Branes and Quivers

In honor of Philippe Di Francesco's @ 60

Amihay Hanany Jun 2024

Branes and Supersymmetric Gauge Theories

Outline

- Branes on Gorenstein singularities
- Brane Tilings
- Cluster Algebras
- Branes and tropical geometry
- Branes and symplectic singularities

Branes on Gorenstein Singularities
1998

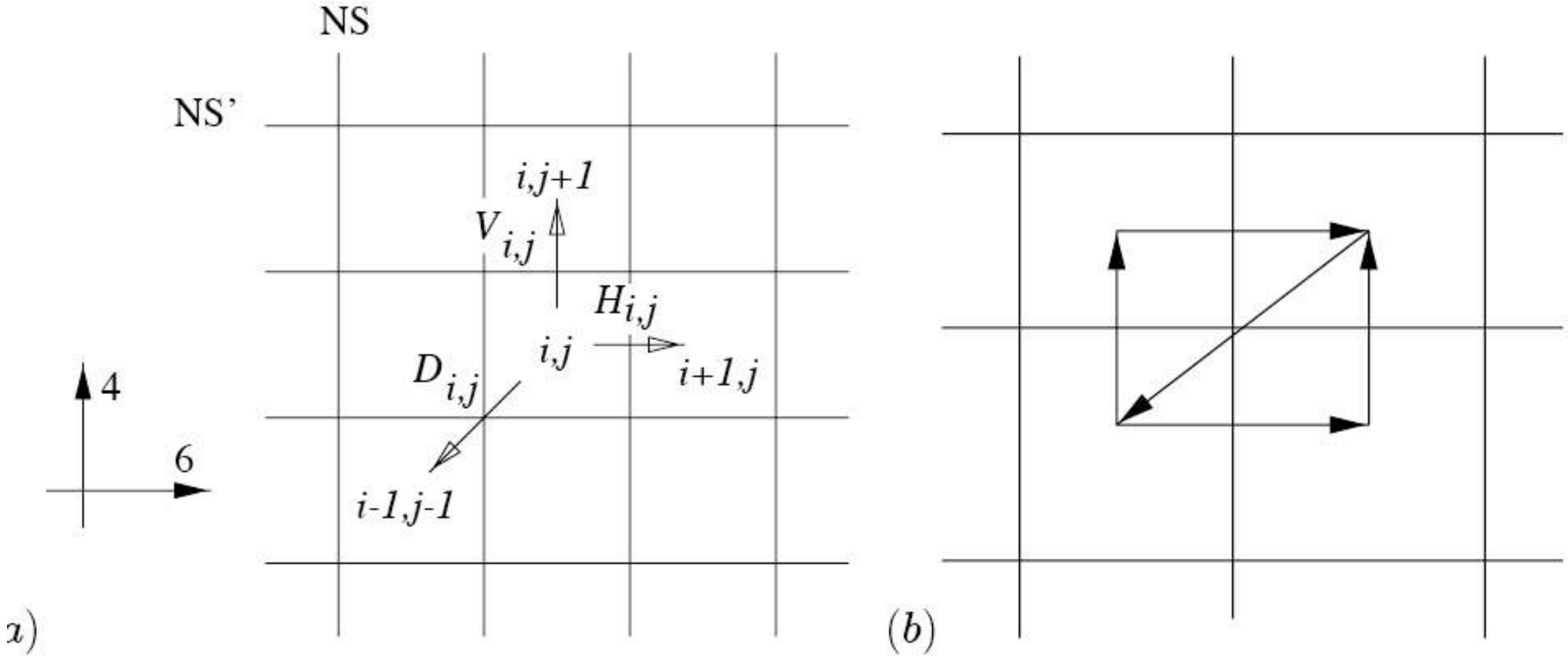
Type IIB superstring (10)

D3 branes (4) on Gorenstein singularities (6)

- N D3 branes on Gorenstein singularities of cplx dim 3
- Fractional branes
- Open strings between the different branes
- Gauge fields, matter fields, interactions — Quiver Q with a superpotential W
- Problem — compute the Q & W for a given singularity

Brane Boxes

1998



$$W = \sum_{i,j} (H_{i,j} V_{i+1,j} D_{i+1,j+1} - V_{i,j} H_{i,j+1} D_{i+1,j+1})$$

Quiver & Superpotential $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$

Graph taken from the work of Philippe & Jean-Bernard (1989)

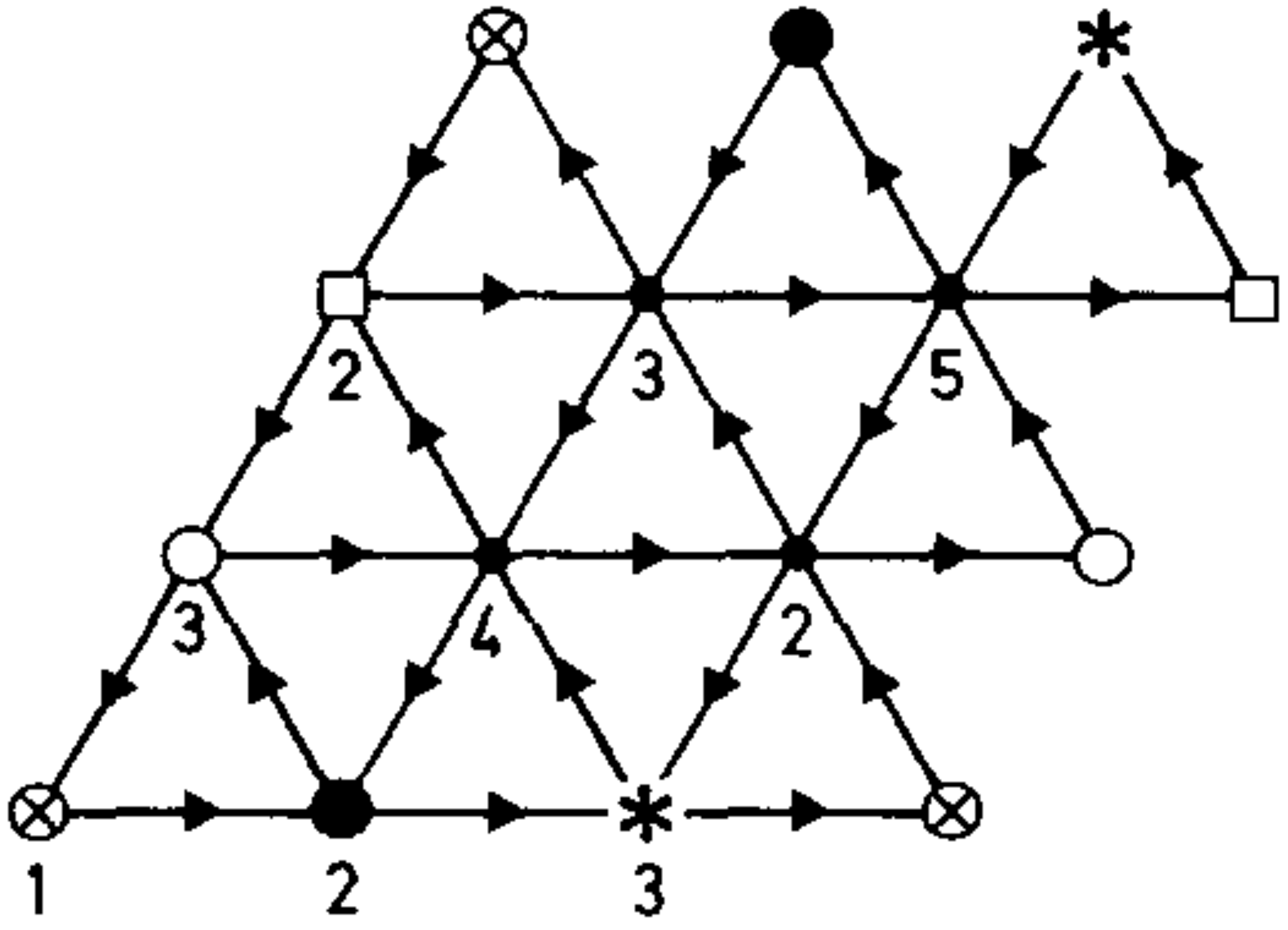
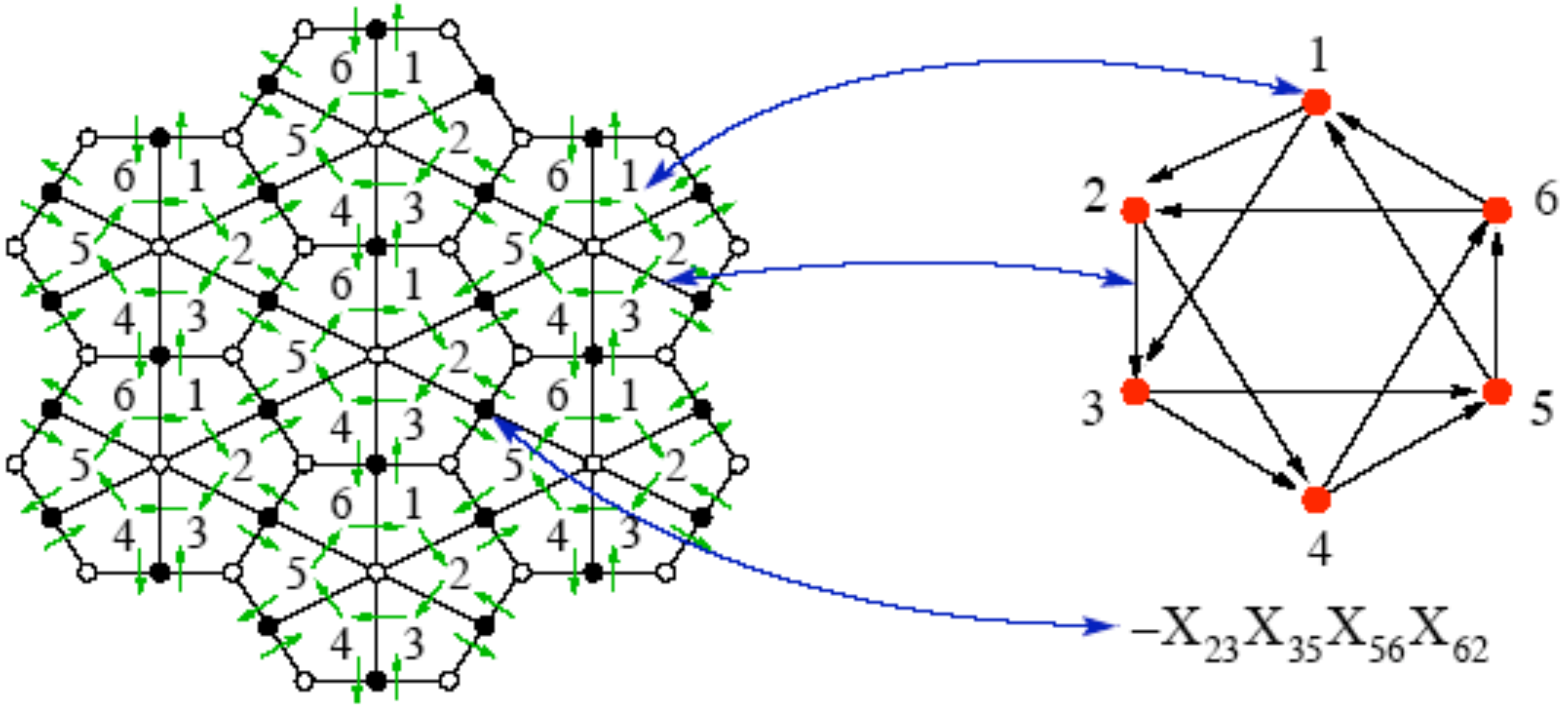


Fig. 9. Graph of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup of SU(3); points with the same symbol (•, ⊗, etc.) have to be identified.

