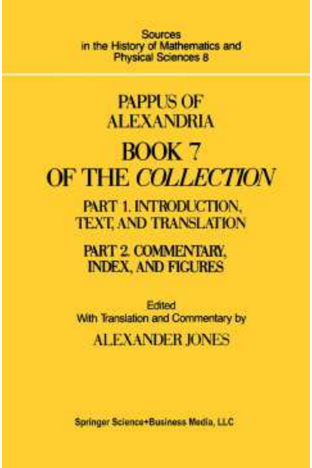
University of Michigan

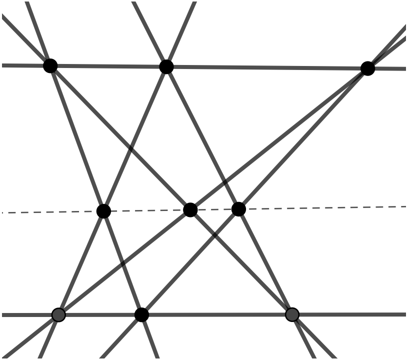
S. Fomin and P. Pylyavskyy, Incidences and tilings, `arXiv:2305.07728`

Linear incidence geometry

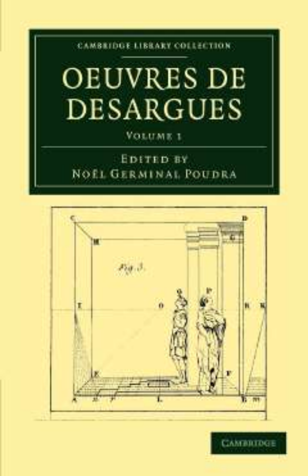


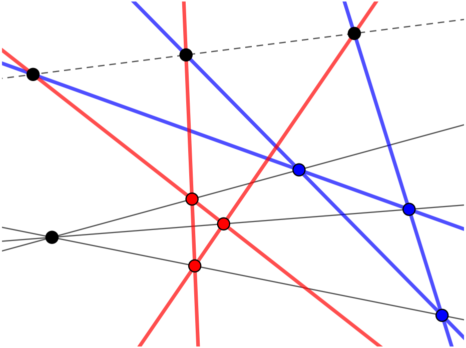
Pappus' Theorem [≈ 340]



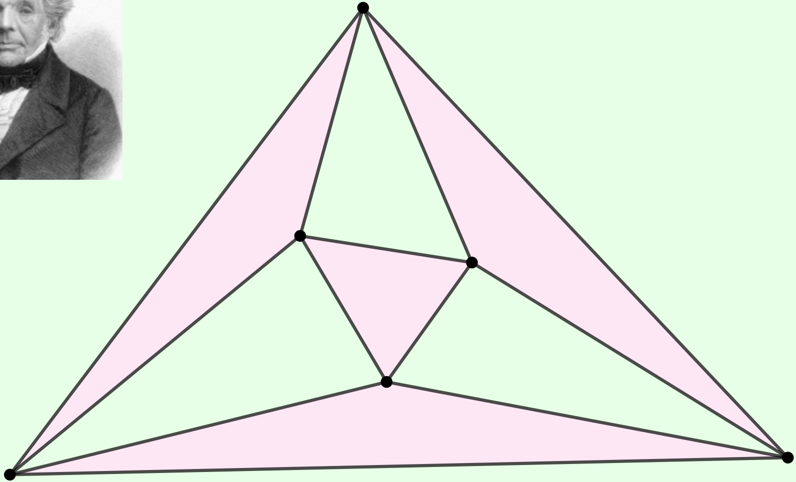


Desargues' Theorem [$\approx$1639]





Möbius' Theorem [1828]



Big questions

Many more incidence theorems have been discovered over the years. . .

- Is there a system behind these incidence theorems?
- What kinds of other mathematics are these theorems related to?

In this talk, I will try to answer these questions.

19^{th} century: The golden age of projective geometry

J.-V. Poncelet

M. Chasles

J. Steiner

K. von Staudt

Among the advances of the last fifty years in the field of geometry, the development of projective geometry occupies the first place.

– F. Klein, *The Erlangen Program*, 1872

Since the 19^{th} century, projective geometry has occupied a central position in geometric research. [. . .] The theorems relating to incidence are by far the most important theorems of projective geometry.

– D. Hilbert and S. Cohn-Vossen, 1932

The rise and fall of classical projective geometry

Projective geometry [...] had its heyday and then gradually faded away. All the more elementary results were worked out and incorporated into textbooks, and there wasn't any new work for mathematicians to do.

– P. A. M. Dirac, 1972

Classical projective geometry was a beautiful field of mathematics. It died, in our opinion, not because it ran out of theorems to prove, but because it lacked organizing principles by which to select theorems that were important. Also, it was isolated from the rest of mathematics.

– R. MacPherson and M. McConnell, 1988

The curse of universality

There is no hope for a reasonable classification of incidence theorems, even in the case of the projective plane. Indeed, the problem of deciding whether a given matroid is representable over $\mathbb{R}$ is NP-hard.

