The sinh-Gordon model and its excited states

Philippe 60

25/6/2024

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1. The basic problem
2. The Ising model
3. Yang-Lee & TBA continuation
4. The sinh-Gordon model
5. Results (including movies)

1. The basic problem

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Often the answer is yes, if we can continue analytically in a parameter: an old idea, see eg Bender & Wu's work* for the Q.M. (anharmonic oscillator) case.

(* "using methods of unknown validity" - B. Simon)

1. The basic problem

Suppose we know the ground-state energy of some quantum system. Can we use this to find the energies of its excited states?

Often the answer is yes, if we can continue analytically in a parameter: an old idea, see eg Bender & Wu's work for the Q.M. (anharmonic oscillator) case.

Big goal: get a similar level of understanding for a QFT.

With Roberto Tateo we made a start, years ago...

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- While we found how the low-lying states were connected, a full picture for the perturbed minimal conformal field theories* we studied remained elusive.

(*see later)

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- While we found how the low-lying states were connected, a full picture for the perturbed minimal conformal field theories we studied remained elusive.

- New angle: look at the sinh-Gordon model, which has an extra parameter – systematic patterns appear as this parameter becomes small.

Toy model:

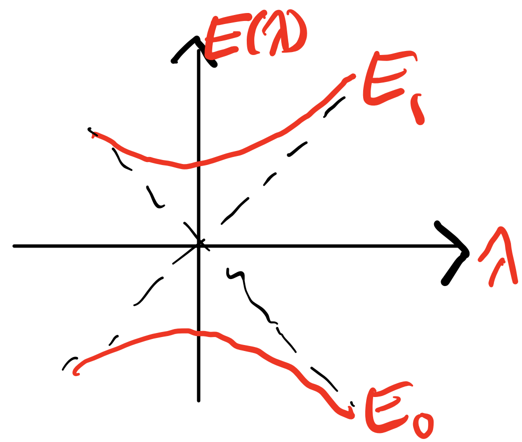
$$H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

"H^{QFT}" = "H^{CFT}" + "V" ~ perturbed C.F.T.

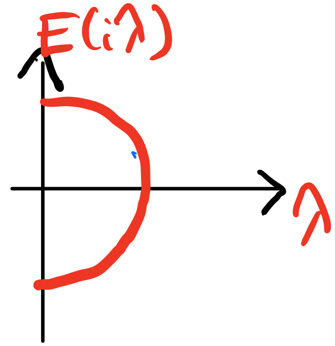
$$H\psi = E\psi \rightsquigarrow \begin{vmatrix} -1-E & \lambda \\ \lambda & 1-E \end{vmatrix} = 0$$

$$\Rightarrow E = \pm\sqrt{1+\lambda^2}$$

So for $\lambda \in \mathbb{R}$, two disconnected energy levels:

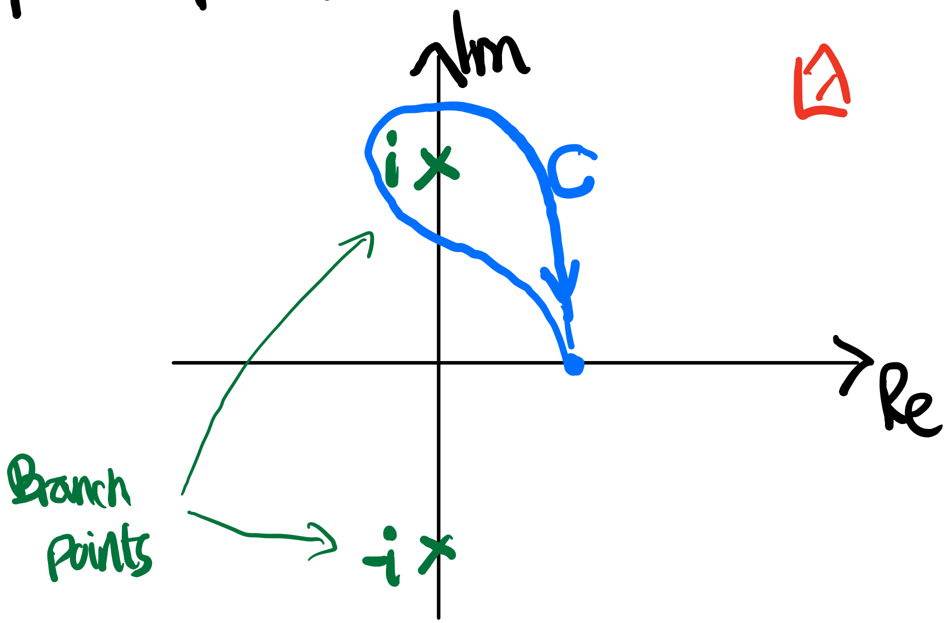


... but $E(;\lambda) = \sqrt{1-\lambda^2}$



– now just one level!

Complex λ plane:



Continuing round C swaps from E_0 to E_1 and back.

Message: in eigenvalue problems, continuing a parameter round a closed contour returns to the same problem, but not necessarily the same eigenvalue. Since the problem hasn't changed, the analytically-continued eigenvalue, if different, must be one of the other eigenvalues of the original problem.

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- In QM or QFT we'll restrict to confining potentials [QM] or put our system in a finite spatial box [QFT] so as to have a discrete spectrum to continue.

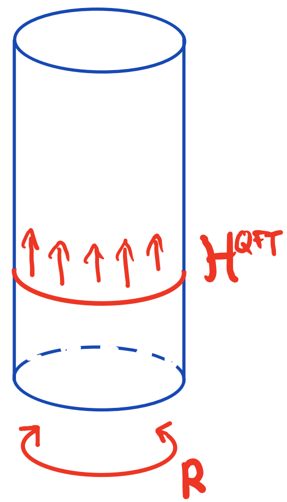
- Even so we might worry [see later]

(NB: we won't necessarily see all other levels - there may be disconnected sectors.)

2. A simple example

Ising field theory on a cylinder

- Off-critical Ising model in thermal direction.
- One mass m, correlation length $\xi = 1/m$.



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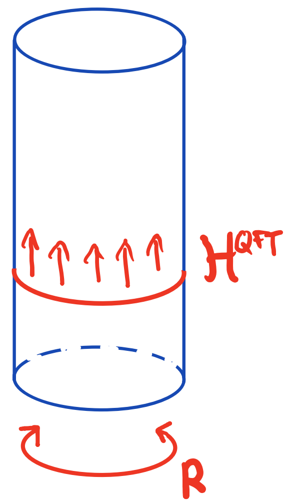

- Off-critical Ising model in thermal direction.
- One mass m, correlation length $\xi = 1/m$.

Ground state energy on a circle is known in closed form:

$$E_0(m,R) = E_{bulk}(m,R) - \frac{\pi}{6R}\, c^{(Ising)}(mR)$$

Sometimes called the "effective central charge".

$mR = r$ (system size in units of correl. length)

with
$$c^{(Ising)}(r) = \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\log\frac{1}{r} + \frac{1}{2} + \ln\pi - \gamma_E\right] + \frac{6}{\pi}\sum_{k=1}^{\infty}\left(\sqrt{r^2+(2k-1)^2\pi^2} - (2k-1)\pi - \frac{r^2}{2(2k-1)\pi}\right)$$

and $E_{bulk} \propto R^2 \log R$ ← bulk term

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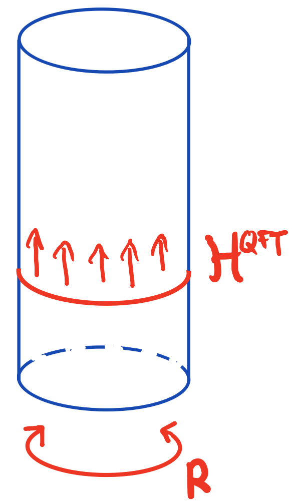

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and $E_{bulk} \propto R^2 \log R$

Note: $c(0) = \frac{1}{2}$ so $E_0(R\to 0) \sim -\frac{\pi}{6R}\cdot\frac{1}{2} = \frac{2\pi}{R}\Big(\underbrace{0+0-\frac{1/2}{12}}_{d+\bar{d}-c/12}\Big) = \overbrace{\frac{2\pi}{R}\left(d+\bar{d}-\frac{c}{12}\right)}$

Correct for the ground state $(d=\bar{d}=0)$ in the $c=1/2$ CFT found as $R\to 0$

Continuation in r:

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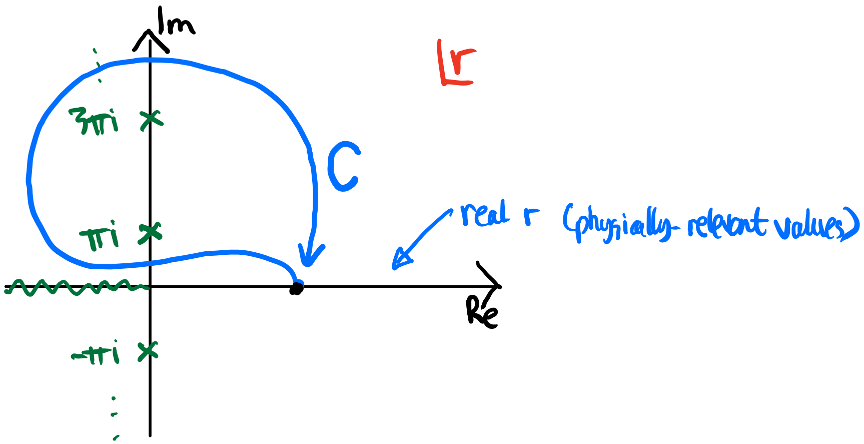
∞ sequence of $\sqrt{\ }$s

Complex r plane:



$\sqrt{\ }$ branch points at $r = (2k-1)\pi i$, $k \in \mathbb{Z}$.