Branes Tilings
2005

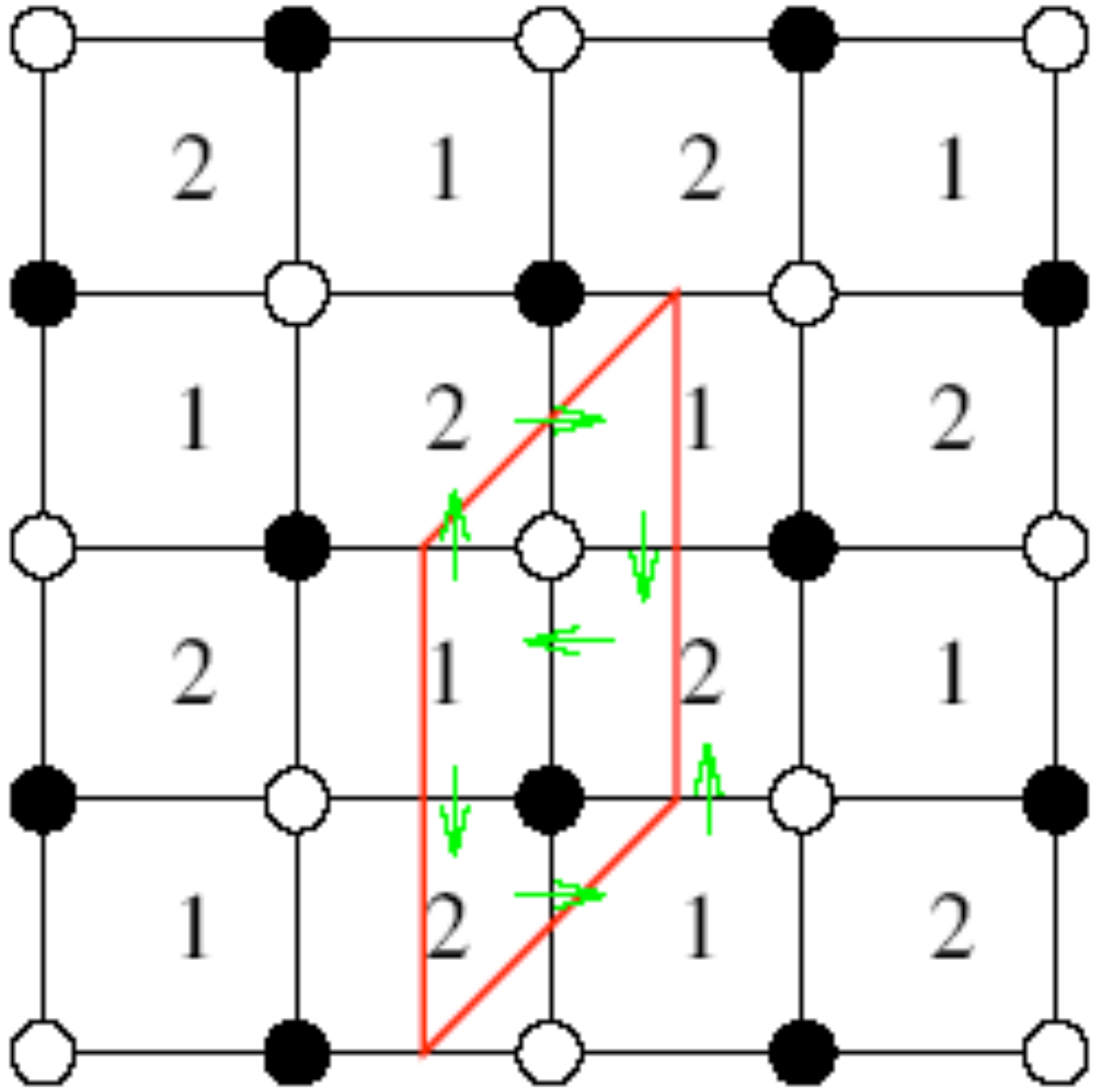
Brane Tilings

2005

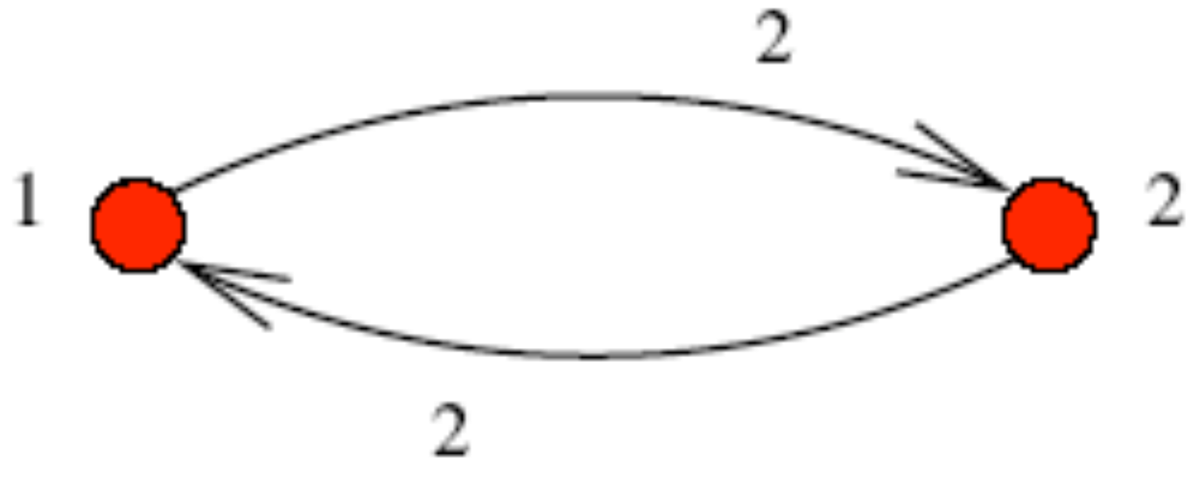


Conifold tiling

2005



brane tiling

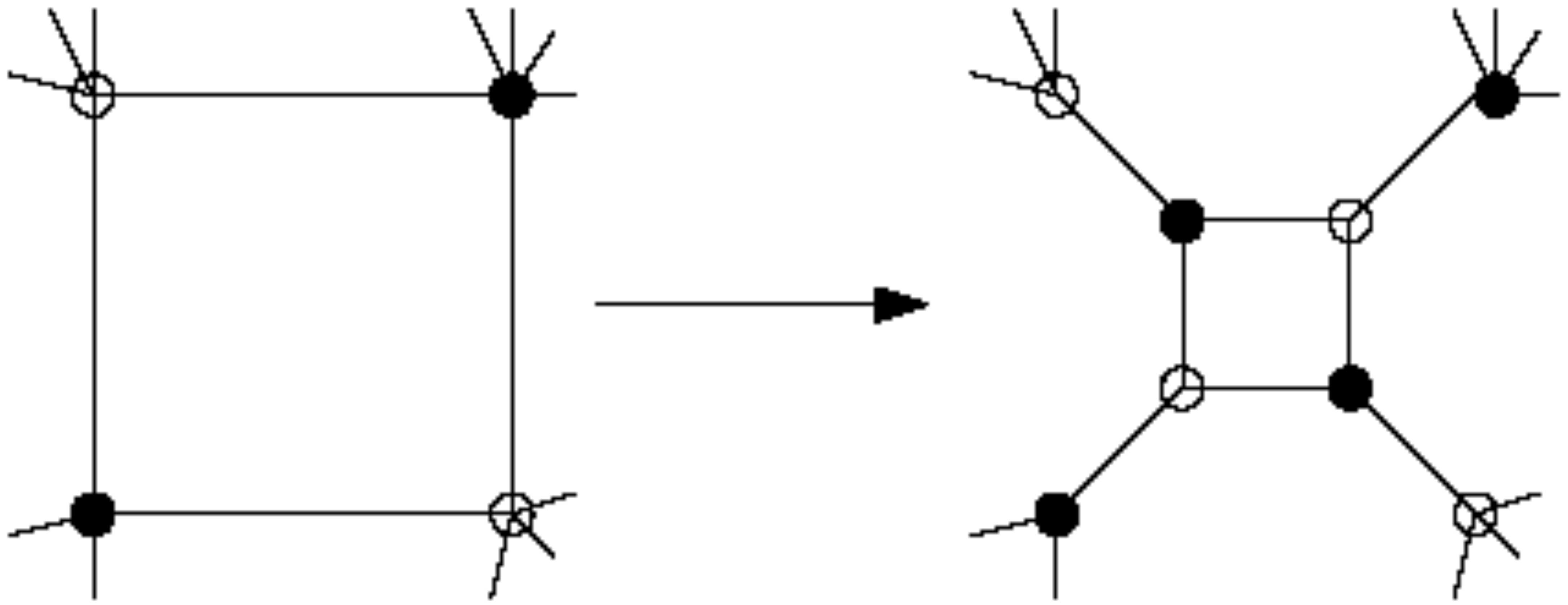


quiver

$$W = X_{12}^{(1)} X_{21}^{(1)} X_{12}^{(2)} X_{21}^{(2)} - X_{12}^{(1)} X_{21}^{(2)} X_{12}^{(2)} X_{21}^{(1)}$$

Seiberg Duality

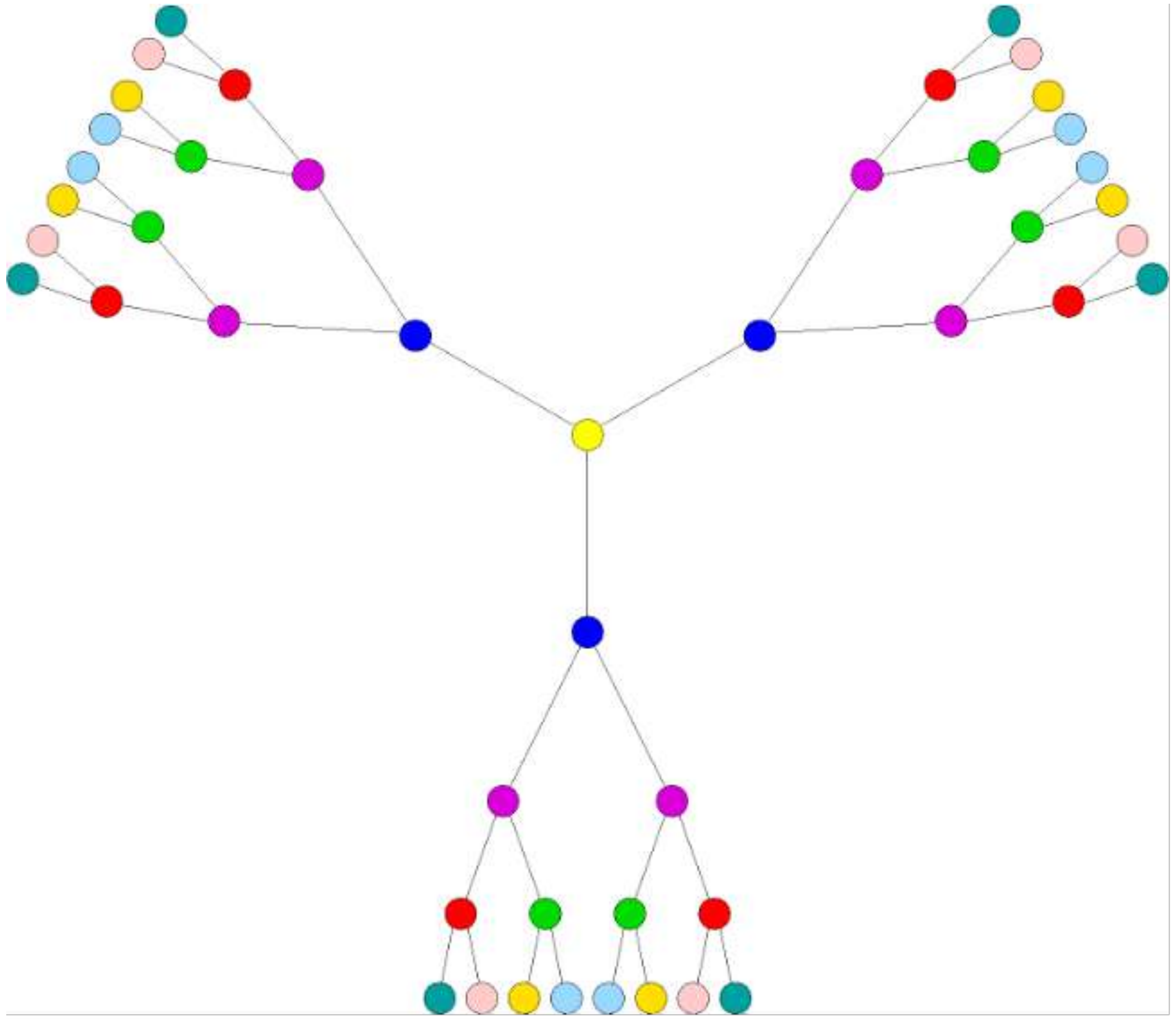
Urban Renewal — quiver mutation

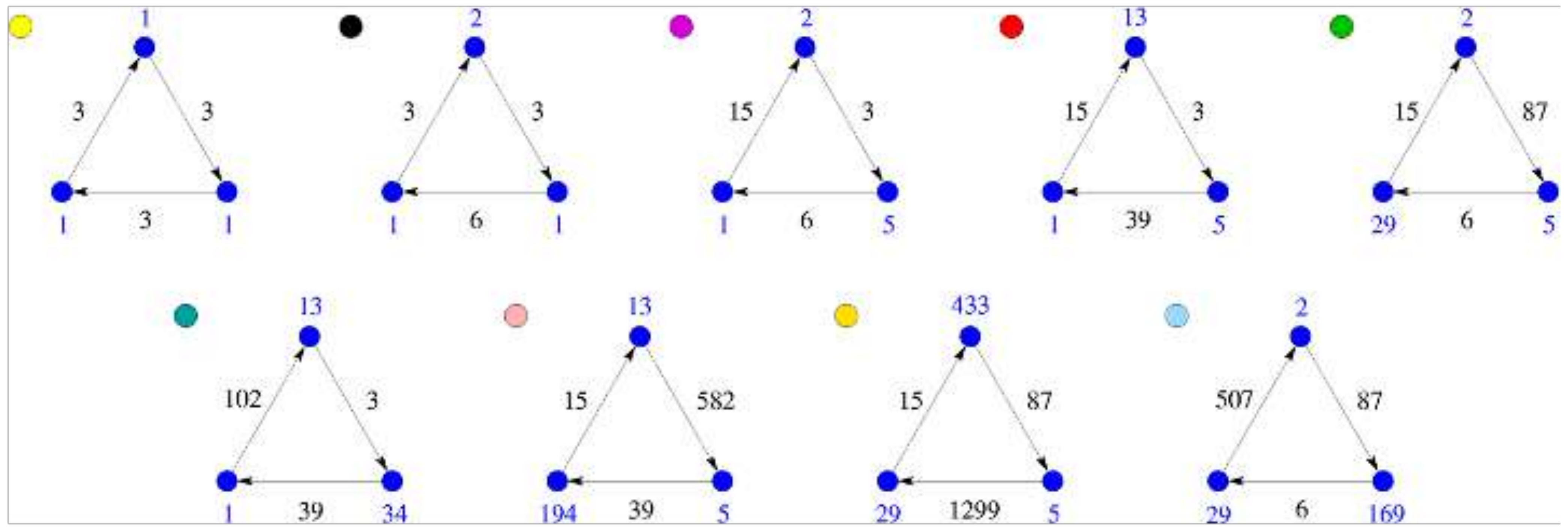


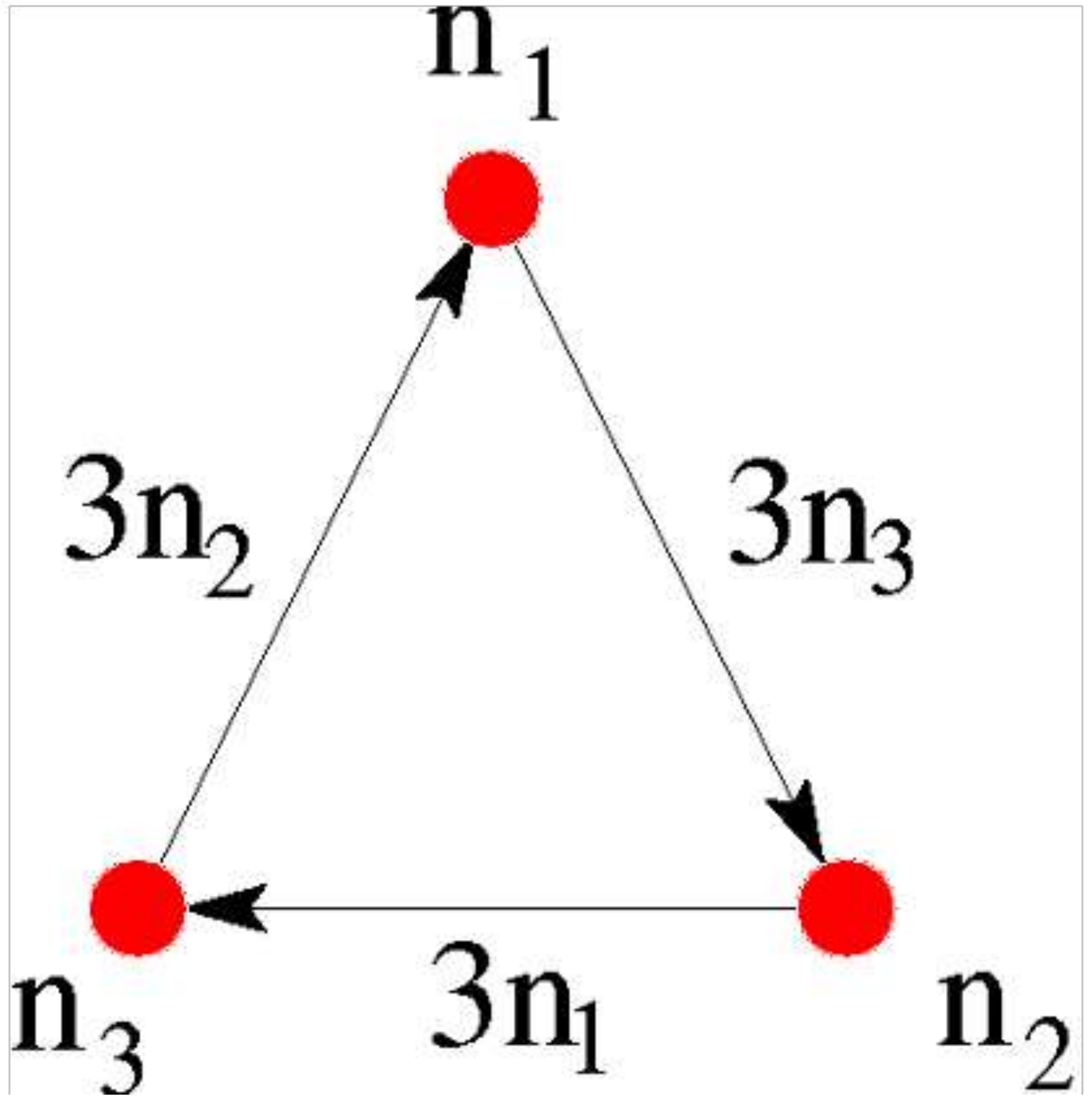
Cluster Algebras

Duality tree for dP_0, also known as $\mathbb{C}^3/\mathbb{Z}_3$

2003



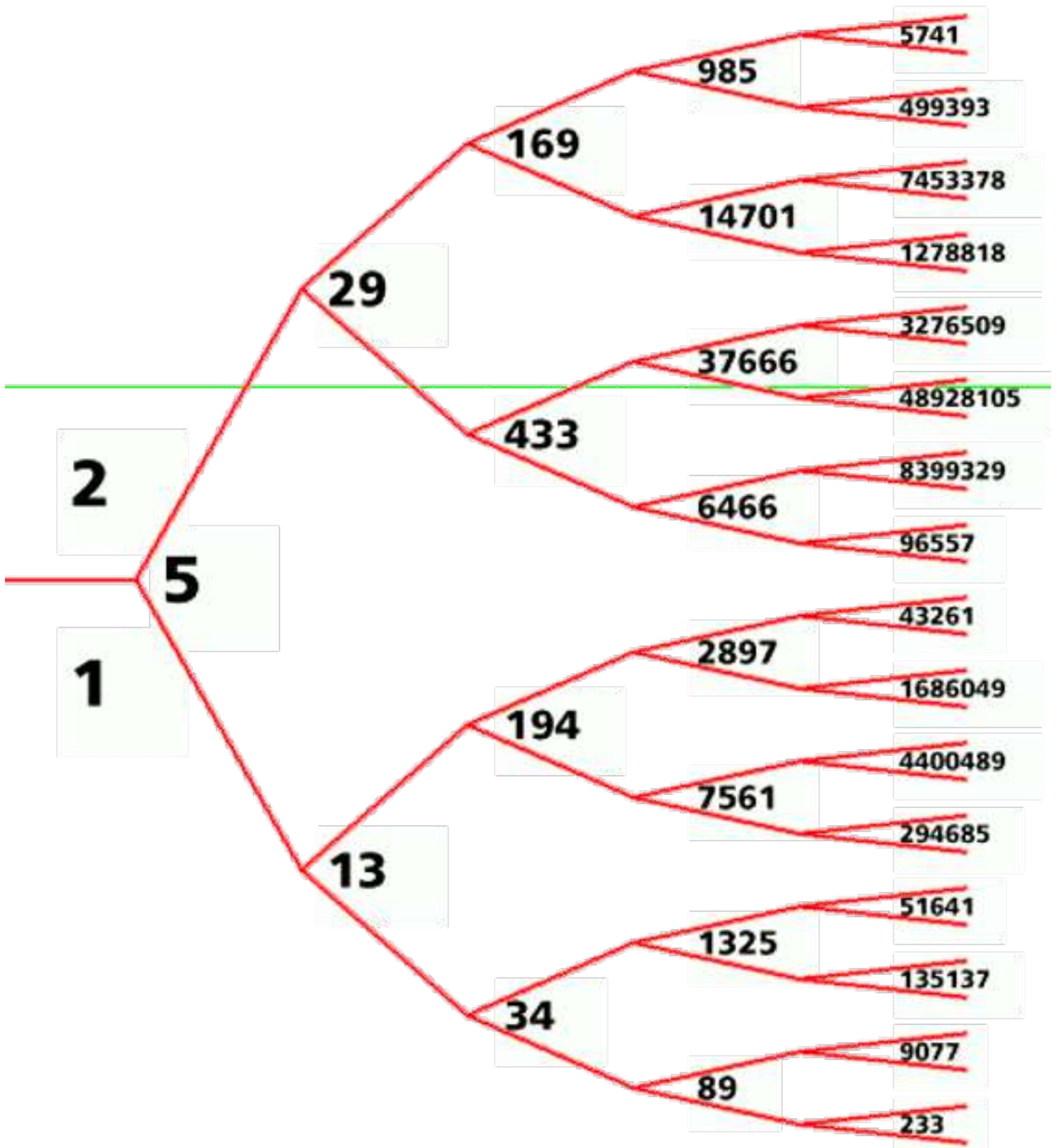




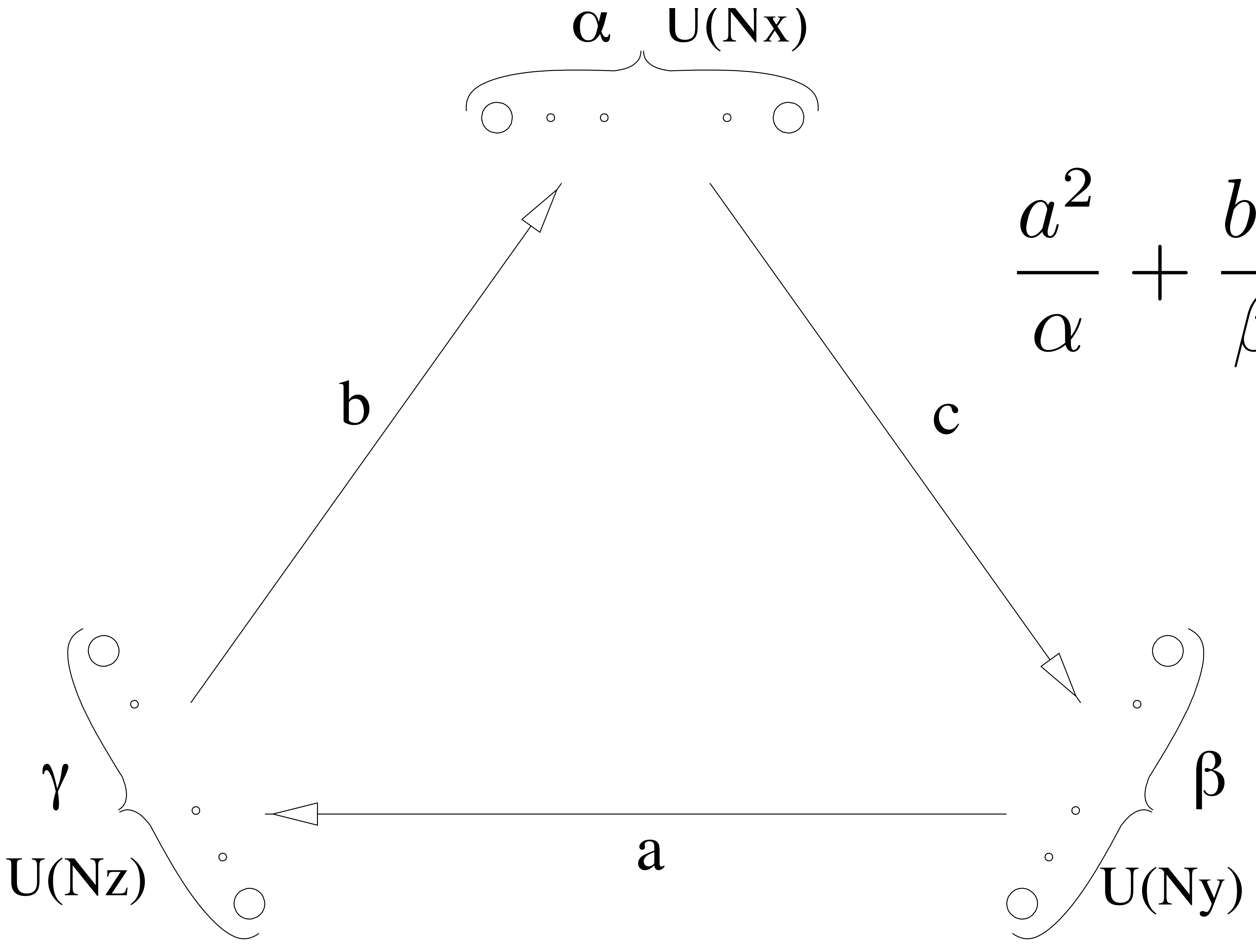
Diophantine (Markov) equation

$n_1^2 + n_2^2 + n_3^2 = 3n_1n_2n_3$

Markov Numbers



Quiver 3 block model



$$\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} = abc$$

$$R_a = \frac{2}{abc}\frac{a^2}{\alpha}; \quad R_b = \frac{2}{abc}\frac{b^2}{\beta}; \quad R_c = \frac{2}{abc}\frac{c^2}{\gamma}$$

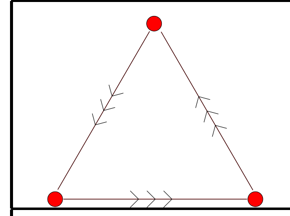