- N. E. Mnëv, Varieties of combinatorial types of projective configurations and convex polyhedra,
  Dokl. Akad. Nauk SSSR **283** (1985), 1312–1314.
- N. E. Mnëv, The universality theorems on the classification problem of configuration varieties and convex polytopes varieties,
  Lecture Notes in Math. **1346** (1988), 527–543.
- P. W. Shor, Stretchability of pseudolines is NP-hard,
  Applied geometry and discrete mathematics, 531–554, AMS, 1991.
- P. Vámos, The missing axiom of matroid theory is lost forever.
  J. London Math. Soc. (2) **18** (1978), 403–408.

Mechanical theorem proving

Algorithms of modern computational commutative algebra provide efficient automated proofs of theorems of linear incidence geometry.

Nowadays any incidence theorem of reasonable complexity can be proved by a computer, with minimal human input.

- H. Li and Y. Wu, Automated theorem proving in incidence geometry—a bracket algebra based elimination method, *Lecture Notes in Comput. Sci.* **2061** (2001), 199–227.
- J. Richter-Gebert, Mechanical theorem proving in projective geometry, *Ann. Math. Artificial Intelligence* **13** (1995), 139–172.
- B. Sturmfels, Computational algebraic geometry of projective configurations, *J. Symbolic Comput.* **11** (1991), 595–618.

Big questions revisited

Questions

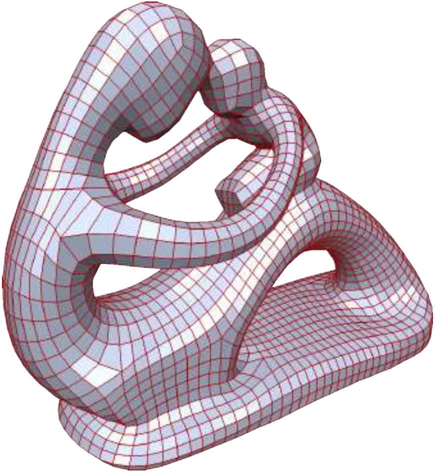
- Where do the classical incidence theorems come from?
- Which of these theorems are important—and why?
- What kind of other mathematics are they related to?

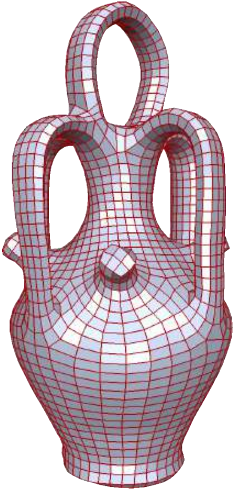
We attempt to answer these questions via a "master theorem" of real/complex linear incidence geometry, from which various—perhaps all—incidence theorems can be obtained as special cases.

As a result, we obtain a unifying perspective on why all these incidence theorems hold.

Tiled surfaces

We will work with tilings of oriented surfaces by quadrilateral tiles.





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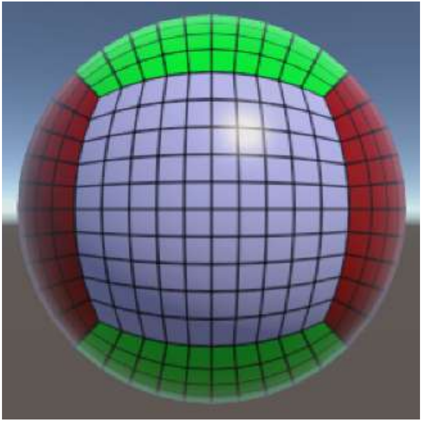
Tiled surfaces



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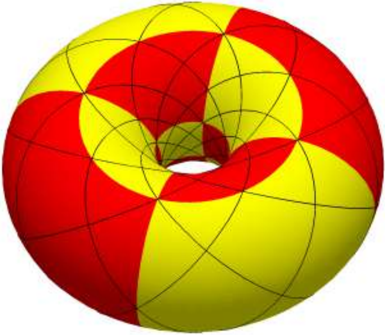

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Any such tiling (endowed with a bipartite labeling of its vertices) will give rise to an incidence theorem.

Coherent tiles

$\mathbb{P}$ = finite-dimensional projective space over $\mathbb{R}$ or $\mathbb{C}$

Definition

Throughout this talk, a tile is a topological quadrilateral

$$\begin{array}{ccc} A & — & \ell \\ | & & | \\ m & — & B \end{array}$$

with vertices labeled by points $A, B \in \mathbb{P}$ and hyperplanes $\ell, m \in \mathbb{P}^*$.

Such a tile is called coherent if

- neither A nor B is incident to either ℓ or m;
- either $A = B$ or $\ell = m$ or else the line (AB) and the codimension 2 subspace $\ell \cap m$ have a nonempty intersection.

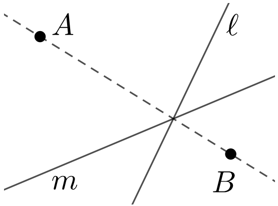
Tile coherence in the projective plane

Simplest case: $\dim \mathbb{P} = 2$ (a real/complex projective plane).

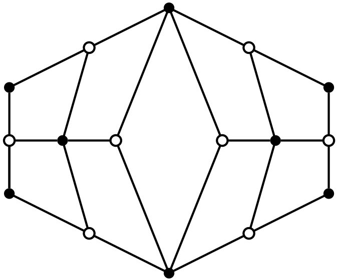
Let $A, B \in \mathbb{P}$ be two distinct points.
Let $\ell, m \subset \mathbb{P}$ be two distinct lines that do not pass through A or B.