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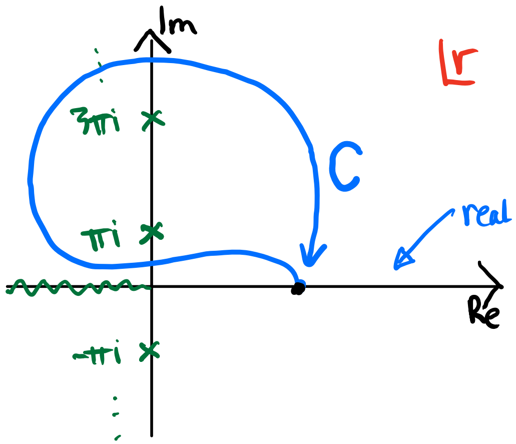

$\boxed{r}$

real r (physically relevant values)

$\sqrt{\ }$ branch points at $r = (2k-1)\pi i,\ k \in \mathbb{Z}$.

Take C around $k_1, k_2, \dots k_n$. This flips the signs of the square roots in E_0 from minus to plus.

Return to real axis to find $E_{k_1\dots k_n}(m,R) = E_0(m,R) + \frac{2}{R}\sum_{i=1}^{n}\sqrt{r^2+(2k_i-1)^2\pi^2}$ ← An excited state!

[+ sign since we flipped $-\sqrt{\ }$ to $+\sqrt{\ }$]

[Correct for $2n$ particles on a circle, momenta $\pm(2k_i-1)$, BA states with S matrix $S=-1$.]

Alternative viewpoint, closer to the Thermodynamic Bethe Ansatz:

- Another expression for $c^{(\mathrm{ising})}(r)$:

$$c^{(\mathrm{ising})}(r) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta \; r\cosh\theta \, \log\left(1+e^{-r\cosh\theta}\right)$$

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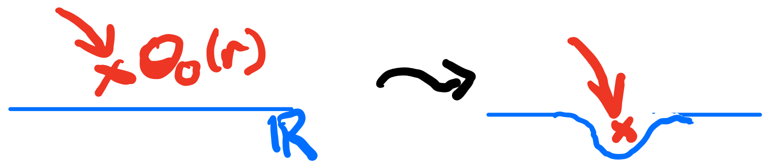
- Usually, you can distort the integration contour ahead of the trouble:



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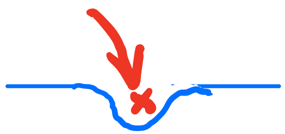

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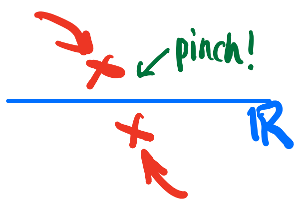
- This fails if two singularities approach the contour from opposite sides: a <u>pinch singularity</u>:



.

This generates the branch points!

3. Yang-Lee and TBA continuation

Problem: in general we don't have a closed form for $E_0(R)$. But for integrable QFTs we do know the TBA equation exactly. This is enough to continue it to an "excited TBA" equation, and use this to explore the connectivity of the finite-size energy levels in the complex r plane.

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Simplest example is the Yang-Lee model, a perturbation of the M_{25} minimal CFT:

$$H^{QFT}_{YangLee} = \underset{\uparrow\, c=-22/5}{H^{CFT}_{M(2,5)}} + \lambda \int \underset{\uparrow\, d=\bar{d}=-1/5}{\varphi(x)}\, d^2x$$

- The ∞ volume theory has just one particle type and a very simple S-matrix $S(\theta) = \dfrac{\sinh\theta + i\sin\pi/3}{\sinh\theta - i\sin\pi/3}$

  (θ = rapidity; momentum = $(p_0, p_1) = m(\cosh\theta, \sinh\theta)$)
- Its mass is $m(\lambda) = (2.642\ldots)(-\lambda)^{5/12}$
- Since $r = m(\lambda)R$ we can equivalently think of our procedure as analytic continuation in the coupling λ.

  (inf. vol. spectrum is real for λ negative)

TBA recipe to find $c(r)$:

Solve $\varepsilon(\theta) = r\cosh\theta - \phi * L(\theta)$ for the auxillary function $\varepsilon(\theta)$ (the pseudoenergy), where

$$L(\theta) = \log(1 + e^{-\varepsilon(\theta)})$$

$$f * g(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta')$$

$$\phi(\theta) = -i\frac{\partial}{\partial\theta}\log S(\theta)$$

[NB: if θ is real, $S(\theta) = e^{i\delta(\theta)}$ is a pure phase so $\phi(\theta)$ is also real]

Then $$c(r) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta\, r\cosh\theta\, L(\theta)$$

and $$E_0(m,R) = E_{bulk}(m,R) - \frac{\pi}{6R}c(r)$$

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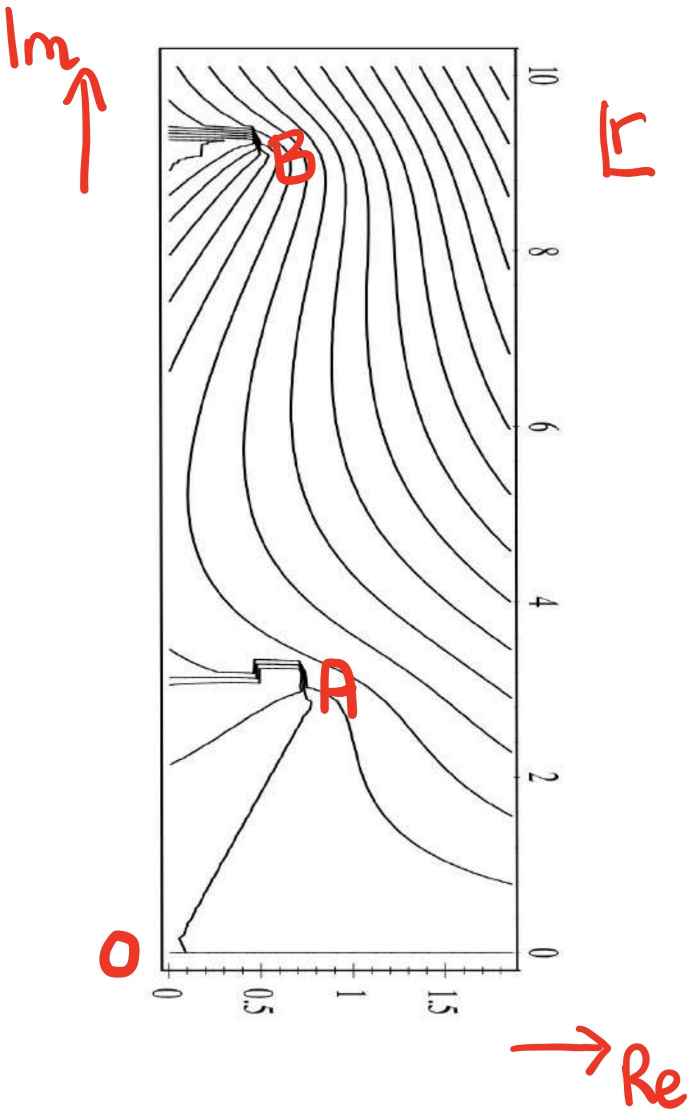
Usually we solve this for real r. But nothing stops us from making r complex, solving on a computer, and plotting the results...

Plot of $\mathrm{Im}(c(r))$ from ground-state YL TBA at complex r



Note A and B look like $\sqrt{\ }$ branch points – can suspect similar causes to the Ising field theory case, but more subtle since the TBA equation, as well as the integral giving $c(r)$, may undergo monodromy.

(NB: $\mathrm{Im}(c(r)) = 0$ on real axis, but also on the line OA. This line corresponds to λ real but positive rather than negative – so $\arg(r) = \frac{5\pi}{12}$ since $r = mR$ and $m \propto (-\lambda)^{5/12}$.)