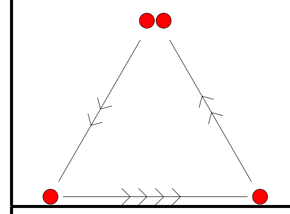
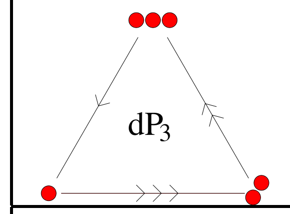
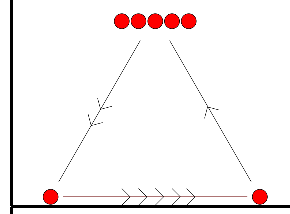
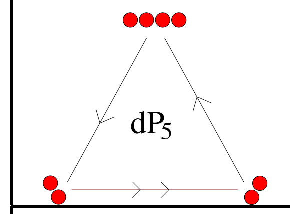
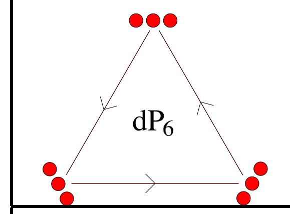
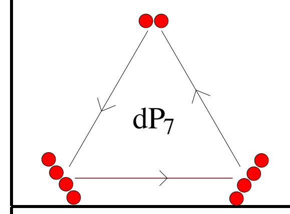
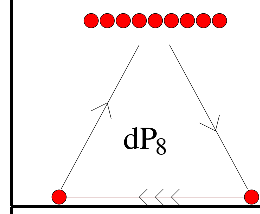
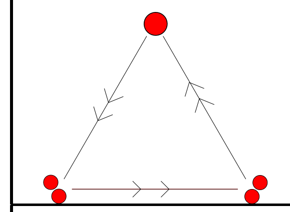
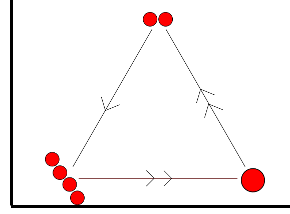
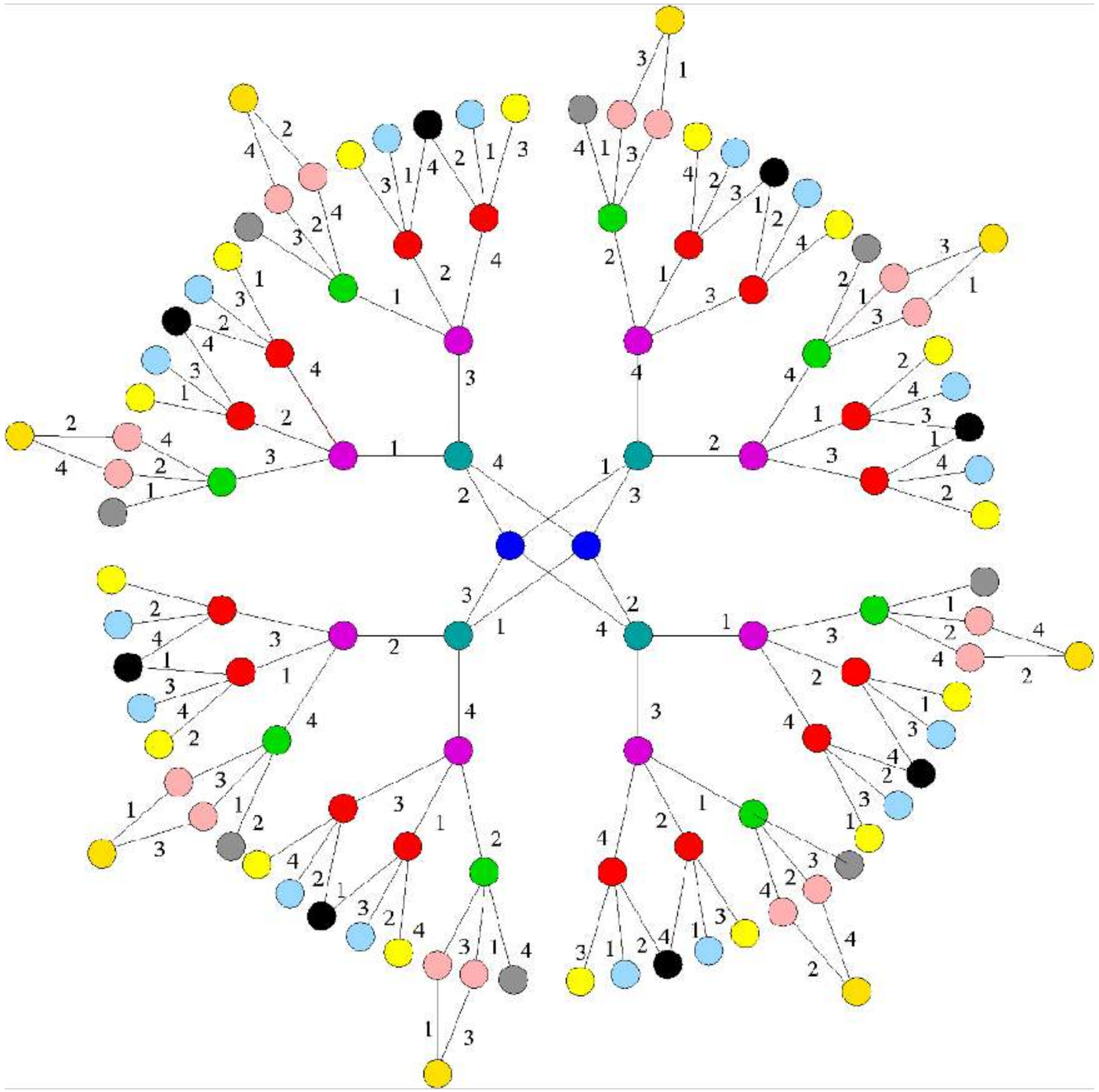
Quiver	α	β	γ	a	b	c	x	y	z	K^2	surface
	1	1	1	3	3	3	1	1	1	9	dP_0
	1	1	2	2	2	4	1	1	1	8	F_0
	1	2	3	1	2	3	1	1	1	6	dP_3
	1	1	5	1	2	5	1	2	1	5	dP_4
	2	2	4	1	1	2	1	1	1	4	dP_5
	3	3	3	1	1	1	1	1	1	3	$dP_6\ I$
	2	1	6	1	1	3	1	2	1	3	$dP_6\ II$
	4	4	2	1	1	1	1	1	2	2	$dP_7\ I$
	1	1	8	1	1	4	2	2	1	2	$dP_7\ II$
	3	1	6	1	1	2	1	3	1	2	$dP_7\ III$
	1	1	9	1	1	3	3	3	1	1	$dP_8\ I$
	8	2	1	2	1	1	1	2	4	1	$dP_8\ II$
	2	3	6	1	1	1	3	2	1	1	$dP_8\ III$
	5	5	1	1	2	1	1	2	5	1	$dP_8\ IV$
	2	2	1	2	2	2	1	1	2	4	$sh\,dP_5$
	2	1	4	2	1	2	2	2	1	2	$sh\,dP_7$

Table 1: The complete list of the minimal models for 3-block chiral quivers.