The following are equivalent, by definition:

- the tile $\begin{array}{ccc} A & \text{—} & \ell \\ | & & | \\ m & \text{—} & B \end{array}$ is coherent;
- the lines $(AB), \ell, m$ are concurrent;
- the points $A, B, \ell \cap m$ are collinear.



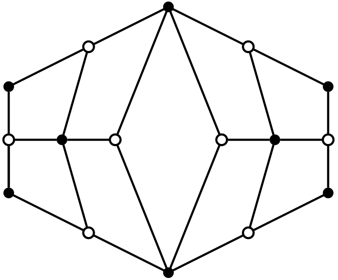
The master theorem of linear incidence geometry



"Master theorem"

Consider a tiling T of a closed oriented surface by quadrilateral tiles, with the vertices colored black and white in bipartite fashion. To each black (resp., white) vertex, associate a point (resp., a hyperplane) in $\mathbb{P}$, so that for each edge $\overset{A}{\bullet}\!\!-\!\!\!-\!\!\overset{h}{\circ}$, the point A does not lie on the hyperplane h. If all tiles of T, with the exception of one, are coherent, then the remaining tile is coherent as well.

Proof of the master theorem



$$\begin{matrix} A & — & \ell \\ | & & | \\ m & — & B \end{matrix}$$

$$(A,B;\ell,m) = \frac{\langle \mathbf{A},\boldsymbol{\ell}\rangle\langle \mathbf{B},\mathbf{m}\rangle}{\langle \mathbf{A},\mathbf{m}\rangle\langle \mathbf{B},\boldsymbol{\ell}\rangle}$$

Proof (sketch)

Coherence of a tile is equivalent to requiring that the corresponding mixed cross-ratio is equal to 1. The product of mixed cross-ratios over all tiles in the tiling is equal to 1. The claim follows.

Special cases of the master theorem

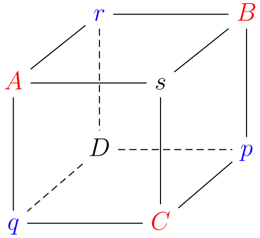
Theorem

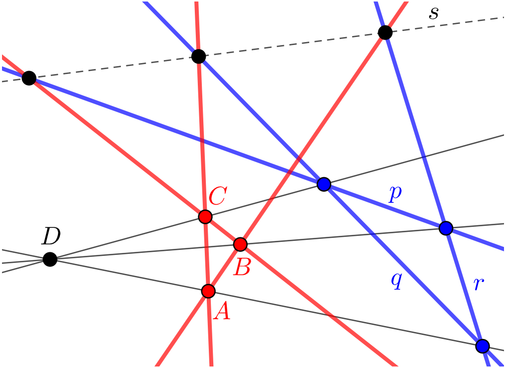
Each of the following results of linear incidence geometry can be interpreted as a special case of our "master theorem:"

- the Desargues theorem;
- the Pappus theorem;
- the complete quadrangle theorem;
- the permutation theorem;
- Saam's theorems;
- the Goodman-Pollack theorem;
- the bundle theorem;
- the sixteen points theorem;
- the Möbius theorem

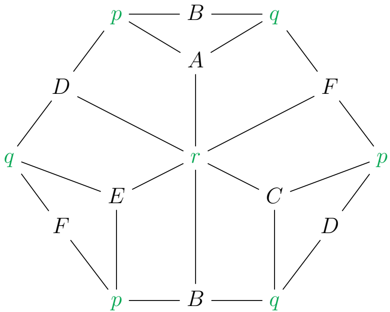
—and there are many more!