Basic mechanism:

$L(\theta) = \log(1 + e^{-\ell(\theta)})$ has singularities in the complex θ plane when $e^{-\ell(\theta)} = -1$ (& also when $e^{\ell(\theta)} = 0$ but these won't be so important)

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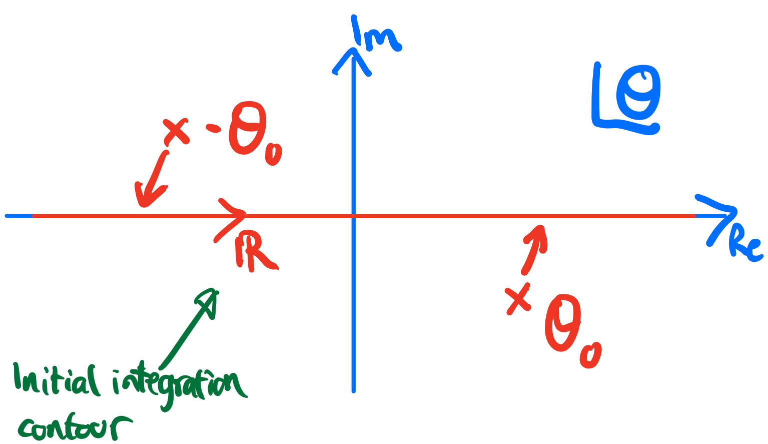
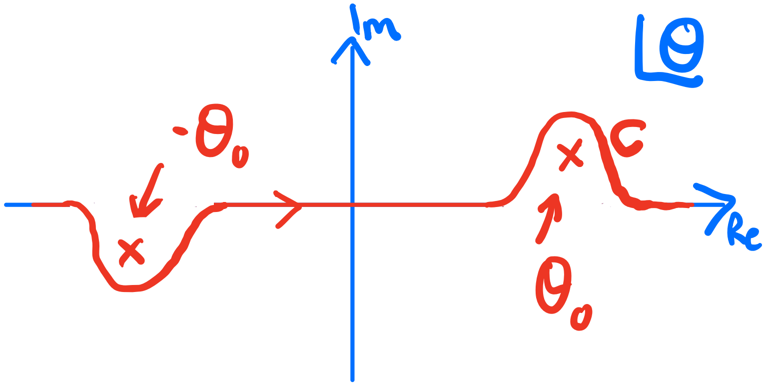


Then the TBA convolution

$$\phi * L(\theta) = \frac{1}{2\pi}\int_{\mathbb{R}} \phi(\theta-\theta')\log(1+e^{-\varepsilon(\theta')})\,d\theta'$$

is in danger...

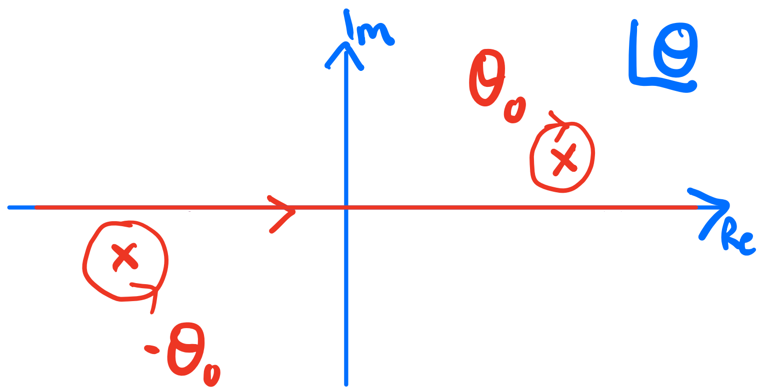
Step 1: avoid the problem by distorting the contour:



$$\phi * L(\theta) \to \phi *_C L(\theta) = \frac{1}{2\pi}\int_C \phi(\theta-\theta')\log\left(1+e^{-\varepsilon(\theta')}\right)d\theta'$$

This gives the correct analytic continuation of the equation.

Step 2: return the contour to the real axis:



The relevant residue terms can be found explicitly...

To find the residue, integrate by parts:

$$\phi *_c L(\theta) = \frac{1}{2\pi} \int_c \phi(\theta-\theta') \log(1+e^{-\varepsilon(\theta')}) d\theta'$$

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$$\phi *_C L(\theta) = \frac{1}{2\pi i}\int_C \log S(\theta-\theta') \, \frac{\varepsilon'(\theta')}{1+e^{\varepsilon(\theta')}} \, d\theta'$$

[Recall $\phi(\theta) = -i\frac{\partial}{\partial\theta}\log S(\theta)$]

[$\frac{\partial}{\partial\theta}\log\left(1+e^{-\varepsilon(\theta)}\right) = \frac{-\varepsilon'(\theta)}{1+e^{\varepsilon(\theta)}}$]

To find the residue, integrate by parts:

$$\phi *_c L(\theta) = \frac{1}{2\pi i}\int_c \log S(\theta-\theta') \frac{\varepsilon'(\theta')}{1+e^{\varepsilon(\theta')}}\, d\theta'$$

Useful fact: if $e^{-\varepsilon(\theta_0)} = -1$, the residue of this at θ_0 is equal to -1.

Hence $$\phi *_c L(\theta) = \underbrace{\frac{1}{2\pi}\int \phi(\theta-\theta')L(\theta')\,d\theta'}_{\phi * L} - \underbrace{\log\frac{S(\theta-\theta_0)}{S(\theta+\theta_0)}}_{\text{Extra term from the two residues at } \theta = \pm\theta_0.}$$

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Likewise $c(r)$ gets an extra bit:

$$c(r) = \frac{12r}{\pi}\cdot i\sinh\theta_0 + \frac{3}{\pi^2}\int_{\infty}^{\infty} r\cosh\theta\, L(\theta)\,d\theta$$

- The new TBA equation has an extra unknown: θ_0, the location of the singularity which crossed the integration contour.

- It's fixed by imposing $e^{\varepsilon(\theta_0)} = -1 \Rightarrow \varepsilon(\theta_0) = (2N+1)\pi i$

- N maps onto the large-volume Bethe Ansatz number(s) for the new state which has been generated.

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A natural question: can this be repeated to build the full surface for $c(r)$, and find how the states with different Ns are connected to each other?

Grand plan:

- Start with the ground-state TBA, continue to find connectivity with first excited-state TBAs, continue those to connect to higher excited-state TBAs and so on, to "bootstrap" to the full Riemann surface of $c(r)$.

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Unfortunately this is hard! Numerical iteration of TBA equations tends not to converge near branch points, and the low-lying levels don't show any obvious patterns, so the full picture for for Yang-Lee is still missing.

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Instead try a different model with more structure: sinh-Gordon.

4. Sinh-Gordon case

Again a single type of particle, but now the S-matrix depends on a parameter p:

$$S(\theta) = \frac{\sinh(\theta) - i\sin\pi p}{\sinh(\theta) + i\sin\pi p}$$

This has zeroes (not poles) at $i\pi p$, $i\pi(1-p)$, and maps to itself under $p \to 1-p$.

The TBA is as before

$$\varepsilon(\theta) = r\cosh\theta - \frac{1}{2\pi}\int_{\mathbb{R}} \phi(\theta.\theta')L(\theta')\,d\theta'$$

where $L(\theta) = \log(1+e^{-\varepsilon(\theta)})$, $\phi(\theta) = -i\frac{\partial}{\partial\theta}\log S(\theta)$.

This provides a representation of $\varepsilon(\theta)$ for $-\pi p < \operatorname{Im}\theta < \pi p$.

This was analysed in detail by Al. Zamolodchikov (JPA 2008).

The continuous parameter p complicates matters!

Al. Z. introduced two functions

$$Y(\theta) = e^{-\varepsilon(\theta)} = \exp\left(-r\cosh\theta + \frac{1}{2\pi}\int_{\mathbb{R}} \phi(\theta-\theta')L(\theta')d\theta'\right)$$

(holds for $-\pi p < \mathrm{Im}(\theta) < \pi p$)

and

$$X(\theta) = \exp\left(-\frac{r}{2\sin\pi p}\cosh\theta + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\cosh(\theta-\theta')} L(\theta')d\theta'\right)$$

(holds for $-\frac{\pi}{2} < \mathrm{Im}(\theta) < \frac{\pi}{2}$)

Note the initial definition of X holds on a wider strip than Y.

X, Y and X·Y systems

Set $a = 1-2p$. Then

$$X\left(\theta+\tfrac{i\pi}{2}\right)X\left(\theta-\tfrac{i\pi}{2}\right)=1+X\left(\theta+\tfrac{ia\pi}{2}\right)X\left(\theta-\tfrac{ia\pi}{2}\right)$$

$$Y\left(\theta+\tfrac{i\pi}{2}\right)Y\left(\theta-\tfrac{i\pi}{2}\right)=\left(1+Y\left(\theta+\tfrac{ia\pi}{2}\right)\right)\left(1+Y\left(\theta-\tfrac{ia\pi}{2}\right)\right)$$

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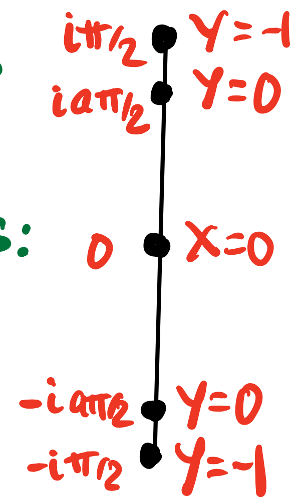
Special values of X and Y form quintets:



Armed with these relations, can extend X and then Y to the whole complex plane, starting from ε on the real axis.