Duality Flower for $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

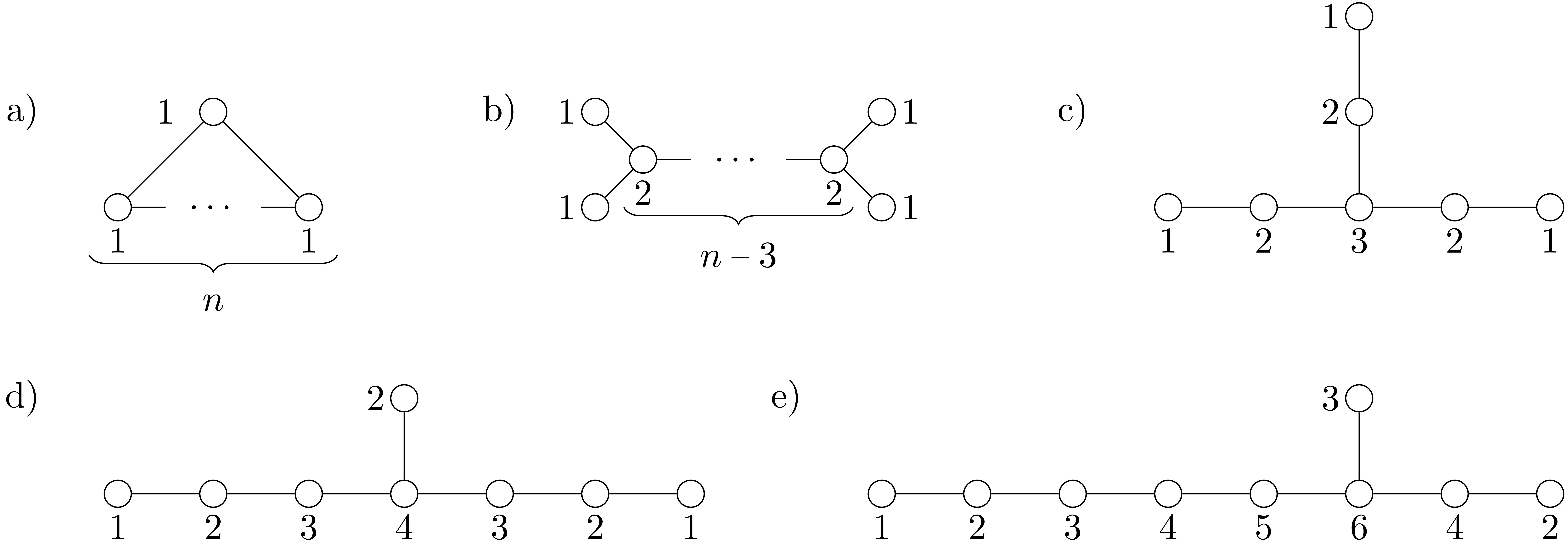
2003



Coulomb Branch
2013

Affine ADE quivers

Coulomb branch — closure of minimal nilpotent orbit



Monopole formula — the ingredients

per each node of label k

- $W = S_k$ — the Weyl group of $GL(k)$
- $\hat{\Lambda}$ — The (Langlands) dual lattice — lattice of coweights
- A set of integer numbers $\hat{\Lambda} = \mathbb{Z}^k \ni m = (m_1, \ldots, m_k)$ — magnetic charges (coweights)
- $\hat{\Lambda}/W$ — Principal Weyl chamber $m_1 \leq \cdots \leq m_k$
- Boundaries of the Weyl chamber — when some m_i coincide
- H_m — stabilizer of m in $GL(k)$ — a Levi subgroup of GL(k)
- d_i^m — degrees of Casimir invariants of H_m

The conformal dimension — $\Delta(m)$

$\mathbb{C}^*$ grading on the Coulomb branch

- Given a quiver with a set of nodes, each with labels k_a
- $\Delta(m)$ is a sum of contributions from nodes and edges:
- For each node with magnetic charges $m_i^a, i = 1\ldots k_a$ there is a negative contribution
- $-\sum_{1\leq i<j\leq k_a} |m_i^a - m_j^a|$ (associated with positive roots of $GL(k_a)$)
- For each edge connecting nodes a, b with magnetic charges m_i^a and m_j^b a positive contribution
- $\frac{1}{2}\sum_{i=1}^{k_a}\sum_{j=1}^{k_b} |m_i^a - m_j^b|$ (associated with bifundamental representation)

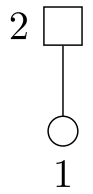
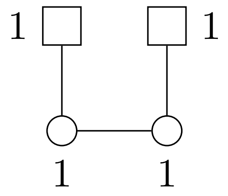
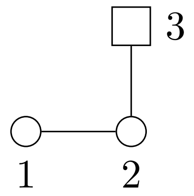
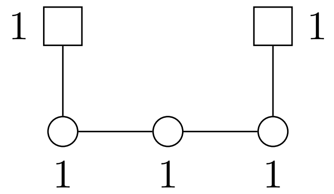
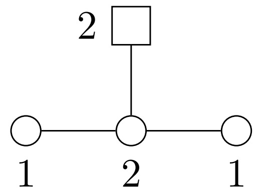
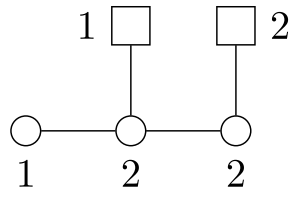
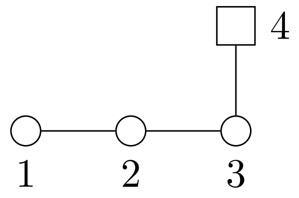
The monopole formula

Hilbert series of the Coulomb branch

- Given a quiver with all the ingredients defined so far
- Introduce a variable t
- The Hilbert series is given by (flavor nodes have fixed m. Set to 0.)
- $$H(t) = \sum_{m \in \hat{\Lambda}/W} t^{2\Delta(m)} P_m(t)$$
- $$P_m(t) = \prod_i \frac{1}{1 - t^{2d_i^m}}$$

Examples — from the world of nilpotent orbits

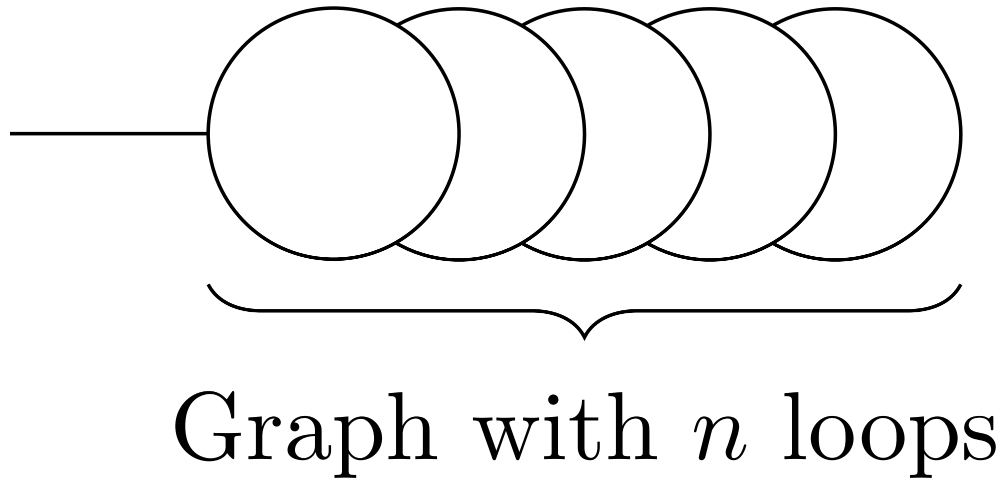
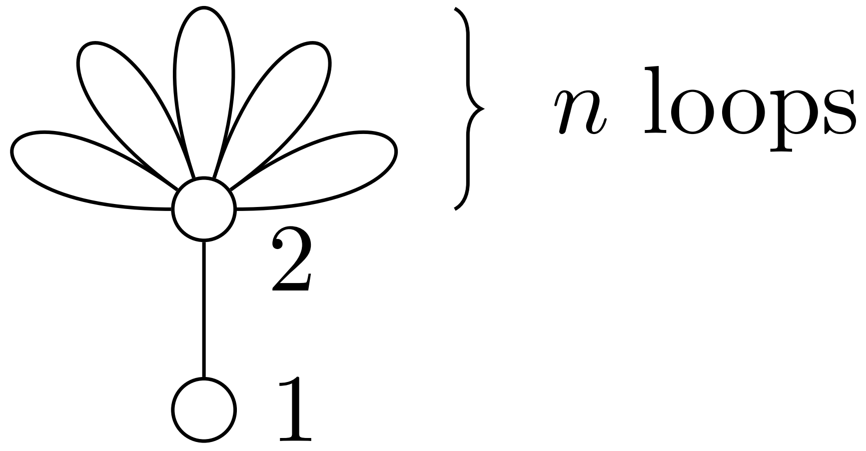
Simple quivers and their Hilbert Series

Nilpotent Orbit	$\text{Dim}_{\mathbb{H}}$	Quiver	HS	HWG
$[1,1]$	0	-	1	1
$[2]$	1		$\frac{(1-t^4)}{(1-t^2)^3}$	$\frac{1}{(1-\mu^2 t^2)}$
$[1,1,1]$	0	-	1	1
$[2,1]$	2		$\frac{(1+4t^2+t^4)}{(1-t^2)^4}$	$\frac{1}{(1-\mu_1\mu_2 t^2)}$
$[3]$	3		$\frac{(1-t^4)(1-t^6)}{(1-t^2)^8}$	$\frac{(1-\mu_1^3\mu_2^3 t^{12})}{(1-\mu_1\mu_2 t^2)(1-\mu_1\mu_2 t^4)(1-\mu_1^3 t^6)(1-\mu_2^3 t^6)}$
$[1,1,1,1]$	0	-	1	1
$[2,1,1]$	3		$\frac{(1+t^2)(1+8t^2+t^4)}{(1-t^2)^6}$	$\frac{1}{(1-\mu_1\mu_3 t^2)}$
$[2,2]$	4		$\frac{(1+t^2)^2(1+5t^2+t^4)}{(1-t^2)^8}$	$\frac{1}{(1-\mu_1\mu_3 t^2)(1-\mu_2^2 t^4)}$
$[3,1]$	5		$\frac{(1+t^2)(1+4t^2+10t^4+4t^6+t^8)}{(1-t^2)^{10}}$	$\frac{(1-\mu_1^3\mu_2^3\mu_3^3 t^{12})}{(1-\mu_1\mu_3 t^2)(1-\mu_2^2 t^4)(1-\mu_1\mu_3 t^4)(1-\mu_1^2\mu_2 t^6)(1-\mu_2\mu_3^2 t^6)}$
$[4]$	6		$\frac{(1-t^4)(1-t^6)(1-t^8)}{(1-t^2)^{15}}$	messy

Quivers for hyper surface symplectic singularities

Slices in Sp(n)

X_n
$\dim_{\mathbb{H}} = 2$

$\mathcal{H}\Bigg($  $\Bigg) = \mathcal{C}\Bigg($ 

$\Bigg)$

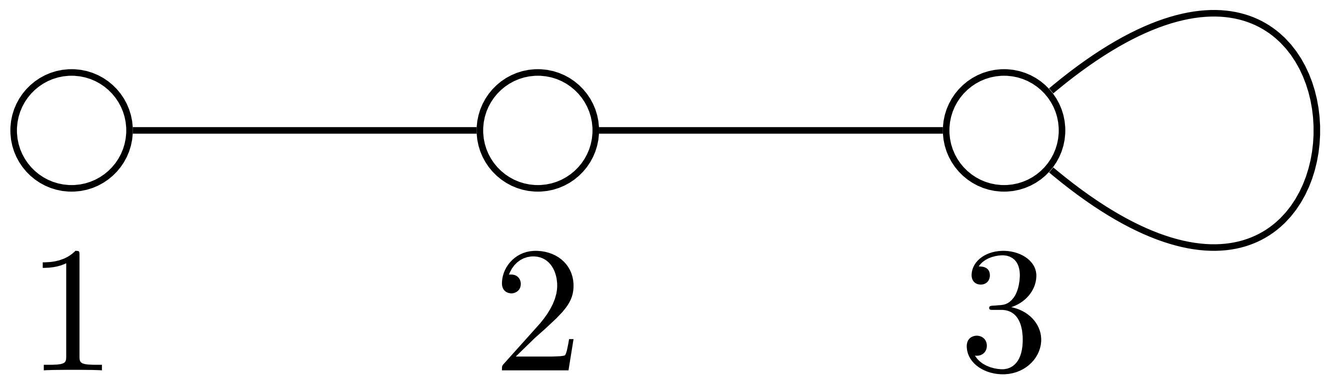
- $PE\left[\mu^2 t^2 + t^4 + \mu\left(t^{2n-1} + t^{2n+1}\right) - \mu^2 t^{4n+2}\right]_{SU(2)}$