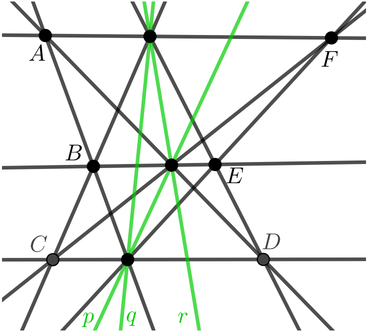
Example 1: Desargues' theorem



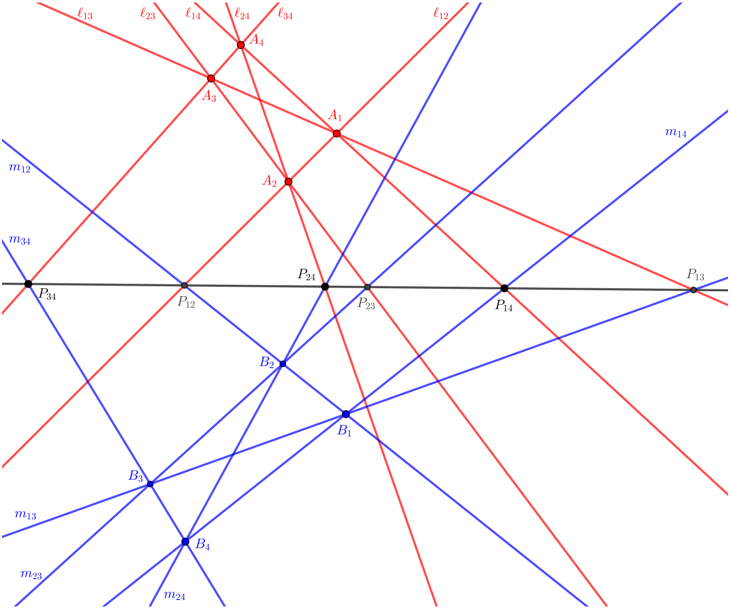


Example 2: Pappus' theorem



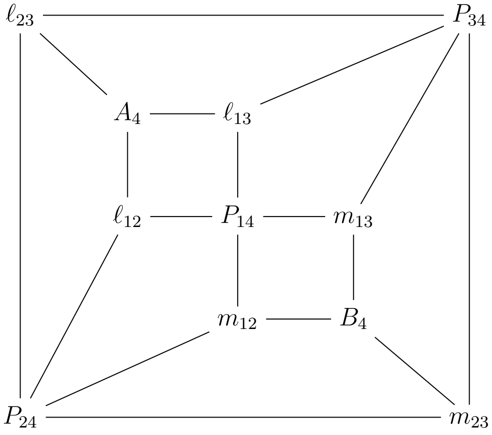


Example 3: The complete quadrangle theorem

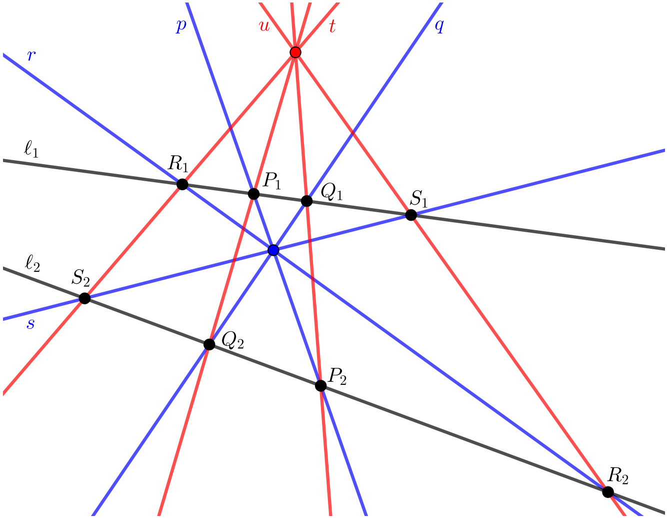


Proof of the complete quadrangle theorem

Apply the master theorem to the following tiling of the sphere:

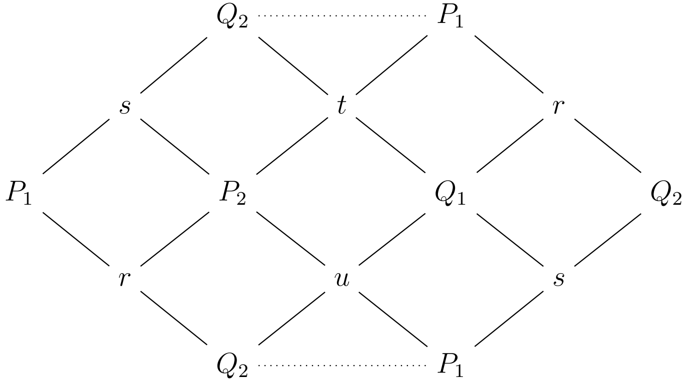


Example 4: The permutation theorem

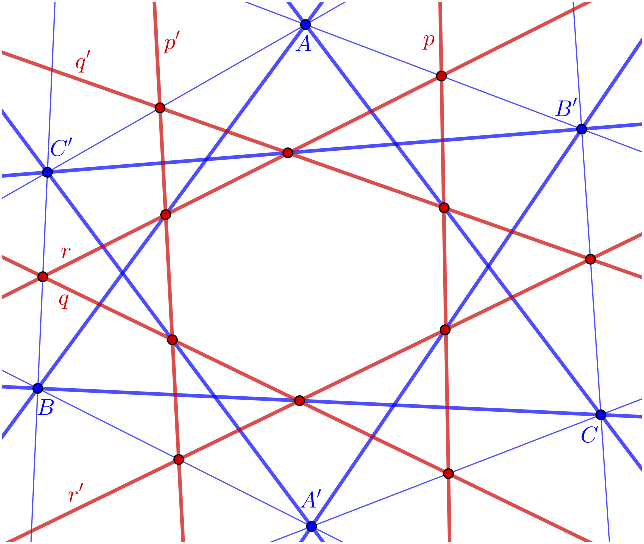


Proof of the permutation theorem

Apply the master theorem to the following tiling of the torus:



Example 5: Twin stars of David [SF-PP]

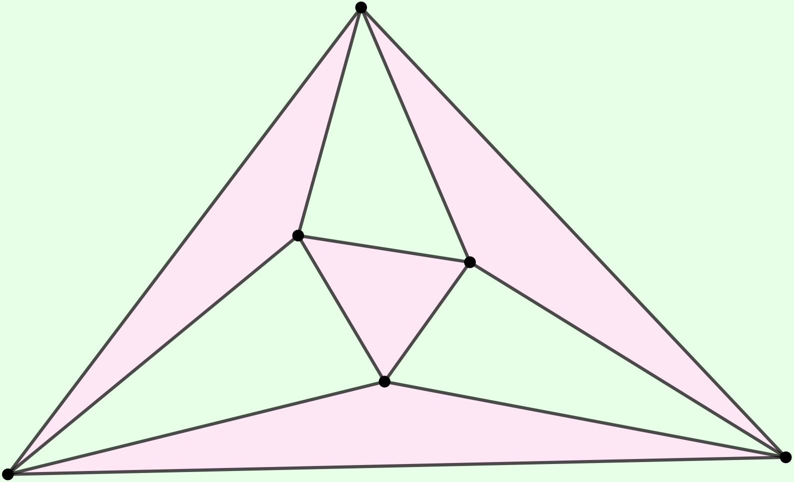


Proof of the twin stars of David theorem

Apply the master theorem to the following tiling of the torus:

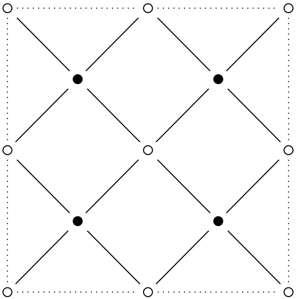
$$
\begin{array}{ccccccccccccccc}
A & \text{—} & p' & \text{—} & A' & \text{—} & p & \text{—} & A & & & & & & \\
 & & | & & | & & | & & | & & & & & & \\
 & & B & \text{—} & r' & \text{—} & B' & \text{—} & r & \text{—} & B & & & & \\
 & & & & | & & | & & | & & | & & & & \\
 & & & & C & \text{—} & q' & \text{—} & C' & \text{—} & q & \text{—} & C & & \\
 & & & & & & | & & | & & | & & | & & \\
 & & & & & & A & \text{—} & p' & \text{—} & A' & \text{—} & p & \text{—} & A
\end{array}
$$

Example 6: Möbius' theorem

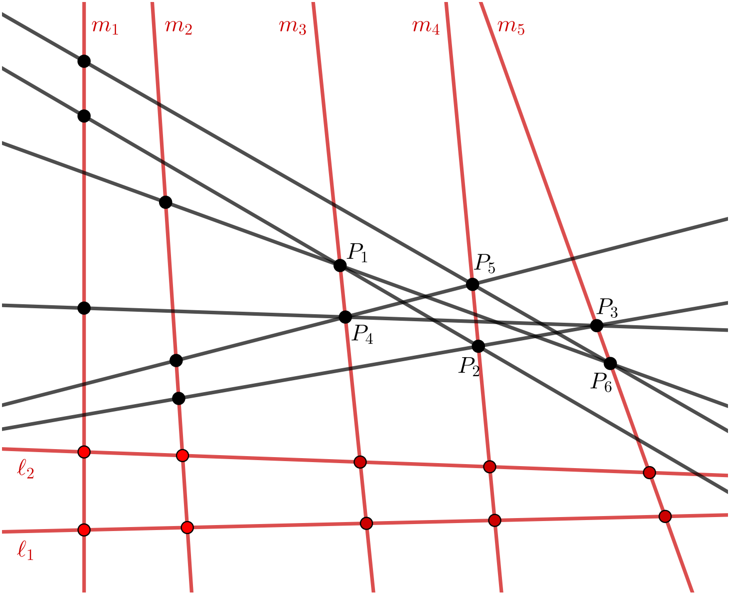


Proof of Möbius' theorem

Apply the master theorem to the following tiling of the torus:

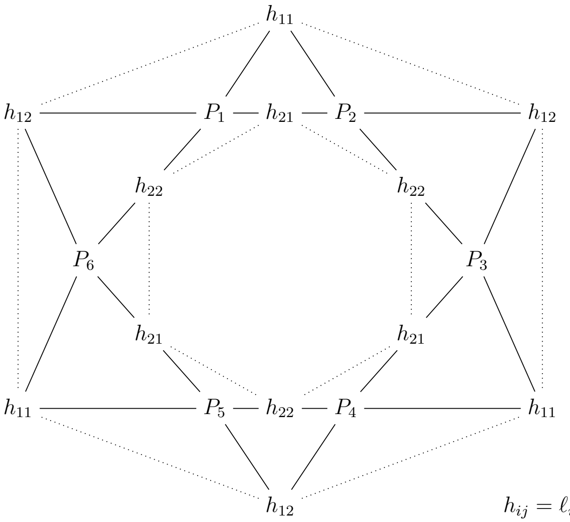


Example 7: The thirteen lines theorem [SF-PP]



Proof of the thirteen lines theorem

Apply the master theorem to this tiling of the genus 2 surface:



$h_{ij} = \ell_i \cap m_j$

From an incidence theorem to a tiling

Problem 1

Can any theorem of linear incidence geometry in the real or complex projective plane be obtained as a special case of our master theorem?

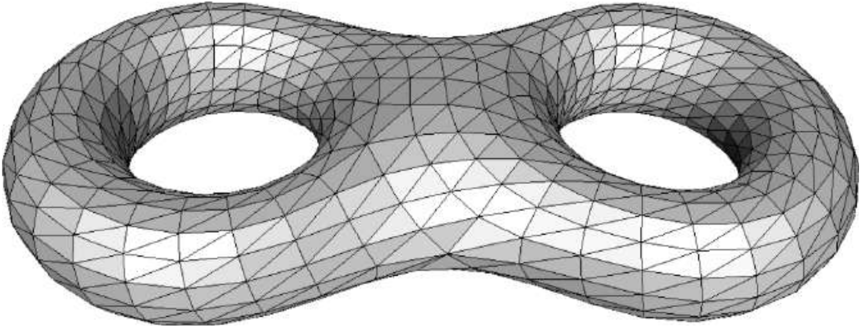
Problem 2

Is there an efficient algorithm for constructing a tiling that delivers a proof of a given incidence theorem?