X, Y and X·Y systems

Set $a = 1 - 2p$. Then

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*Aside: for $p = \frac{1}{2}$ (the self-dual point) the X-system is

$X\left(\theta+\frac{i\pi}{2}\right)X\left(\theta-\frac{i\pi}{2}\right) = 1 + X(\theta)^2$ which is the recurrence for the cluster algebra of type $A_1^{(1)}$ [see Zelevinsky arXiv:math/0606775]

The $\rho\to 0$ limit

As $\rho\to 0$ the kernel $\phi(\theta)$ concentrates near $\theta=0$ and $\phi(\theta-\theta')$ can be replaced in the TBA by $2\pi\delta(\theta-\theta')$. Then the equation becomes

$$\varepsilon(\theta)=r\cosh\theta-\int_{\mathbb{R}}\delta(\theta-\theta')L(\theta')d\theta'=r\cosh\theta-L(\theta)$$

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$$\varepsilon(\theta) = r\cosh\theta - \int_{\mathbb{R}} \delta(\theta-\theta') L(\theta') d\theta' = r\cosh\theta - L(\theta)$$

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This resembles $c^{(\text{ising})}(r)$ but with some sign flips – it is $c_0(r)$, the effective central charge of a free boson on a circle, with $c_0(0) = 1$.

Maybe not surprising, but there's a problem here...

The problem:

Just as with $c^{(\text{ising})}(r)$, there's an alternative formula for $c_0(r)$ [see Saleur & Itzykson 1987, Klassen & Melzer 1991]

$$c_0(r) = 1 - \frac{3r}{\pi} + \frac{3r^2}{2\pi^2}\left[\ln\frac{1}{r} + \frac{1}{2} + \ln 4\pi - \gamma_E\right]$$
$$- \frac{6}{\pi}\sum_{k=1}^{\infty}\left(\sqrt{(2k\pi)^2 + r^2} - 2k\pi - \frac{r^2}{4k\pi}\right)$$

cf: $$c^{(\text{ising})}(r) = \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\log\frac{1}{r} + \frac{1}{2} + \ln\pi - \gamma_E\right]$$
$$+ \frac{6}{\pi}\sum_{k=1}^{\infty}\left(\sqrt{r^2 + (2k-1)^2\pi^2} - (2k-1)\pi - \frac{r^2}{2(2k-1)\pi}\right)$$

- This time the branch points are at even multiples of $i\pi$, not odd ones.
- But more crucially, the signs of the square roots are reversed!

Why might this be a problem?

- When continuing $c^{(\text{ising})}(r)$ around branch points from the ground state, the flipped signs of the square roots led to an <u>increase</u> in $E(m, R)$ and excited states with higher energies than the ground state. [good!]

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- But the square roots in $c_0(r)$ start with the opposite sign, so flipping any of them decreases $E(m,R)$, leading to "states" with lower energy than the ground state. [bad!]

Why might this not be a problem?

- Toy example: SHO (free boson in a universe with one point)

$$\left(-\frac{d^2}{dx^2}+\nu^2x^2\right)\psi = H_\nu\psi = E\psi \quad (*)$$

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- Usually demand $\psi \in L^2(C)$, where $C=\mathbb{R}$,

$$\rightarrow \quad E=(2n+1)\omega \quad (n=0,1,2,\ldots)$$

- But if we set $\nu = re^{i\phi}$ (r real) and continue ϕ from 0 to π, then the eigenvalues change sign even though $H_\nu \rightarrow H_{-\nu} = H_\nu$.

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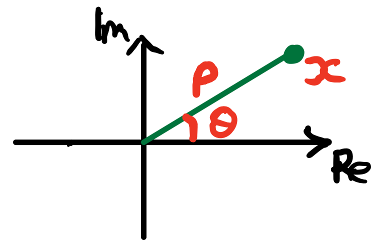
  - one of $\psi_\pm$ grows as $\rho \to \infty$ (is dominant)
  - while the other shrinks (is subdominant)



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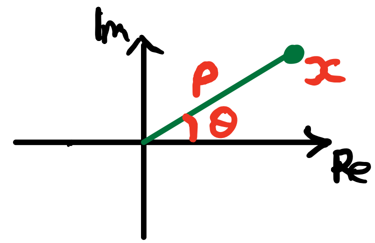

- But if $\mathrm{Re}(\nu x^2/2) = \mathrm{Re}(r e^{i\phi} \rho^2 e^{2i\theta}/2) = 0$ then both oscillate.

$(\nu = r e^{i\phi})$

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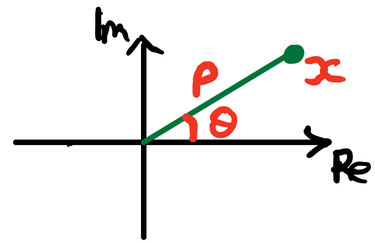

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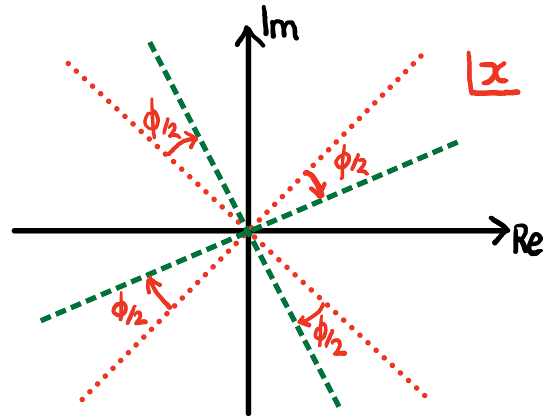
- But if $\mathrm{Re}(v x^2/2) = \mathrm{Re}(r e^{i\phi} \rho^2 e^{2i\theta}/2) = 0$ then both oscillate.

- Such θ define the anti-Stokes lines, and split the complex plane into Stokes sectors:



- If ν is such that the quantisation contour C coincides with an anti-Stokes line at $\underline{t}\infty$, the eigenvalue problem will be in trouble.

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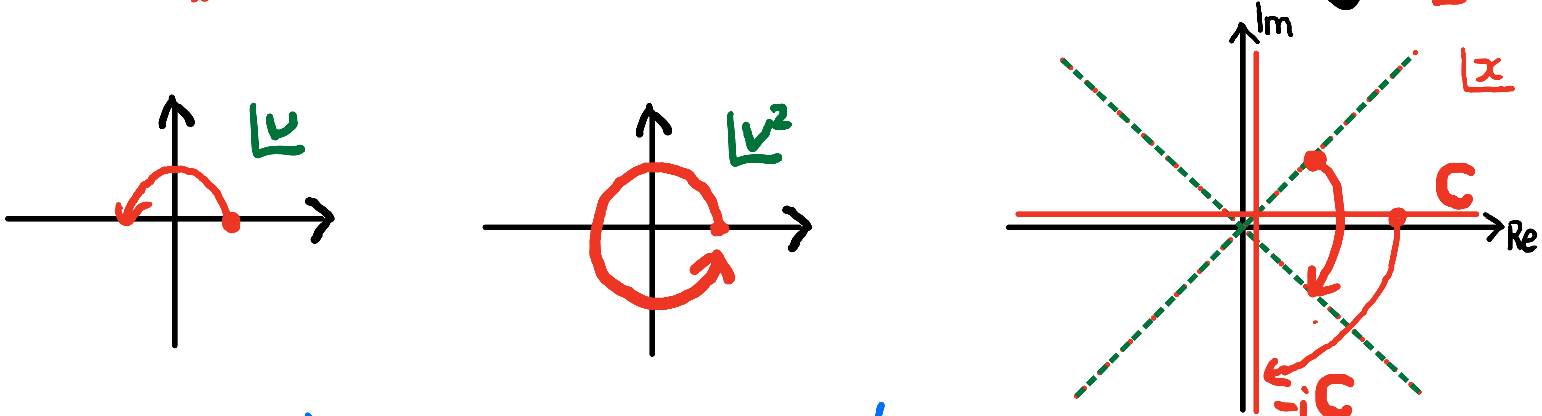
- To keep out of trouble during continuation, C must be distorted so as to track the same pair of Stokes sectors at $\pm\infty$, to avoid anti Stokes lines being crossed.

- For the case in hand, as ϕ increases from 0 to π, $\nu = re^{i\phi} \to -\nu$, $\mathcal{H}_\nu = \left(-\frac{d^2}{dx^2} + \nu^2 x^2\right) \to \mathcal{H}_{-\nu} = \mathcal{H}_\nu$, but the Stokes sectors rotate by $-\pi/2$ so $C \to -iC$



... so the b.c.'s for ψ have changed.

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<u>Claim</u>: this is what is happening for the free boson: the continuation is messing with the boundary conditions at infinity. We would not expect to see this for sinh-Gordon, since the growth of the potential $\cosh(\phi(x))$ at each point x is much stronger.

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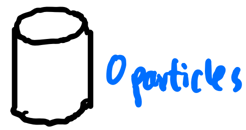
Confirmation comes from numerical work (next section) which shows that for $p>0$ continuation round branch points & back to the real axis does indeed lead to states of higher energy.