Quivers for hyper surface symplectic singularities

Slice in G2

$$\hat{X}$$

$$\dim_{\mathbb{H}} = 3$$

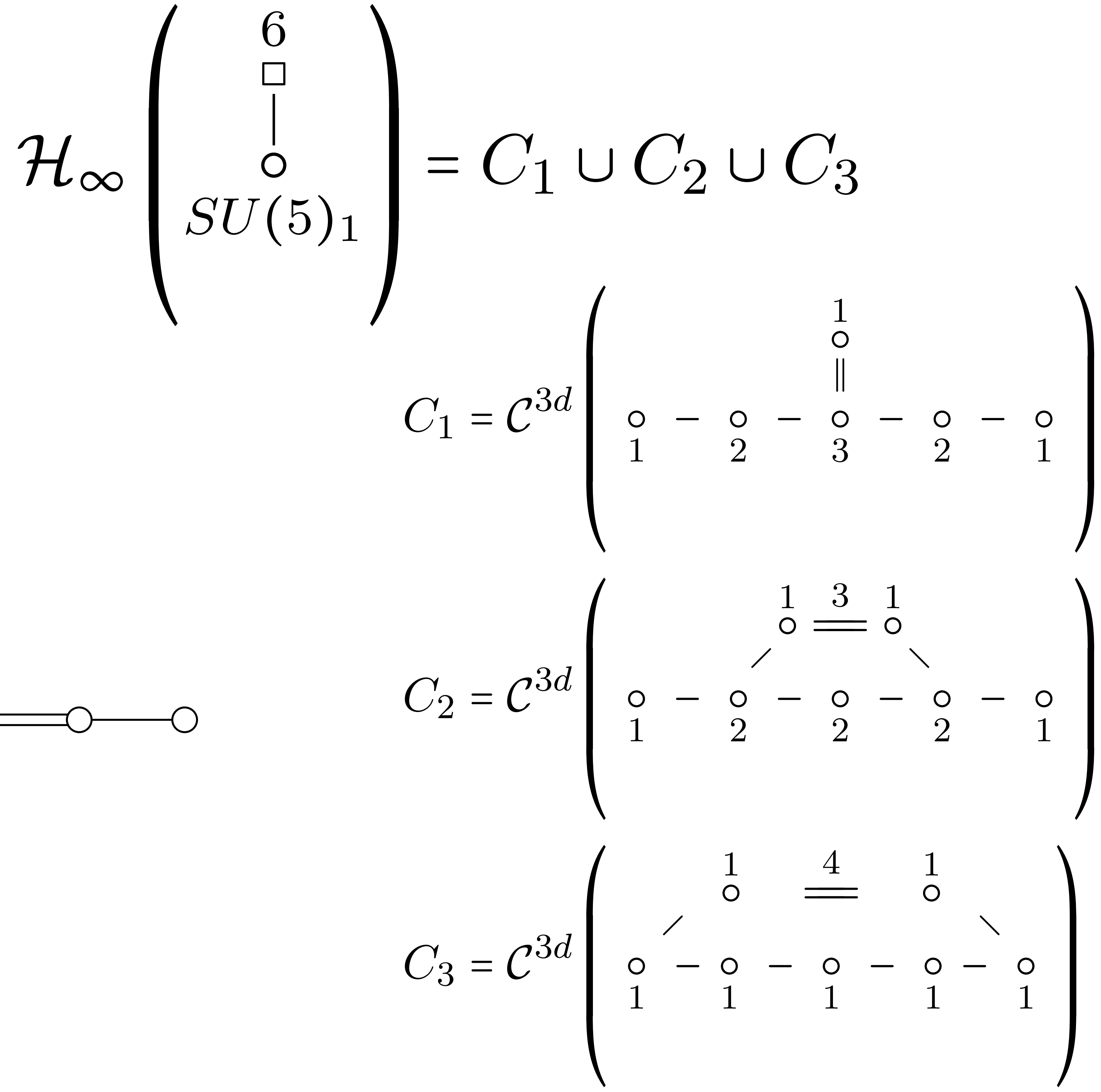


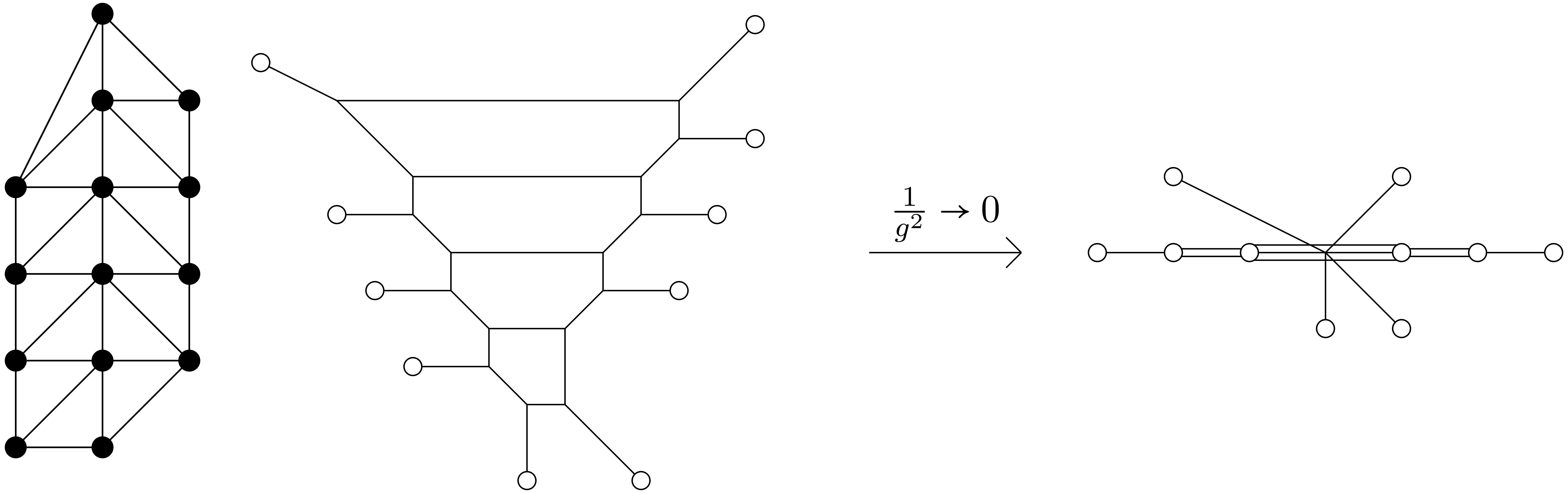
- $H(t,x) = PE\left[[2]t^2 + [3]t^3 - t^{12}\right]$
- Not polynomial PE

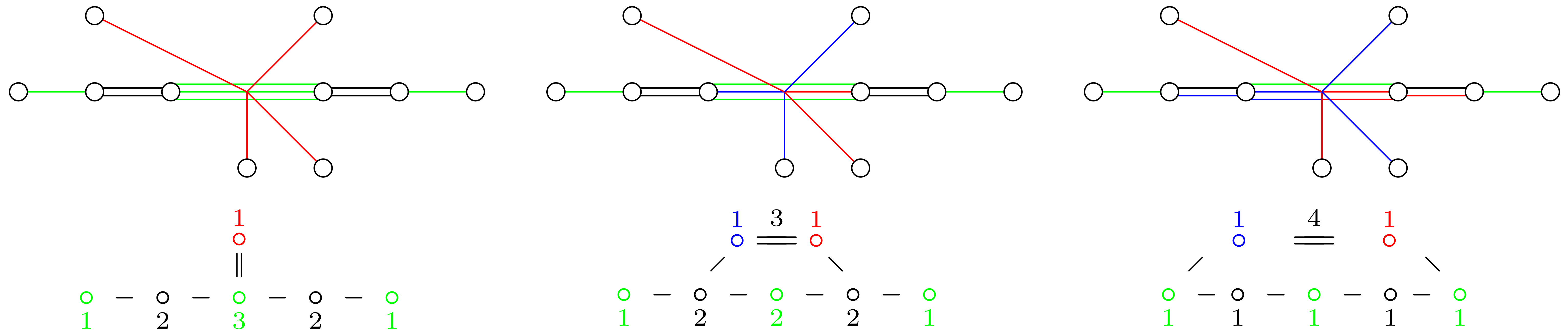
Branes

Union of 3 cones

new physics



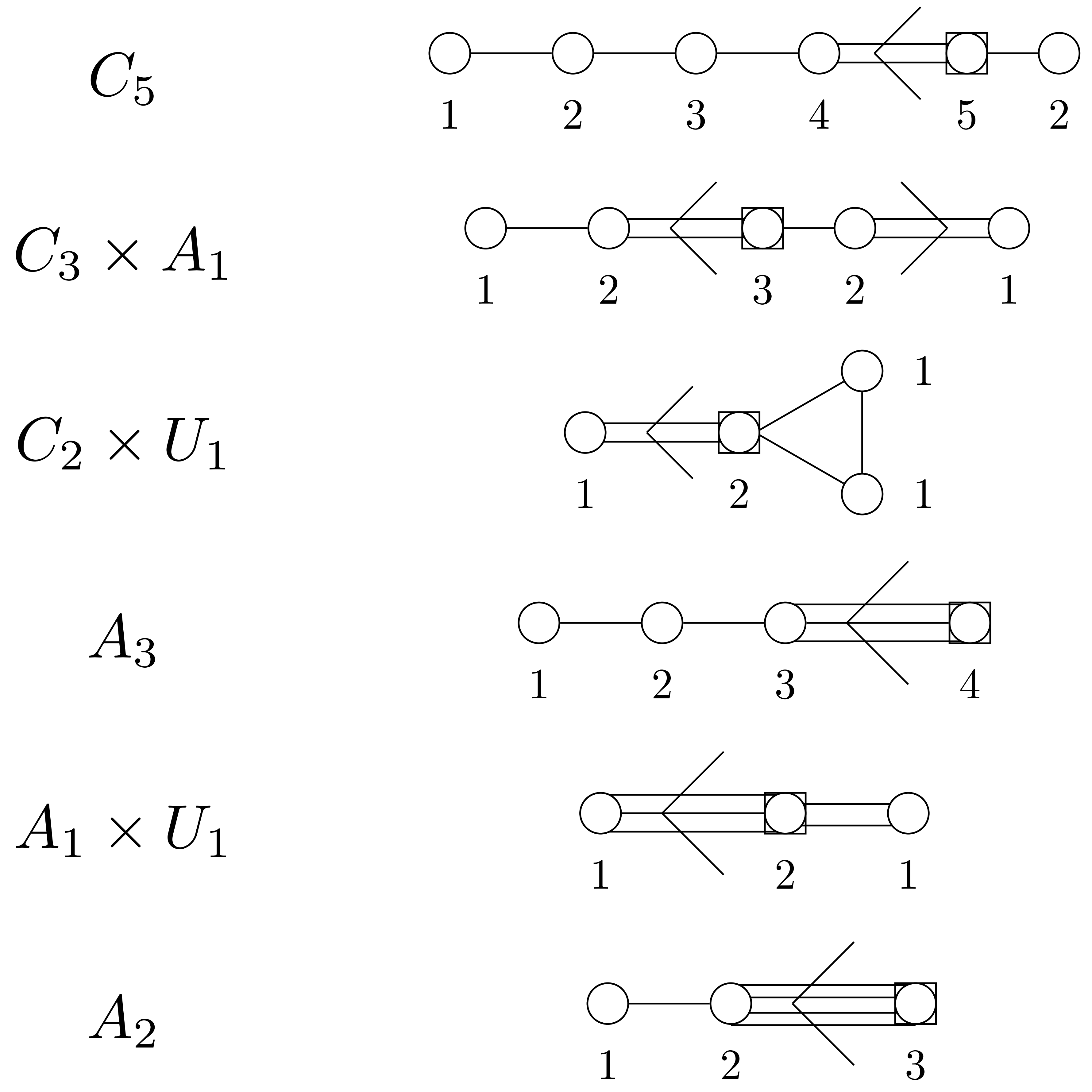




Non simply laced quivers

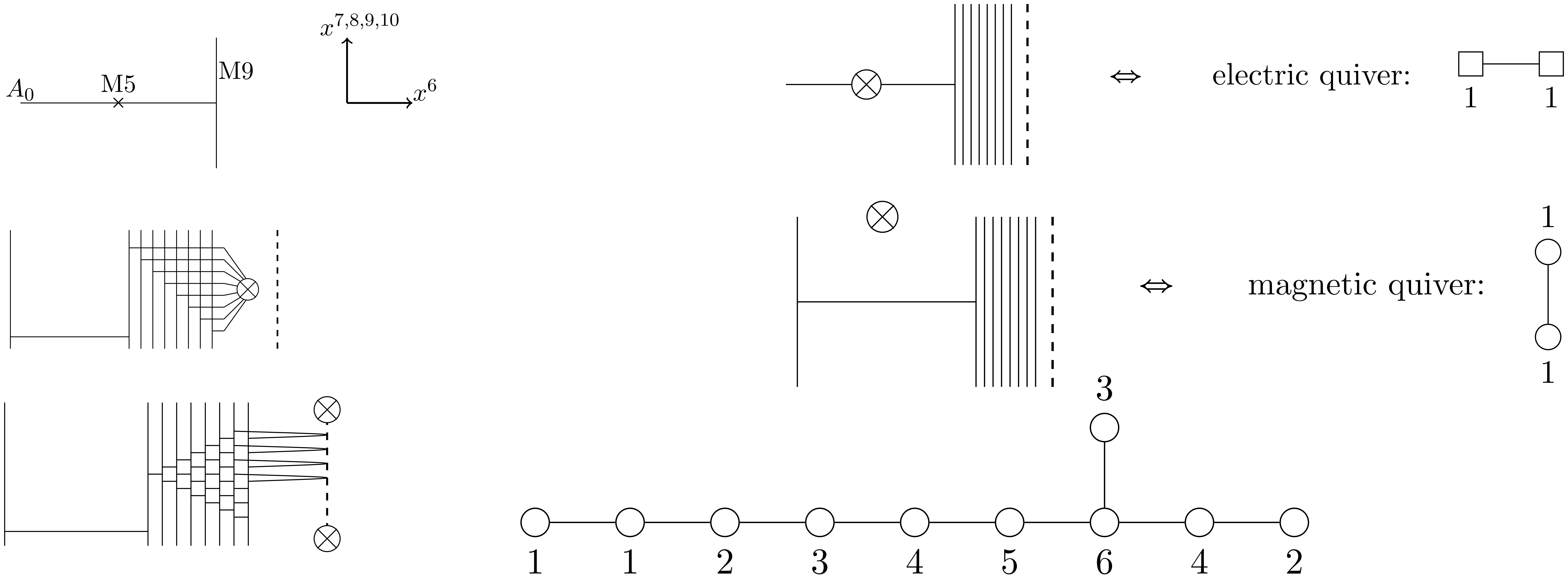
No known Lagrangian or path integral

- A small modification of the monopole formula
- A whole new set of moduli spaces
- A window to exotic moduli spaces
- like rank 1 4d theories



physical effects in 6d

Small instanton transition: 1T <—> 29 H



Hasse (phase) diagrams

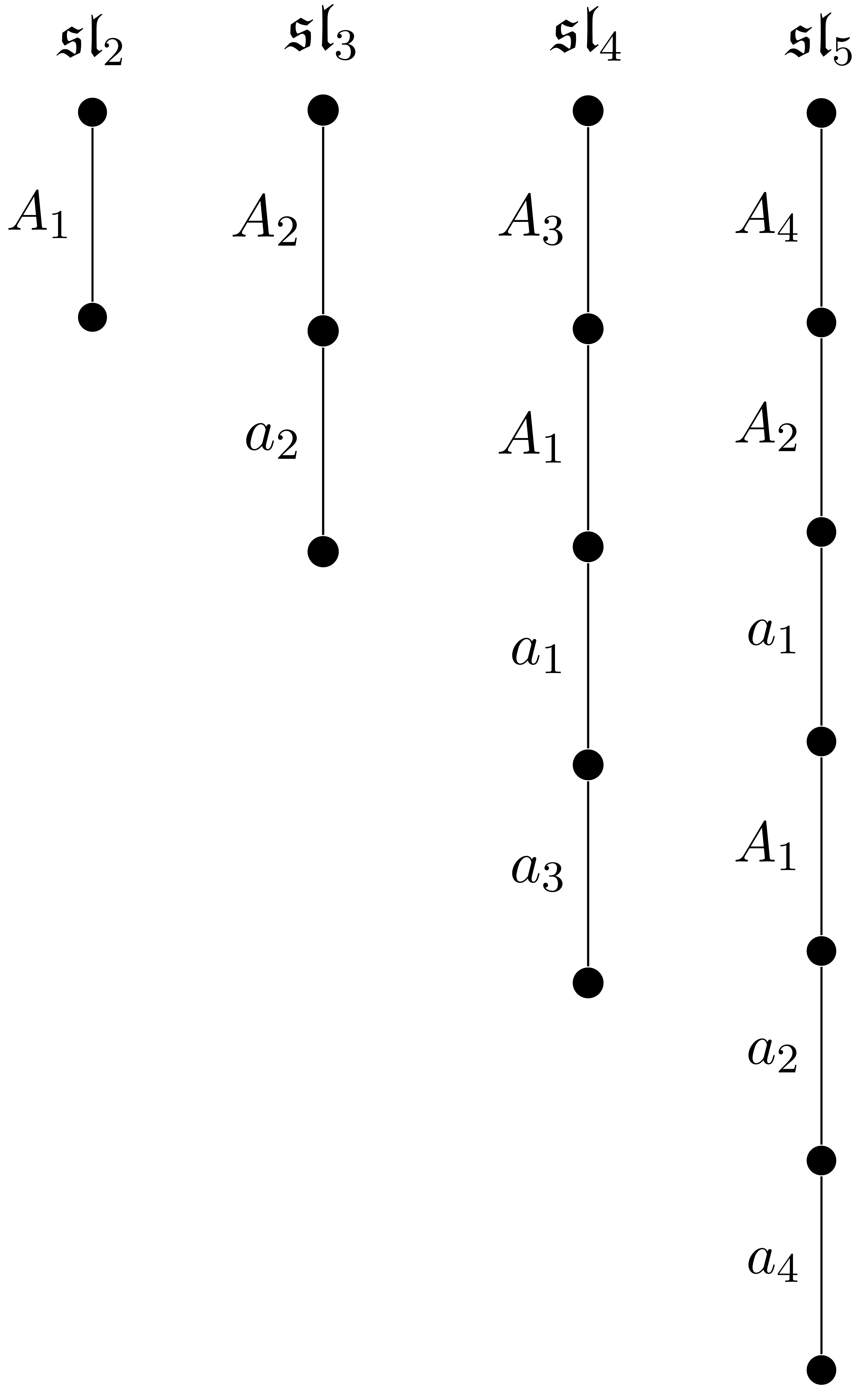
Hasse diagrams

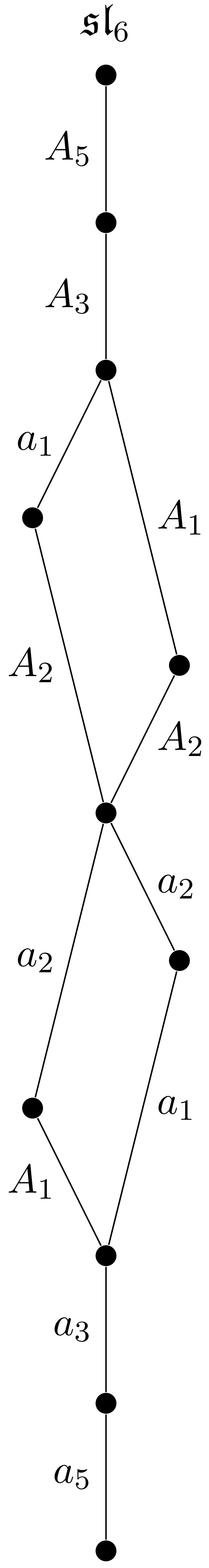
Quiver subtraction

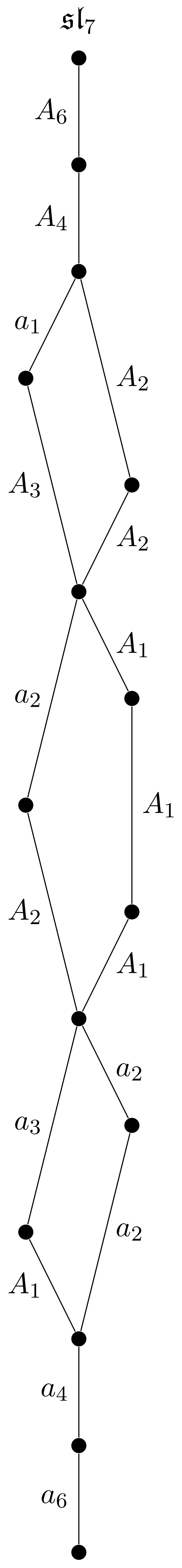
- Recall the work of Kraft and Procesi who classified degenerations in closures of nilpotent orbits
- Minimal degenerations are of two types
- Klein singularity (ADE) — denoted by capital letters
- closure of a minimal nilpotent orbit of some algebra — denoted lower case
- This is reproduced and generalized with the Coulomb branch