Problem 3

For each incidence theorem, determine the minimal genus of a tiling that proves the theorem. Can this minimal genus be arbitrarily large?

Triangulated surfaces



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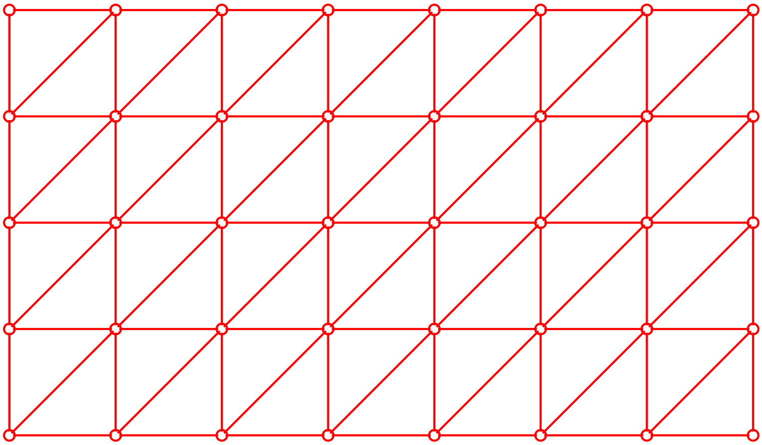
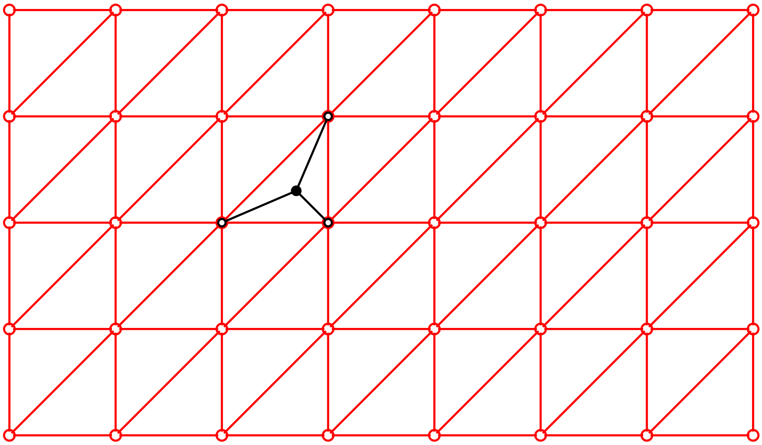
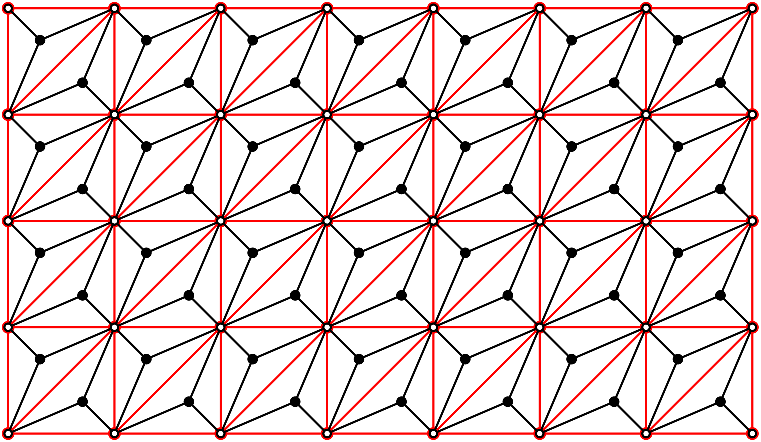
From triangulations to quadrilateral tilings



From triangulations to quadrilateral tilings



From triangulations to quadrilateral tilings



Corollary

Let T be a triangulation of a closed oriented surface.

For each vertex v in T, choose a point P_v on the real/complex plane.
For each edge $u \overset{e}{\text{———}} v$ in T, choose a point P_e on the line $(P_u P_v)$.
Assume that all the chosen points are distinct.

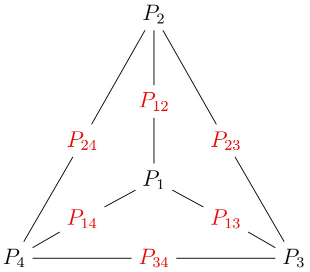
For each triangle in T with sides a, b, c, consider the condition

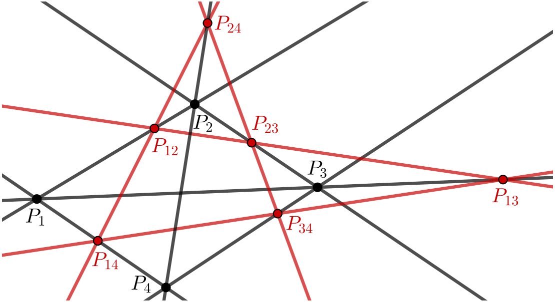
$(*)$ the points P_a, P_b, and P_c are collinear.

Suppose that condition $(*)$ is known to hold for all triangles in the triangulation T but one. Then it holds for the remaining triangle.