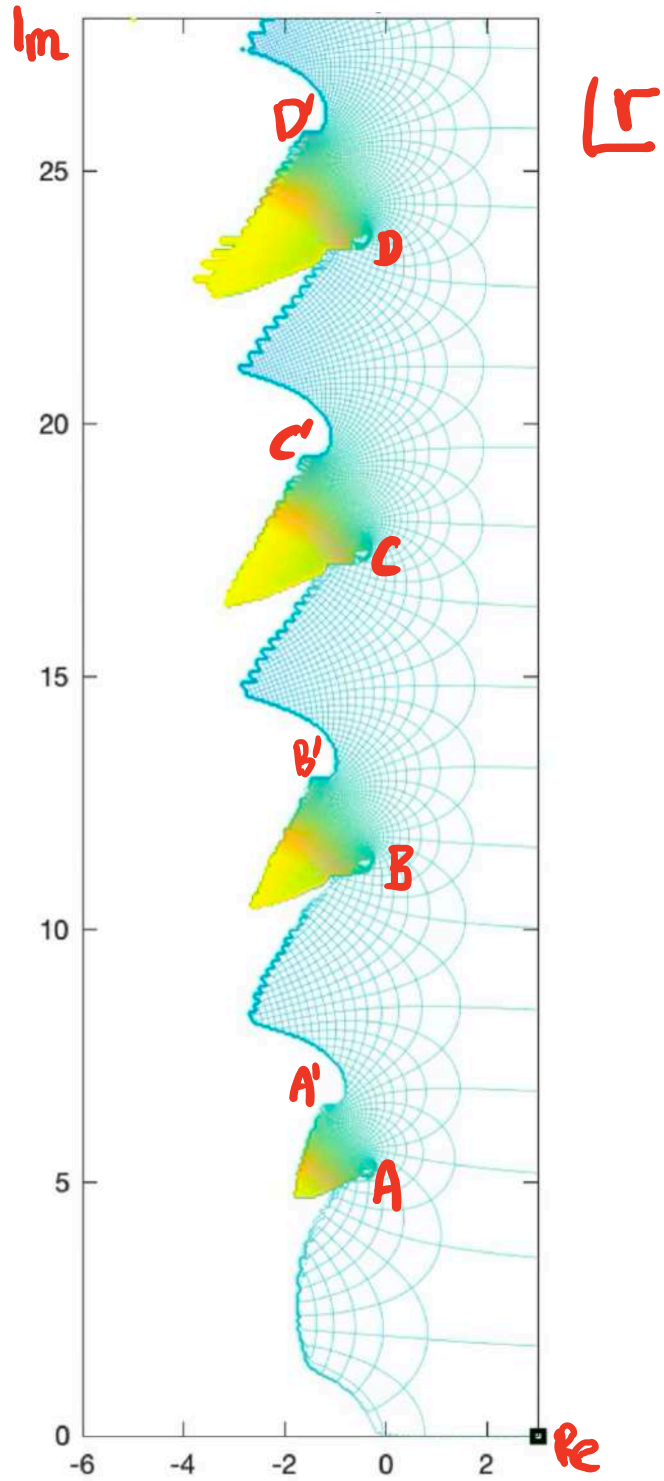
5 Results

Ground state TBA solution for $\rho = 0.04$:

contours of $\mathrm{Re}(c(r))$ and $\mathrm{Im}(c(r))$.



Note paired branch points at $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$.



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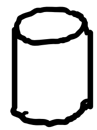

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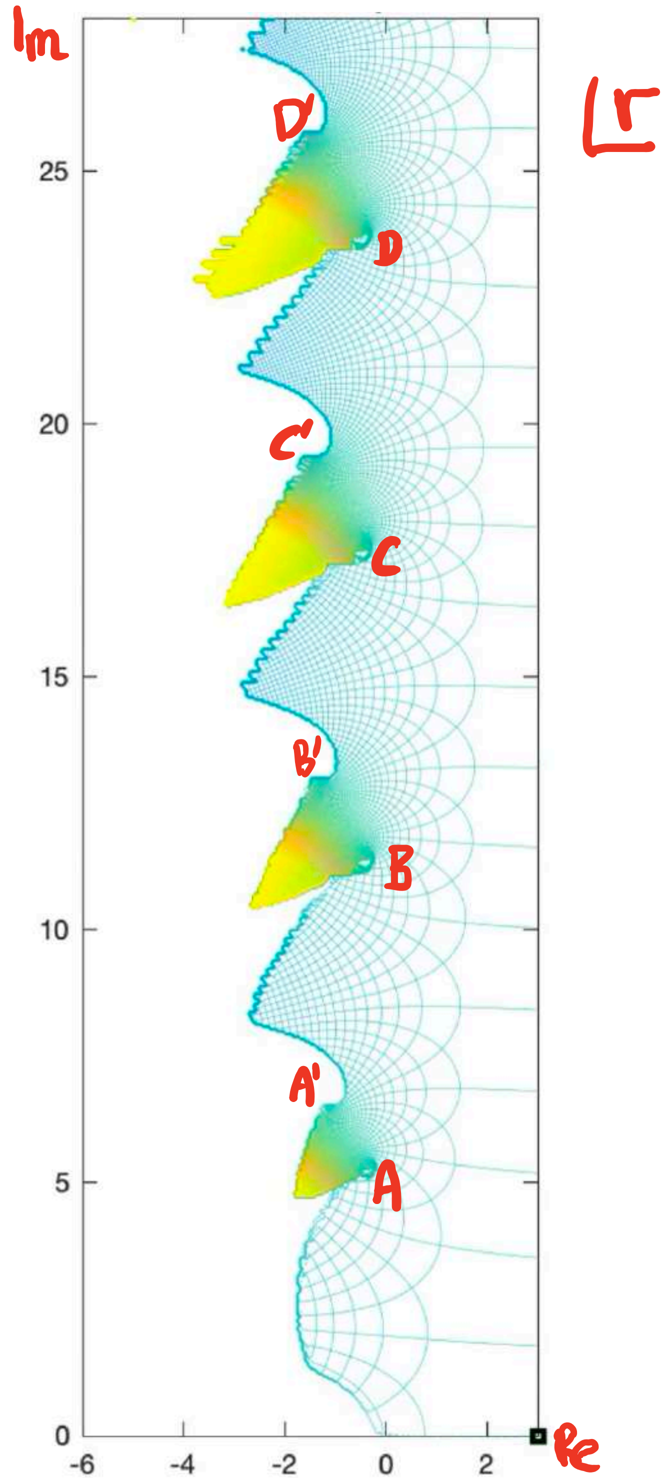
As $p \to 0$,

$A \,\&\, A' \to 2\pi i \qquad B \,\&\, B' \to 4\pi i$

$C \,\&\, C' \to 6\pi i \qquad D \,\&\, D' \to 8\pi i$



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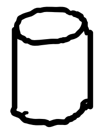

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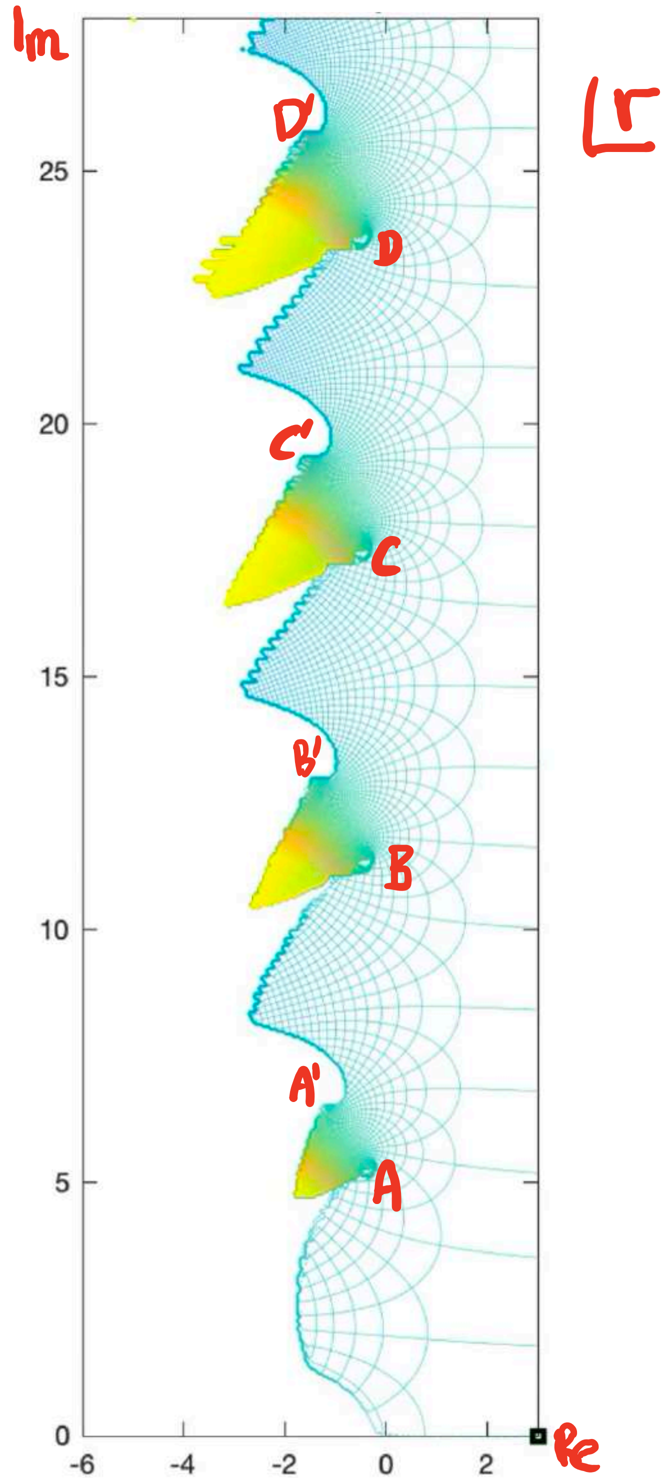
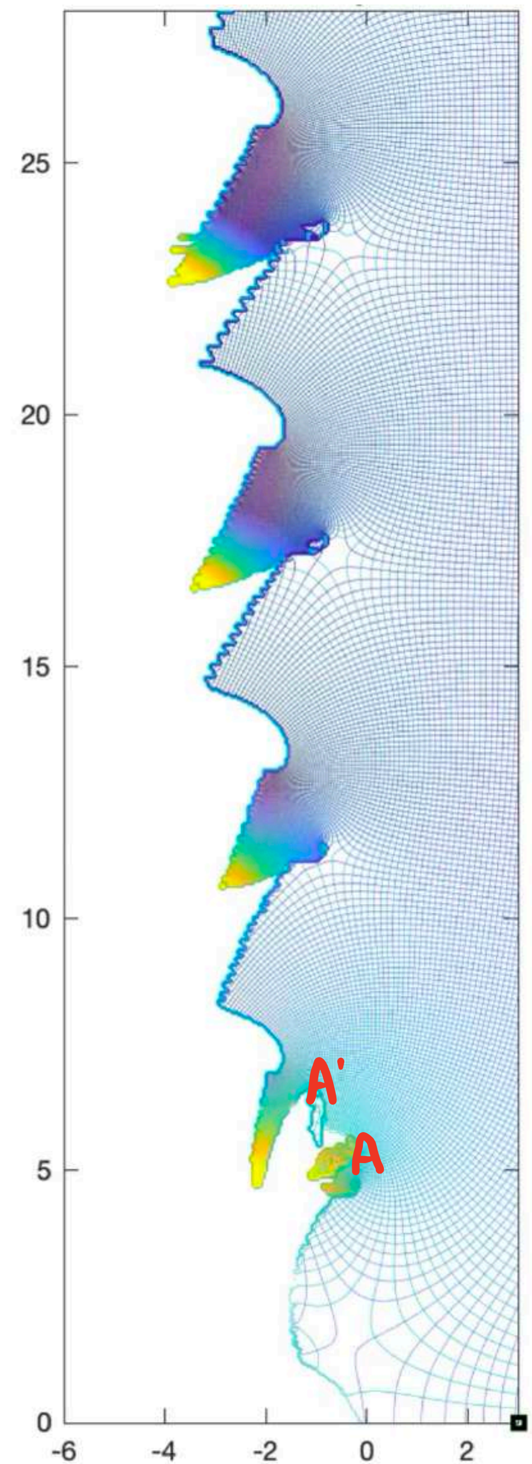
Each connects to a 2-particle TBA solution, with Bethe number $N=1$ (A&A'), $N=2$ (B&B'), $N=3$ (C&C') & $N=4$ (D&D') and so on...