Hasse diagrams for nilpotent orbits

taken from KP



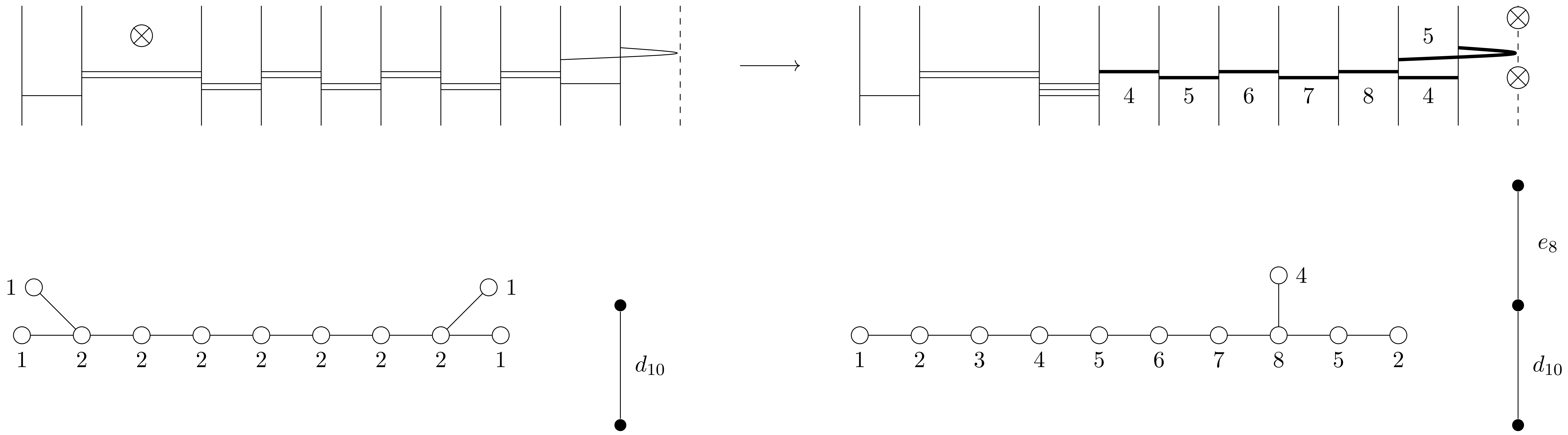




6d — small instanton transition

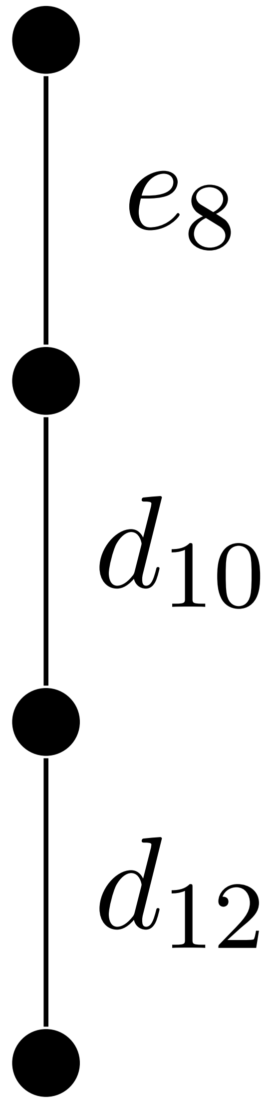
SU(2) with 10 flavors

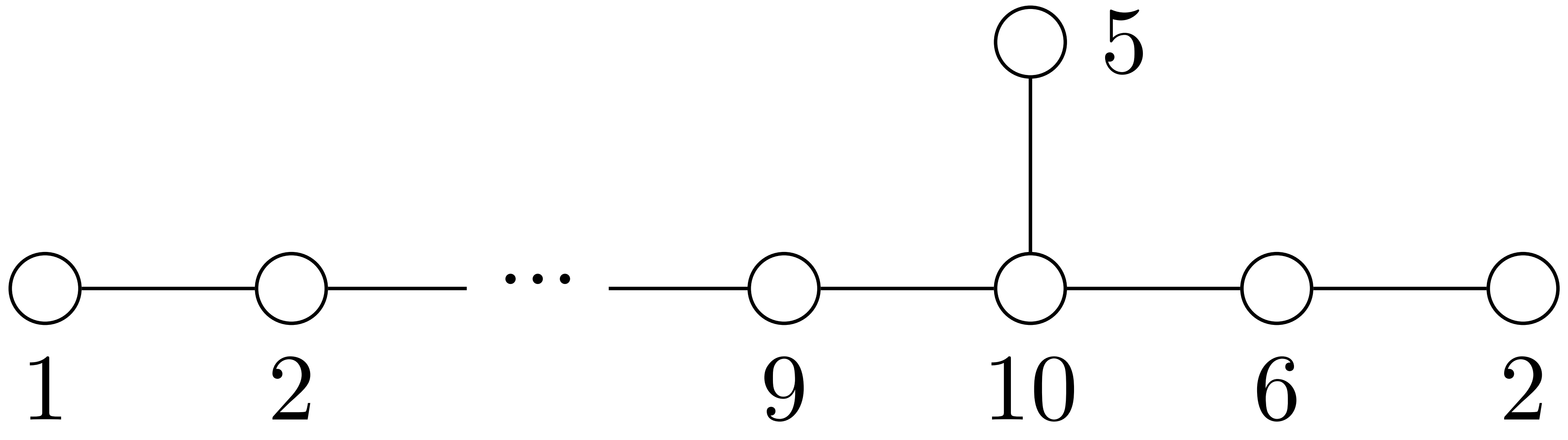
- The Classical Higgs branch — minimal nilpotent orbit of SO(20)
- The moduli space of 1 SO(20) instanton on $\mathbb{C}^2$

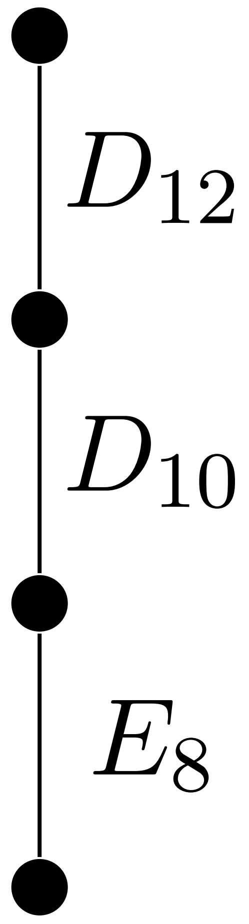


E_8 family $k=2$ — shows up in 5d, 6d

Structure of symplectic leaves — left: Coulomb branch; right: Higgs branch



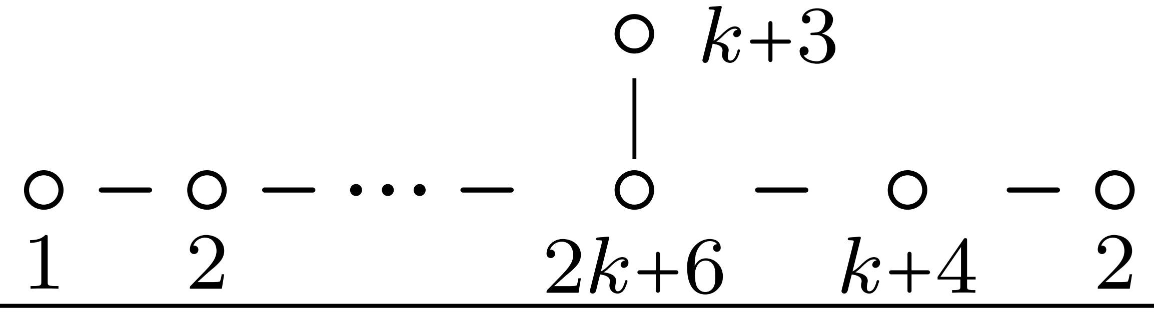
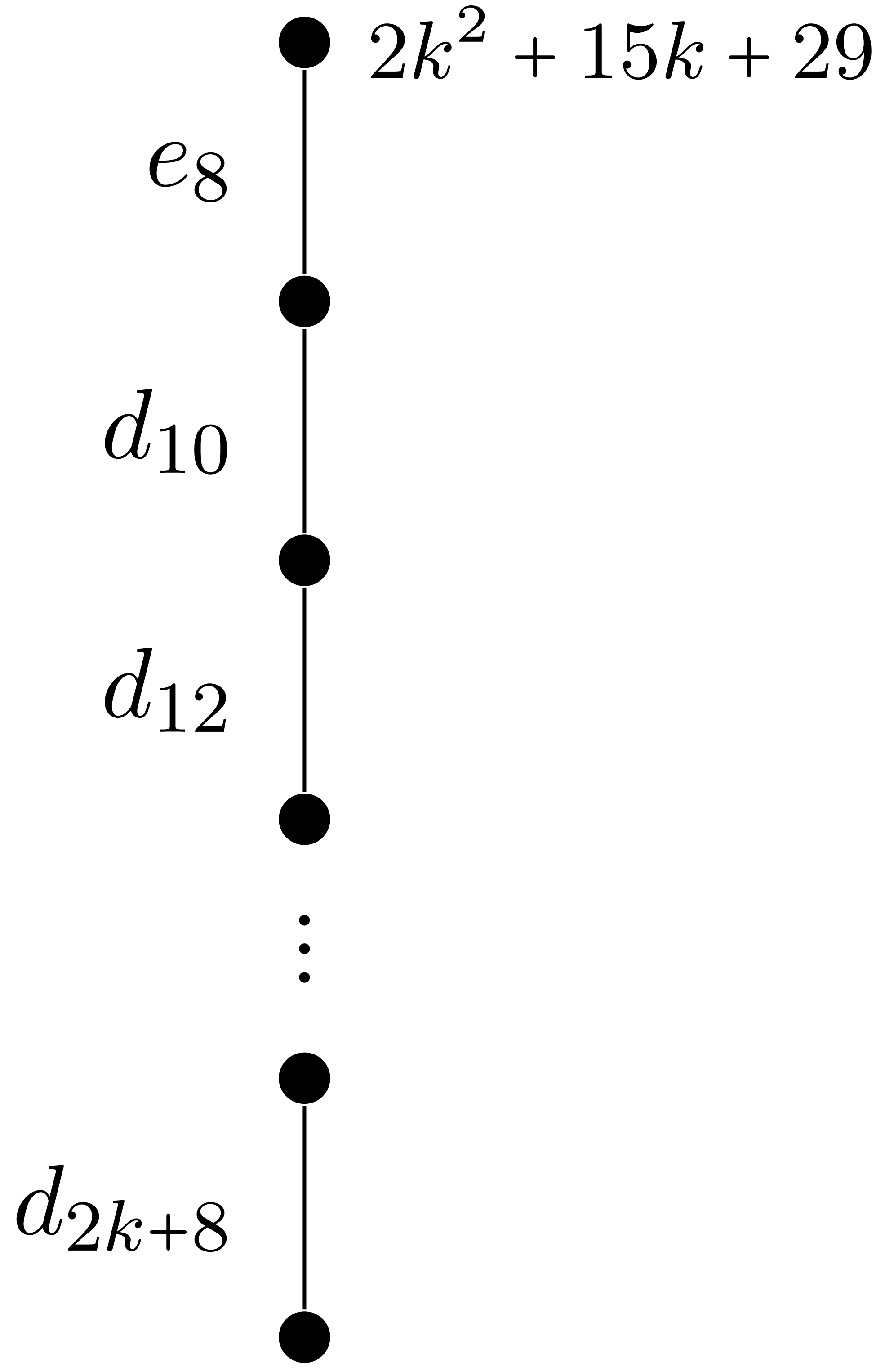
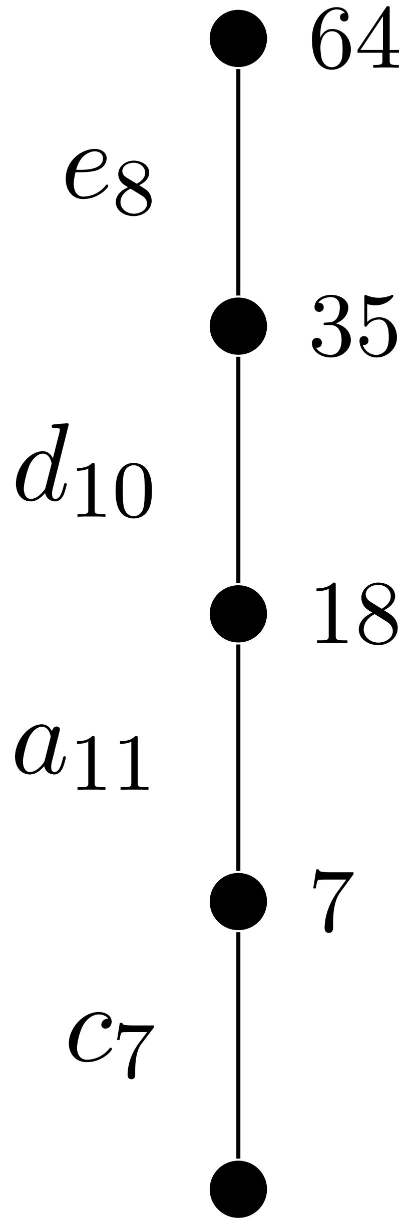