Desargues' theorem from a triangulated surface

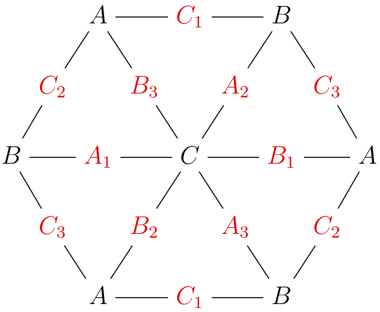
Applying the last corollary to the triangulation of the sphere shown below, we obtain Desargues' theorem.

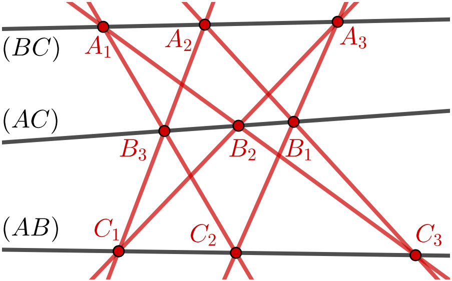




Pappus' theorem from a triangulated surface

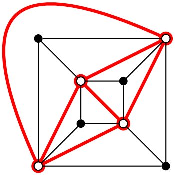
Applying the last corollary to the triangulation of the torus shown below, we obtain Pappus' theorem.

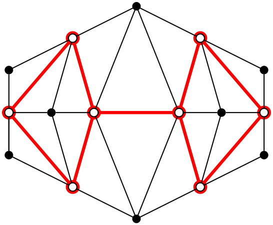




Master theorem from triangulated surface version

We can in fact deduce the tiling version of the master theorem from the triangulated surface version:





$$\begin{tikzcd}[arrows=dash]
B \arrow[r] \arrow[d] & \ell \arrow[r] \arrow[d] & C \arrow[d] \\
p \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & q \arrow[d] \\
E \arrow[r] & m \arrow[r] & D
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}[arrows=dash]
B \arrow[rr] \arrow[d] & & \ell \arrow[rr] \arrow[dl] \arrow[dr] & & C \arrow[d] \\
p \arrow[r] \arrow[d] & A \arrow[dr] & & A' \arrow[r] \arrow[dl] & q \arrow[d] \\
E \arrow[rr] & & m \arrow[rr] & & D
\end{tikzcd}$$

Equivalence of triangulations modulo Desargues

The following transformations of triangulations can be interpreted as applications of Desargues' theorem:

Any two triangulations of a given surface can be connected to each other by these transformations.

Proposition

Any two incidence theorems that come from triangulations of the same surface are equivalent to each other modulo Desargues.

Deducing the master theorem from Desargues and Pappus

Corollary

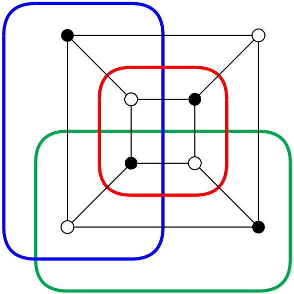
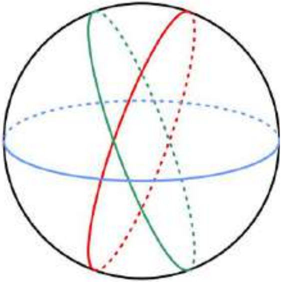
All theorems that come from triangulations of surfaces are equivalent to each other modulo Desargues and Pappus.

Proof

We can deduce theorems corresponding to surfaces of genus g from the theorems for genus $g-1$, by applying the Pappus theorem.

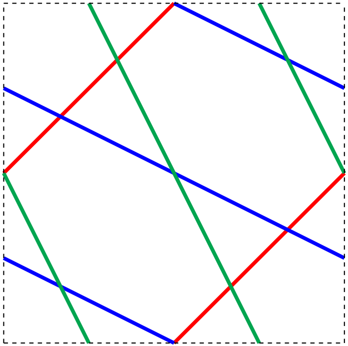
Nodal (multi-)curves

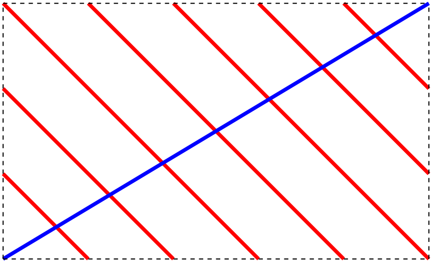
The master theorem can be reformulated in terms of nodal curves instead of tilings:

Desargues' theorem

Nodal curves for the Pappus and permutation theorems

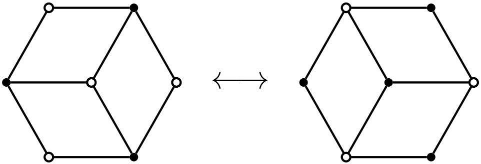




Any nodal curve on an oriented surface yields an incidence theorem.

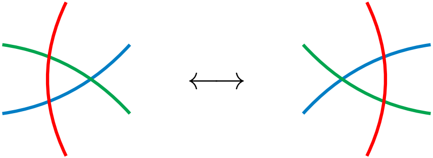
Desargues flips

A flip in a tiling



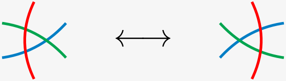
corresponds to an application of Desargues' theorem.