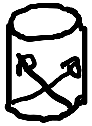
Two-particle sinh-Gordon TBA with $N=1$

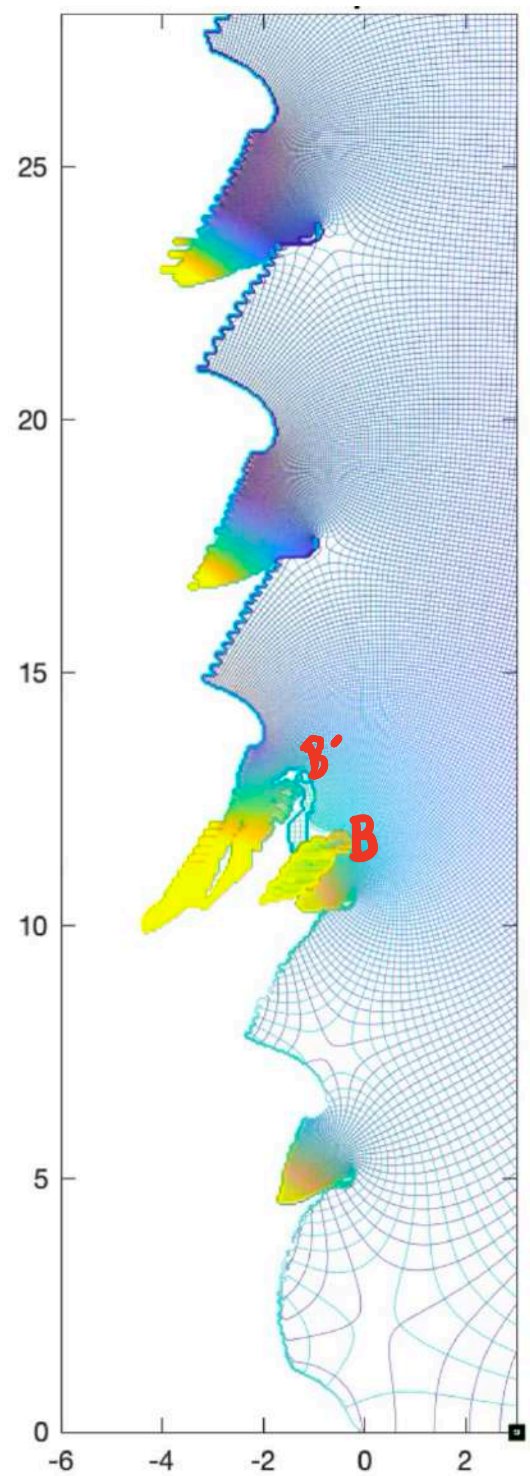
2 particles:



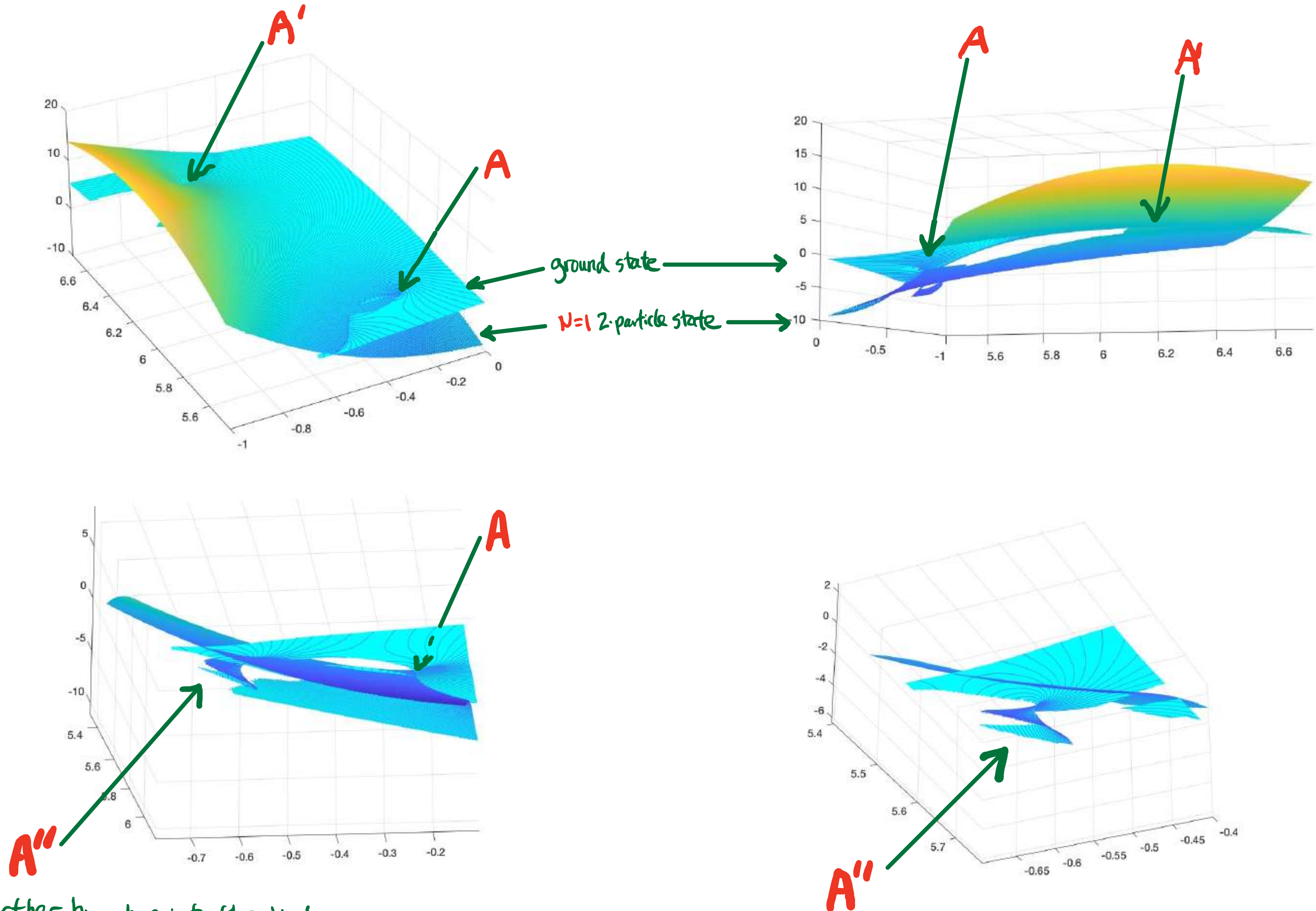
Two-particle sinh-Gordon TBA with N=2

2 particles, going a bit faster:





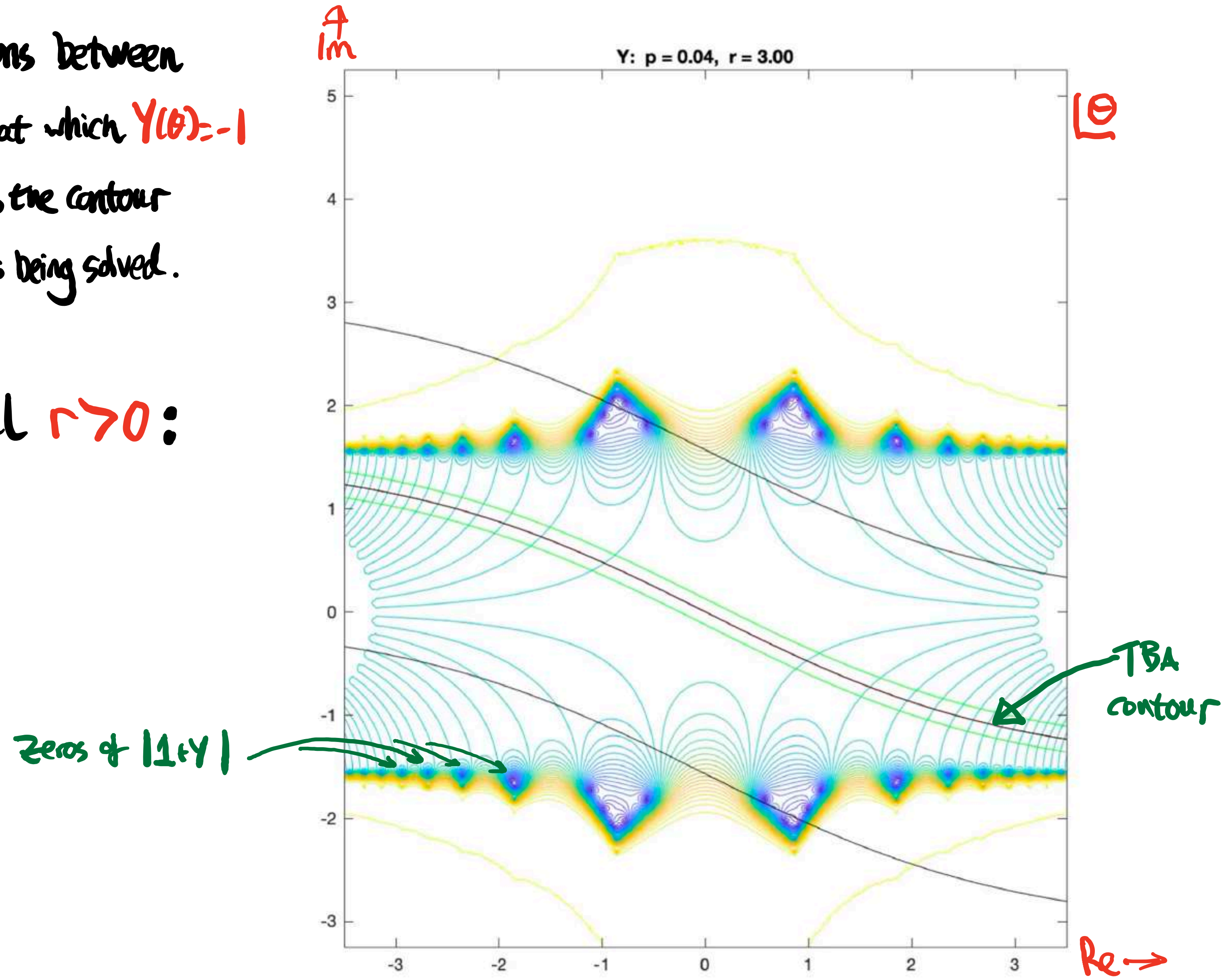
Some 3d plots of Re(c(r)) (p=0.02 this time)



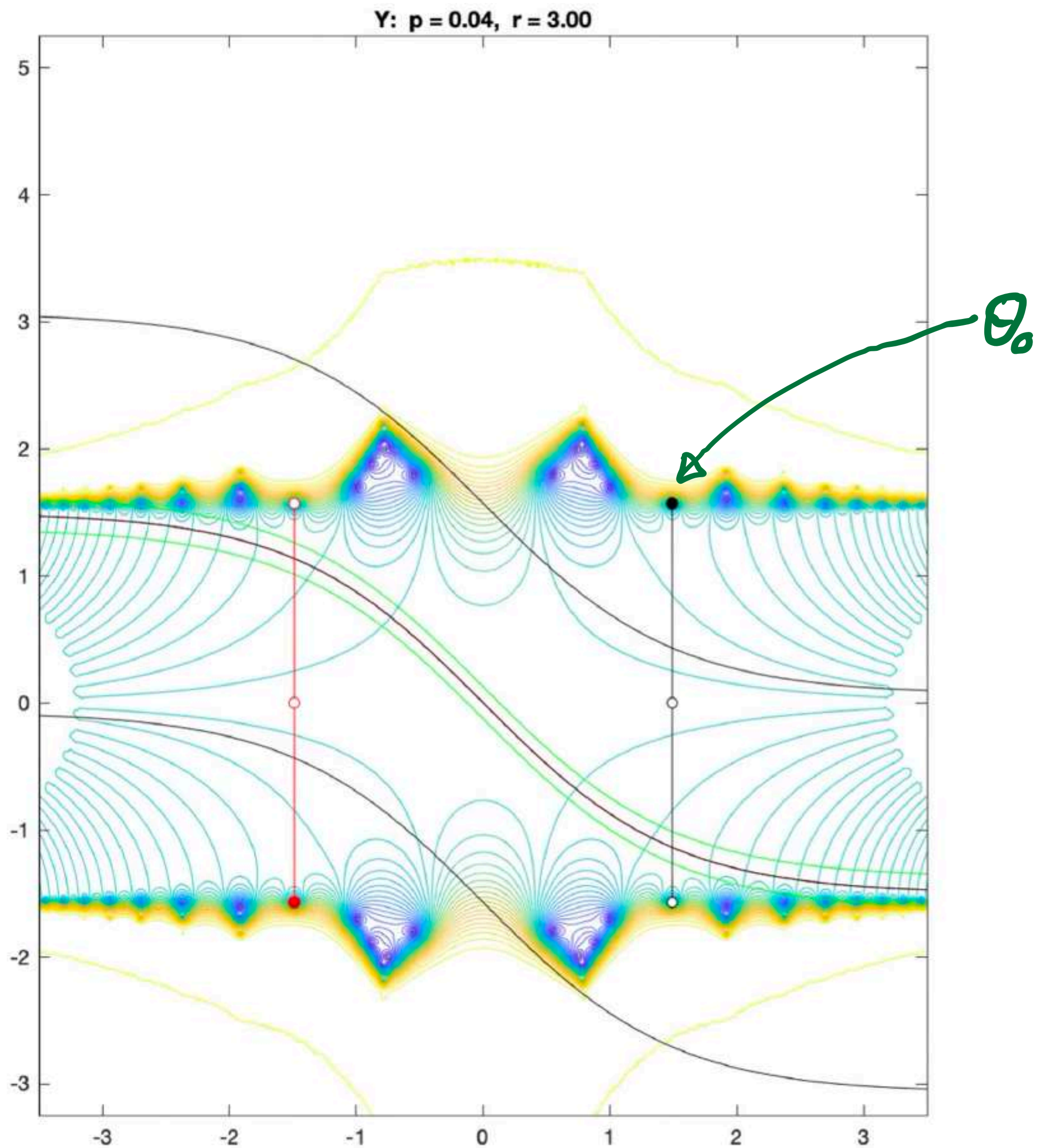
(a further branch point, the first of a sequence, near to A)

To track transitions between TBAs, monitor points at which $Y(\theta)=-1$ to see when they cross the contour along which the TBA is being solved.