- $$PE\left[\sum_{i=1}^{k+3}\mu_{2i}t^{2i}+t^4+\mu_{2k+8}\left(t^{k+2}+t^{k+4}\right)\right]_{SO(4k+16)}$$

small instanton transition

finite - infinite coupling

<table>
<tr><td>6d SCFT</td><td>Sp(k) with $N = 4k + 16$ flavours</td><td>G_2 with 7 flavours</td></tr>
<tr><td>Magnetic quiver</td><td>
</td><td>Not known</td></tr>
<tr><td>Hasse diagram</td><td>
</td><td>
</td></tr>
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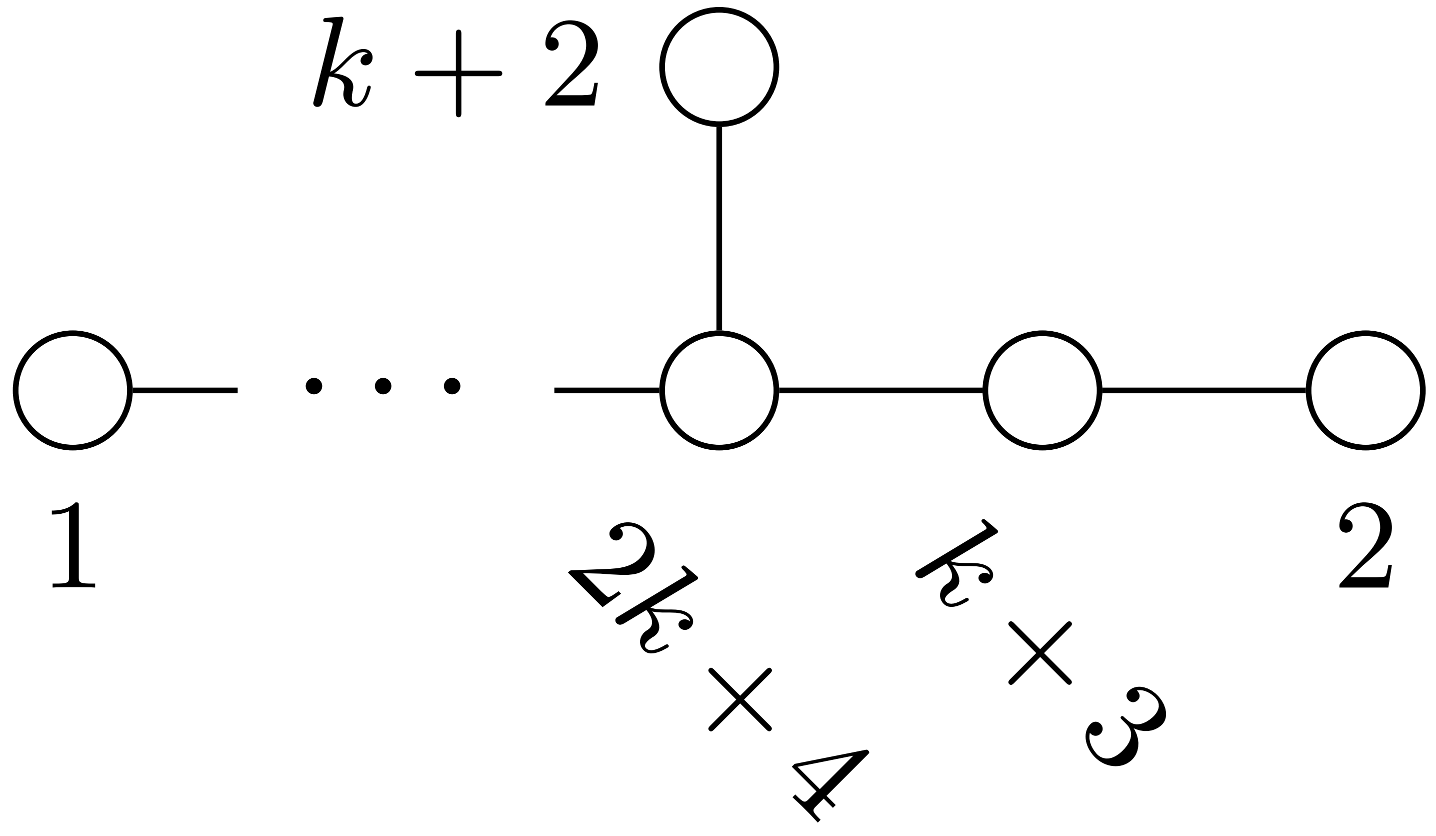
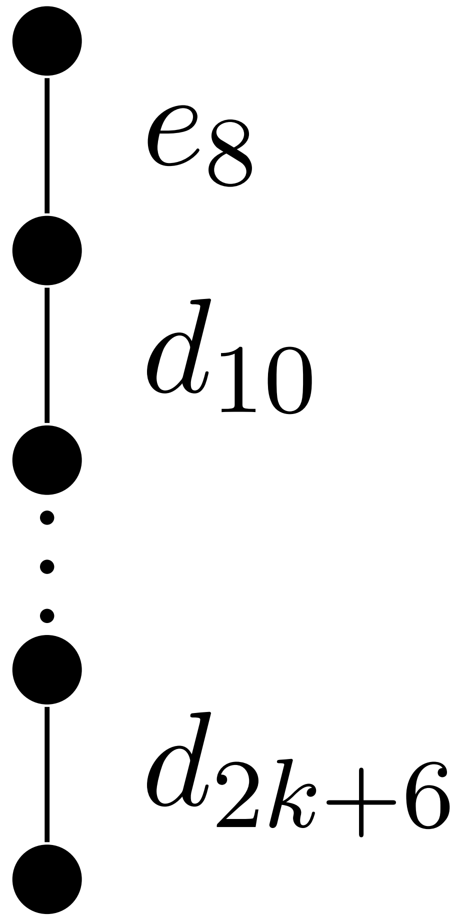
6d — small instanton transition

a family of Sp(k-1) theories with 2k+6 flavors

- The Classical Higgs branch — a nilpotent orbit of SO(4k+12)
- The Higgs branch at infinite coupling with magnetic quiver and Hasse diagram. The HWG is PE of the polynomial below

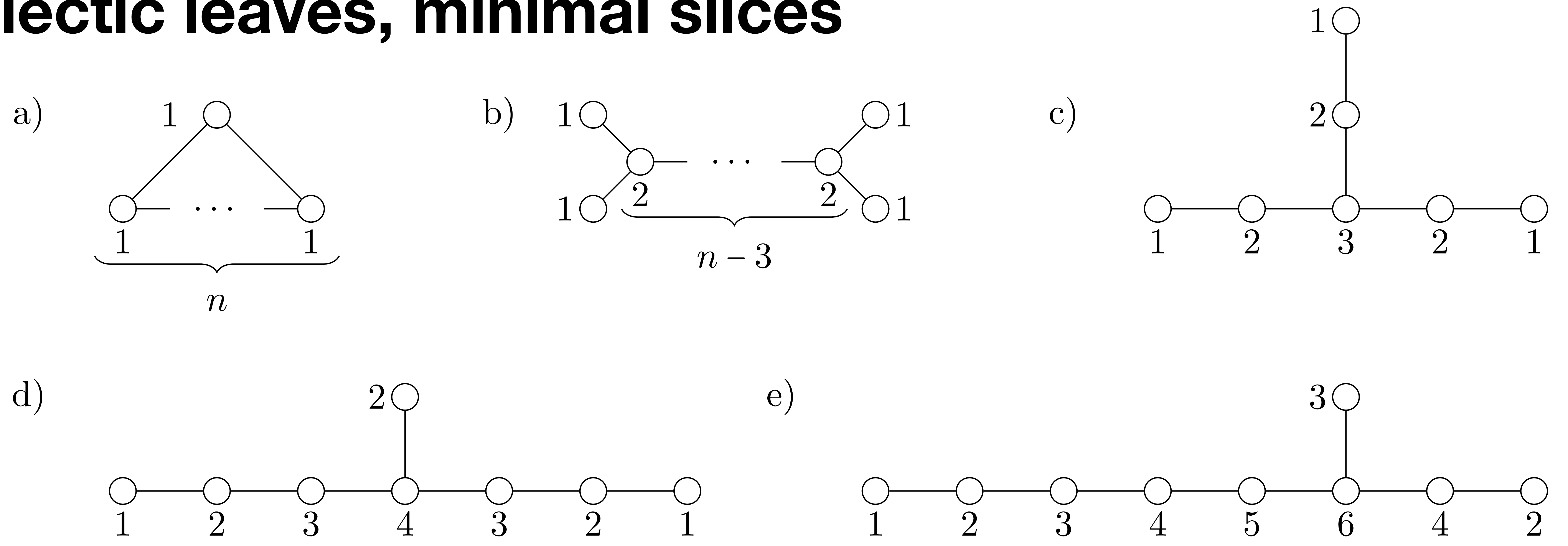


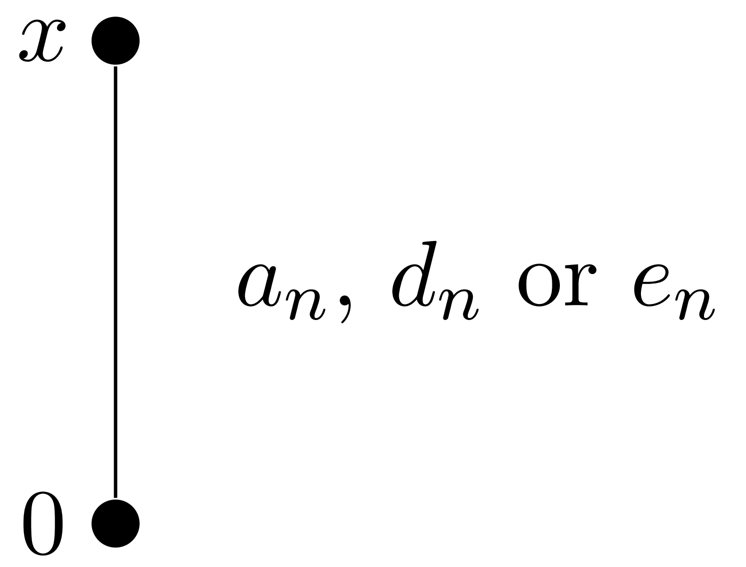


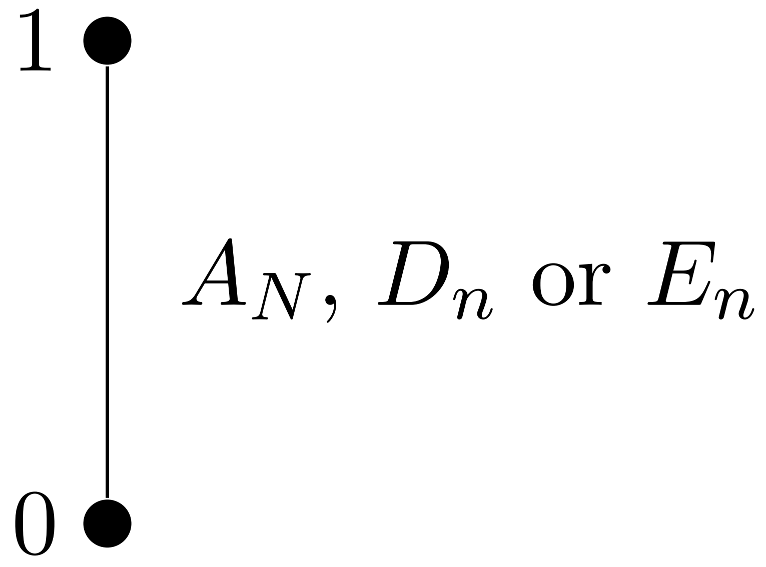
$$\left(\sum_{i=1}^{k+2}\mu_{2i}t^{2i}\right)+t^4+\mu_{2k+6}(t^{k+1}+t^{k+3})$$

Basic Hasse diagrams - affine ADE quivers

2 symplectic leaves, minimal slices



$\mathfrak{H}_C =$ 

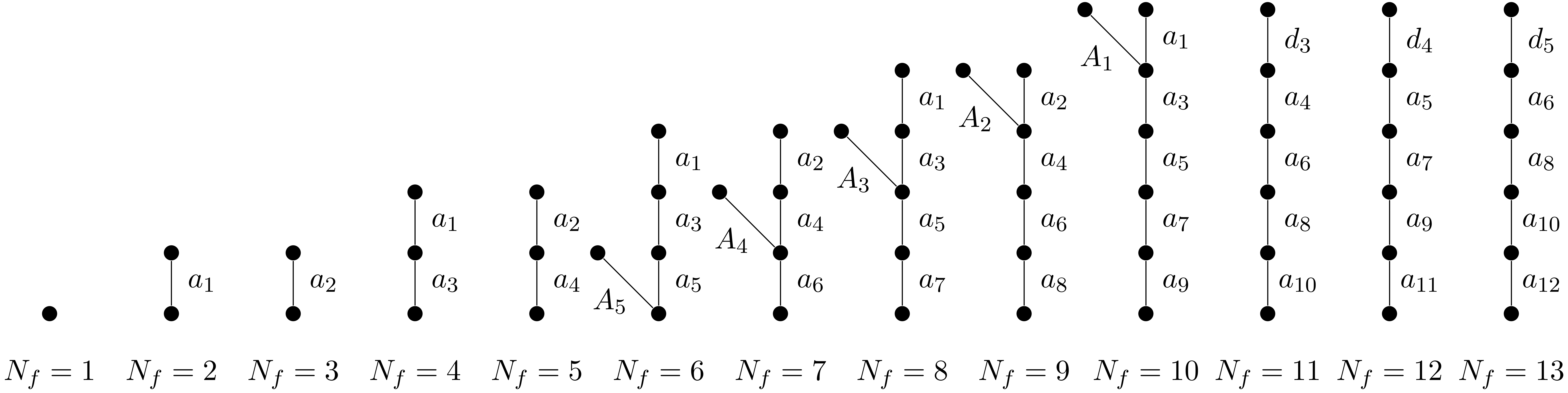
$\mathfrak{H}_H =$ 

.

$$x = \begin{cases} n & \text{for } a_n \\ 2n-3 & \text{for } d_n \\ 11 & \text{for } e_6 \\ 17 & \text{for } e_7 \\ 29 & \text{for } e_8 \end{cases}$$