These Desargues flips translate into local moves on nodal curves:

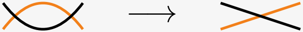


Classifying nodal curves up to local moves

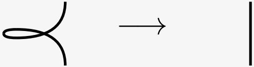
Local moves on nodal curves



Desargues flip



simplification

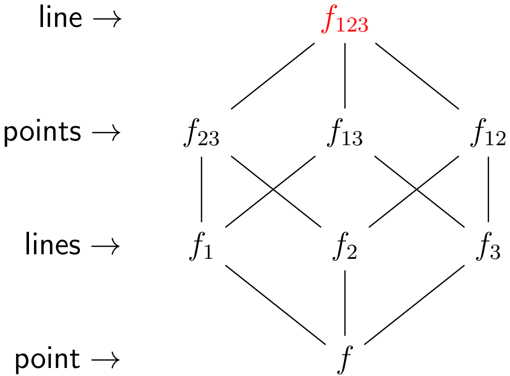


simplification

For a given closed oriented surface, equivalence classes of nodal curves modulo these local moves correspond to equivalence classes of incidence theorems modulo Desargues.

Desargues' theorem as 3D consistency

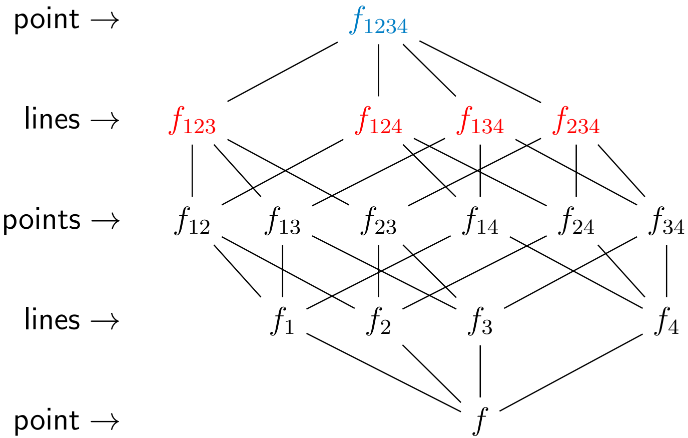
Desargues' theorem can be interpreted as 3D consistency/integrability of tile coherence in the sense of A. Bobenko and Yu. Suris.



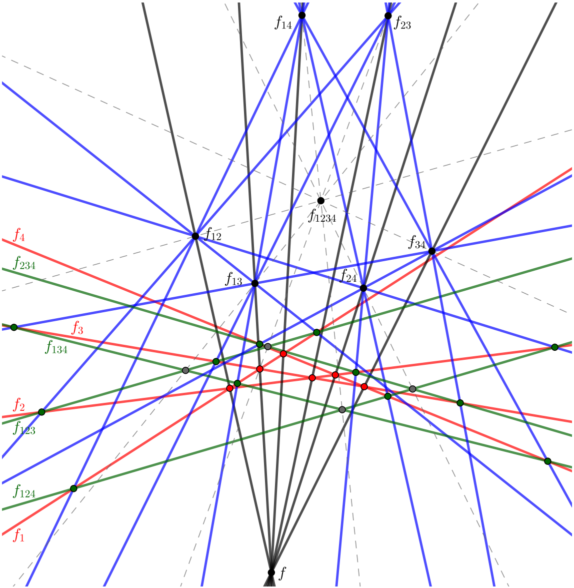
4D consistency of tile coherence

Theorem

The dynamics of Desargues flips exhibits 4D consistency.



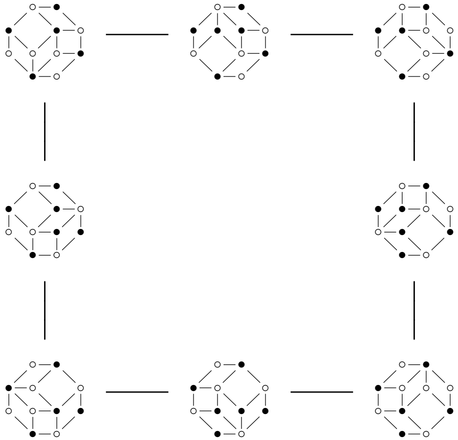
4D consistency of tile coherence



Tetrahedron equation for Desargues flips

Theorem

Desargues flips satisfy Zamolodchikov's tetrahedron equation.



Insights provided by the master theorem

The master theorem helps explain, in a conceptual way:

- why Pappus' theorem is more powerful than Desargues';
- why Pappus (and Desargues) imply all other theorems of planar linear incidence geometry;
- why Desargues holds over noncommutative skew-fields whereas Pappus does not;
- how to construct generalizations of existing incidence theorems;
- which incidence theorems can be formulated as "closure porisms" (Schließungssätze) and which ones cannot;

and many other things.

Further directions

Tiling-based techniques can be used to study:

- R. Schwartz's pentagram map and it variations;
- S. Tabachnikov's skewers;
- incidence theorems for circles and lines on the Möbius plane;
- incidence theorems for conics and algebraic curves of higher degree;
- incidence theorems involving tangency conditions;
- J.-V. Poncelet's closure phenomena;
- incidence theorems for surfaces in 3D;
- incidence theorems over fields of finite characteristic;
- incidence theorems over noncommutative skew-fields;
- incidence theorems in elliptic and hyperbolic geometry,

and undoubtedly a lot more.

Congratulations, Philippe!