4d $\mathcal{N} = 2$ SU(6) with fundamental matter

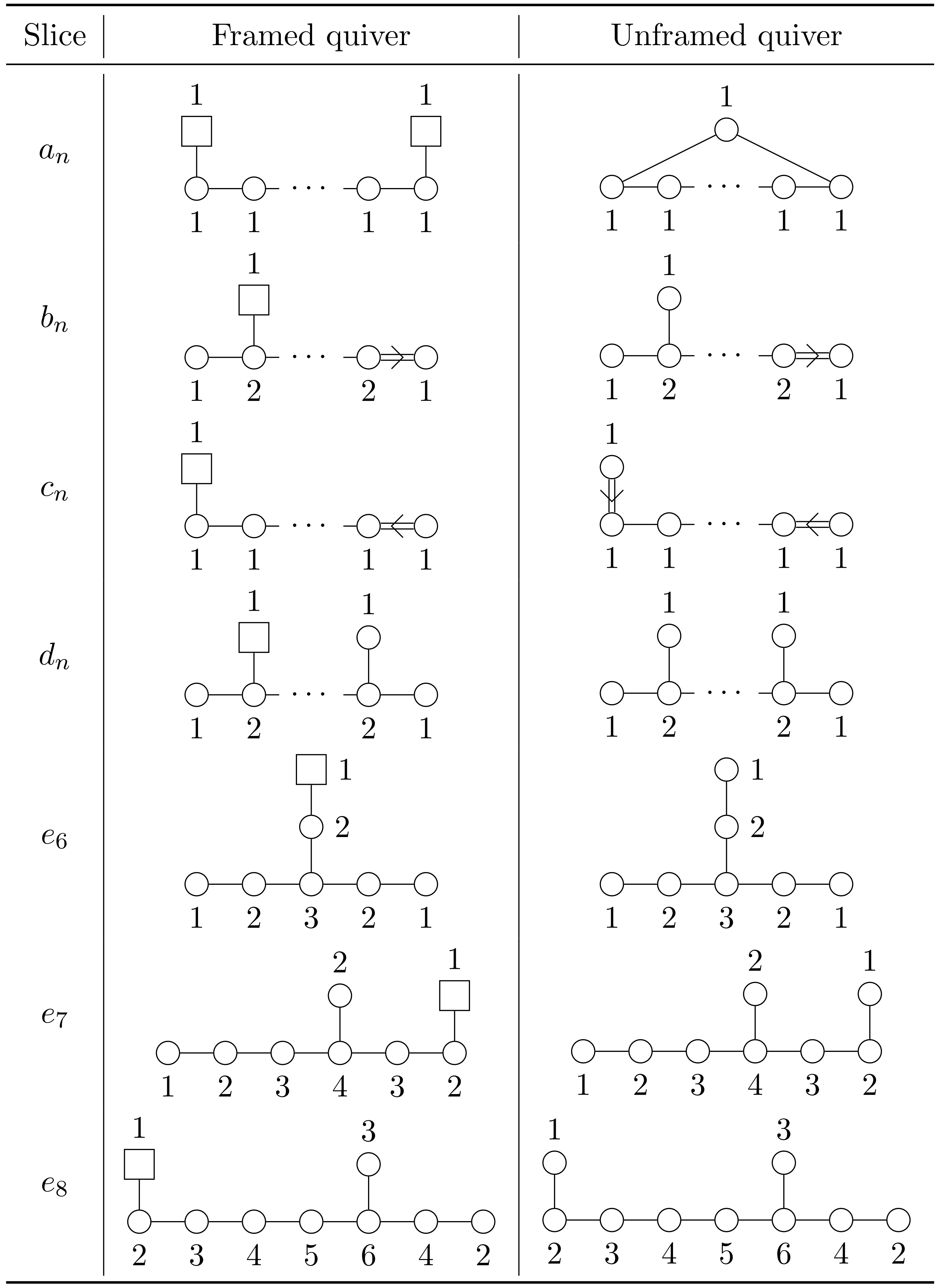
union of 2 cones

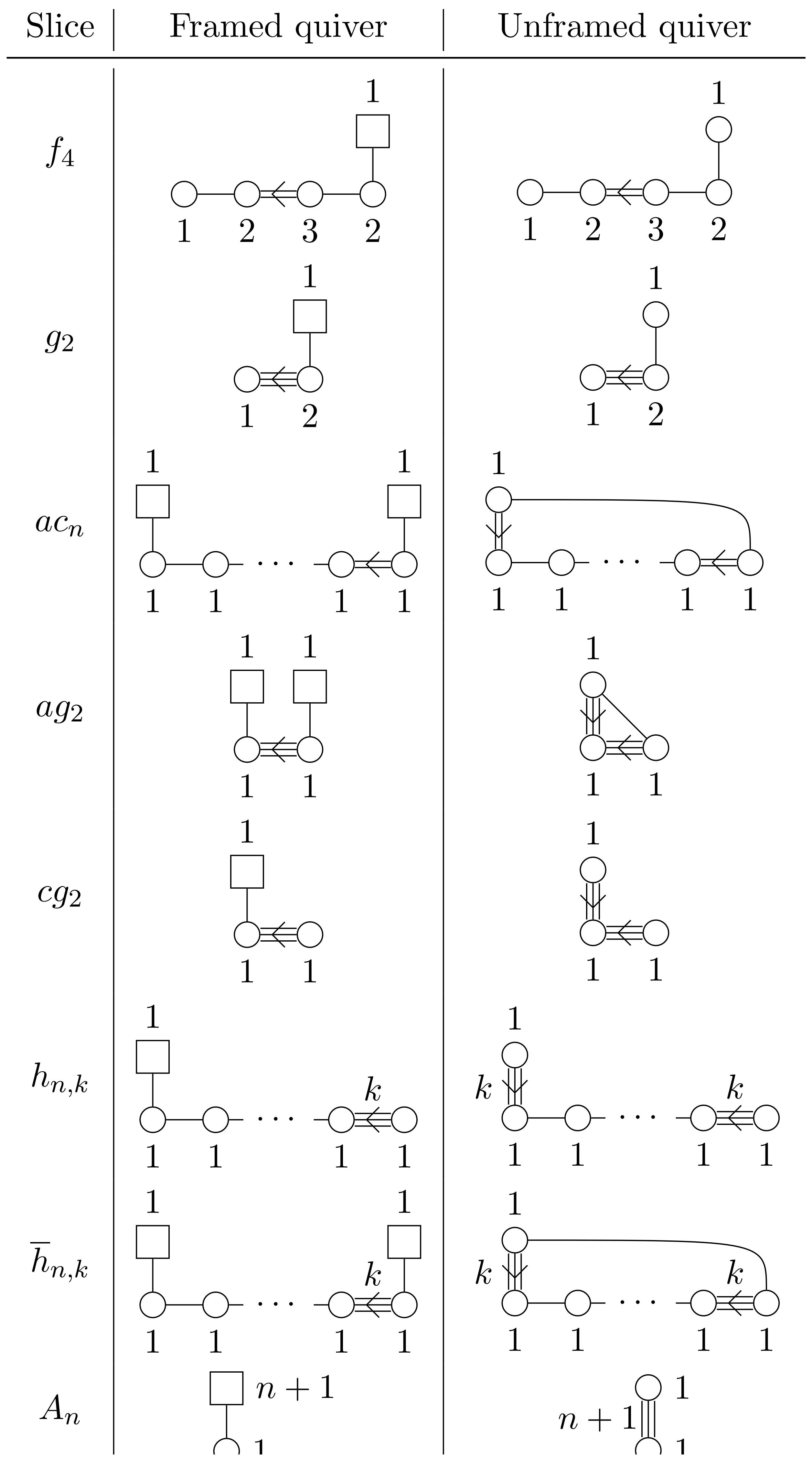


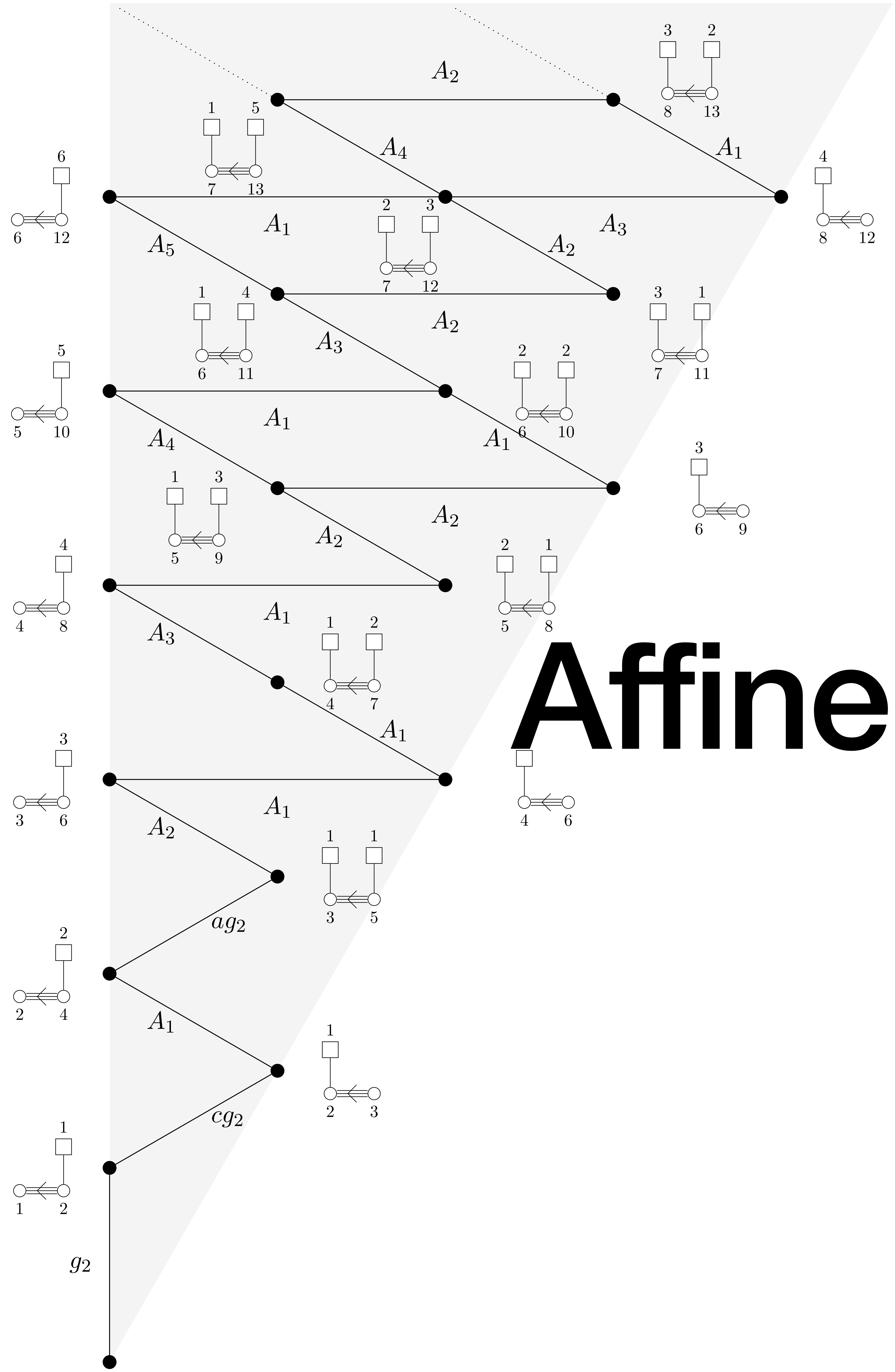
Minimal transverse slices

Symplectic singularities

- A list of minimal transitions



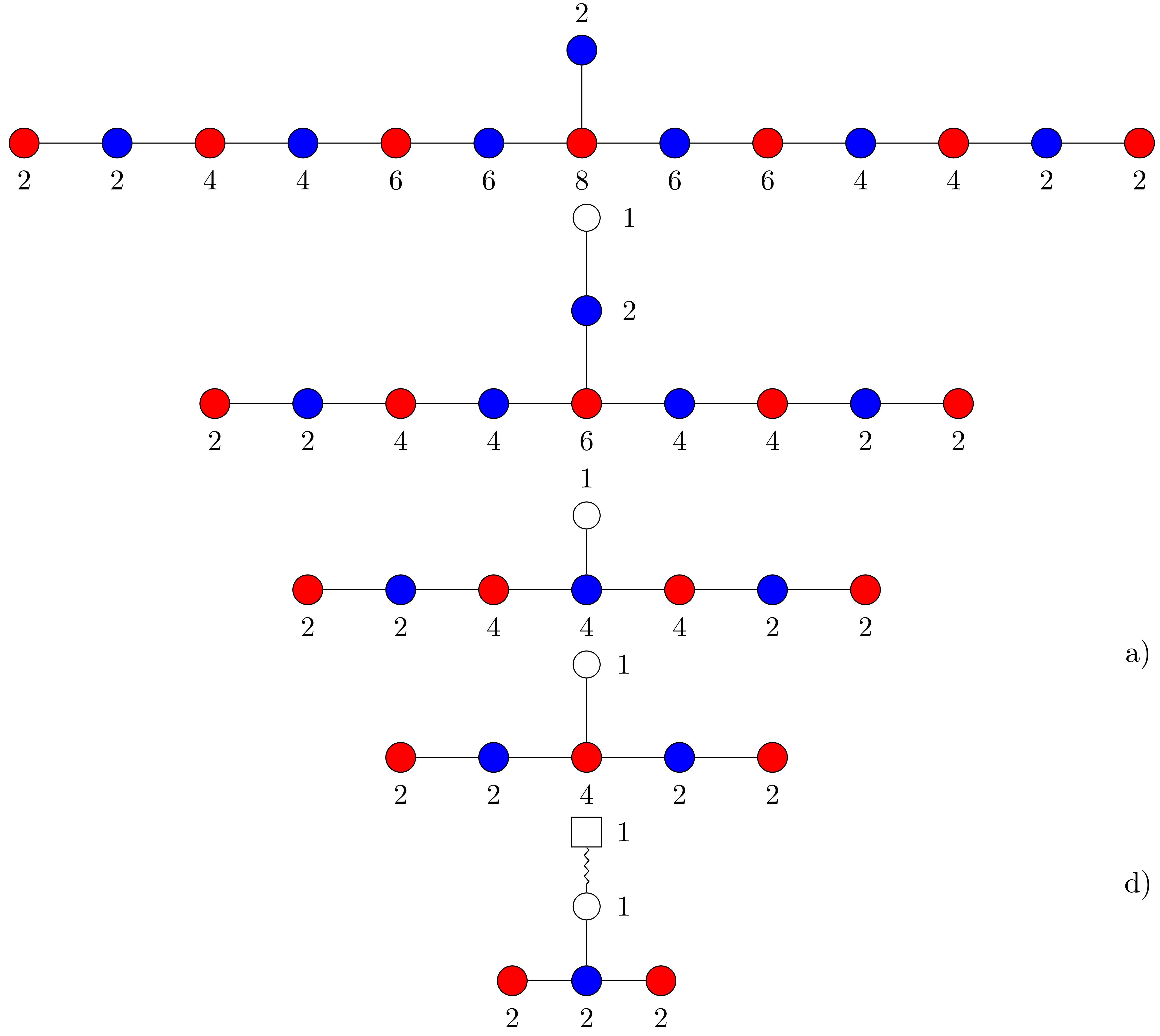


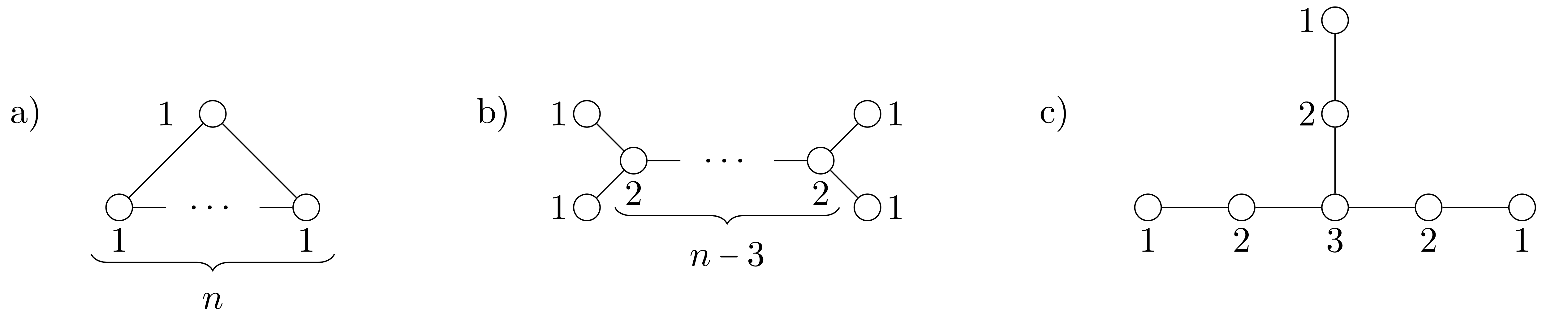


Affine Grasmanian G_2

Orthogonal Symplectic

closures of minimal nilpotent orbits of type E_n for $n = 8,7,6,5,4$





For a future of fruitful discussions!