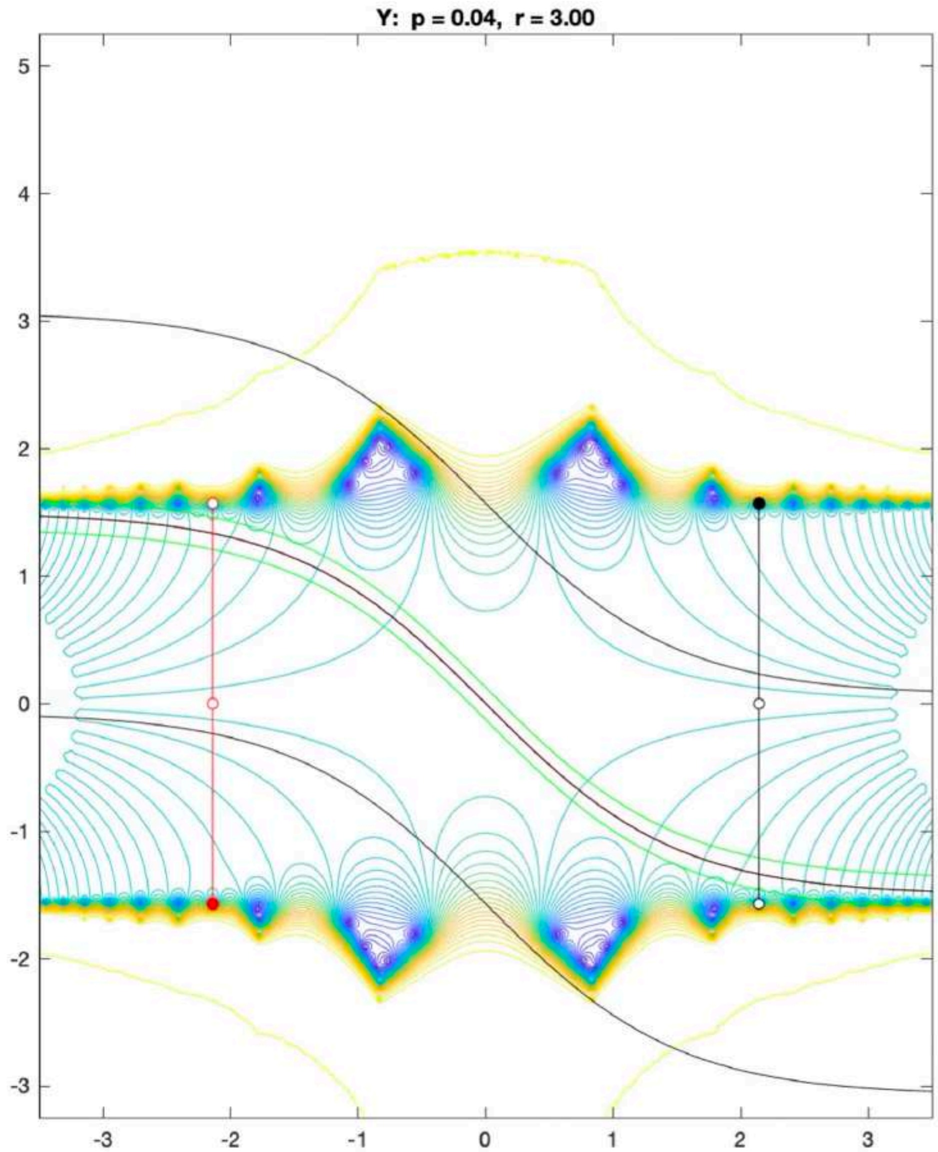
Situation for real $r>0$:



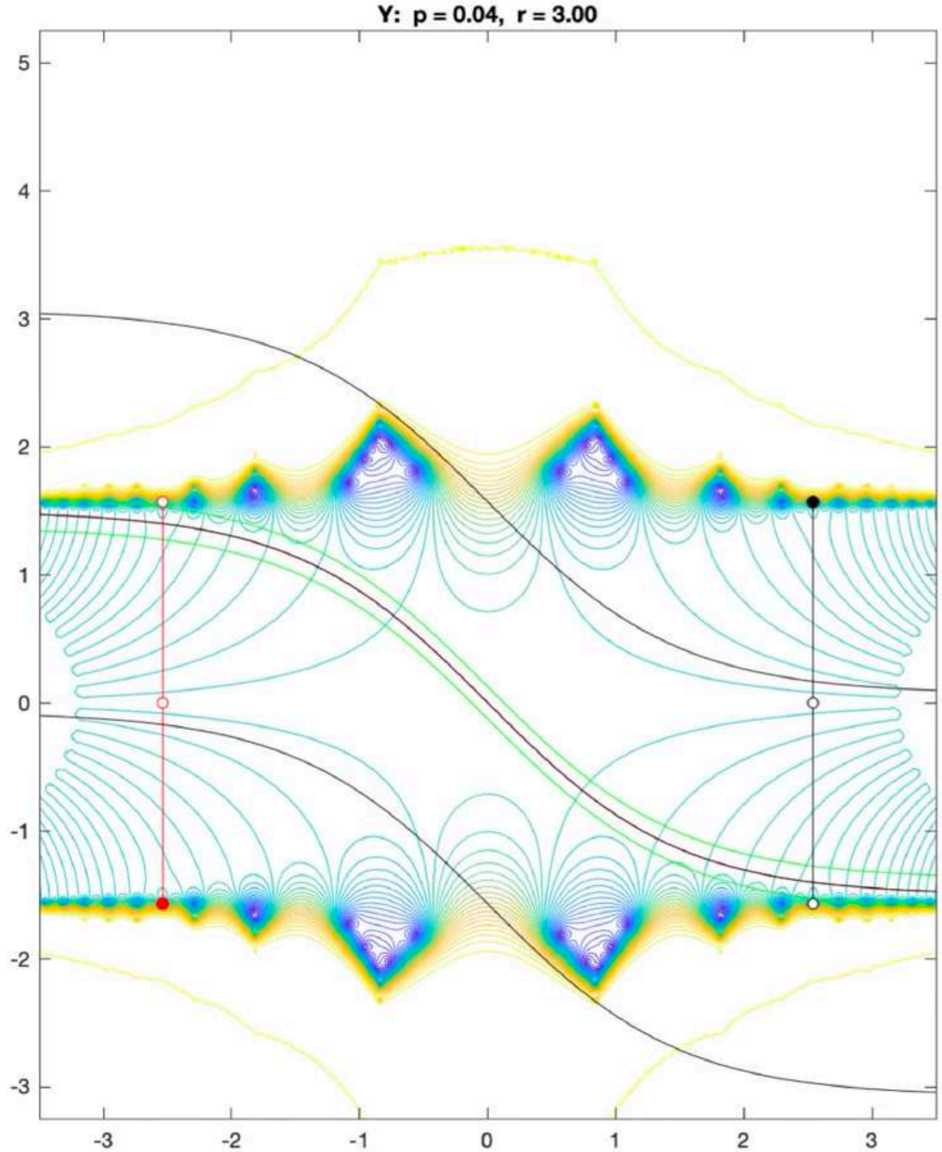
Plot of $|1+Y(\theta)|$ for complex θ: ground state



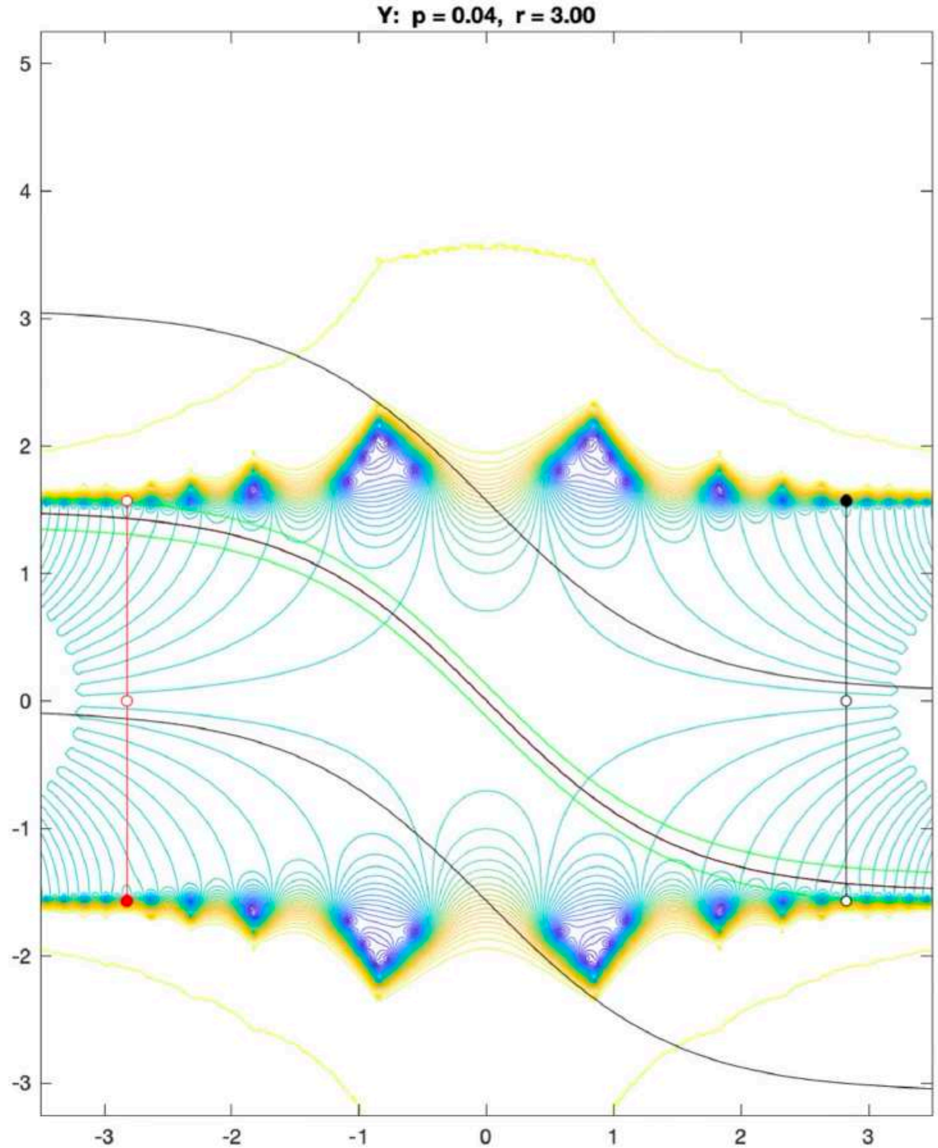
$\|1+Y\|$: 2 particles, $N=1$



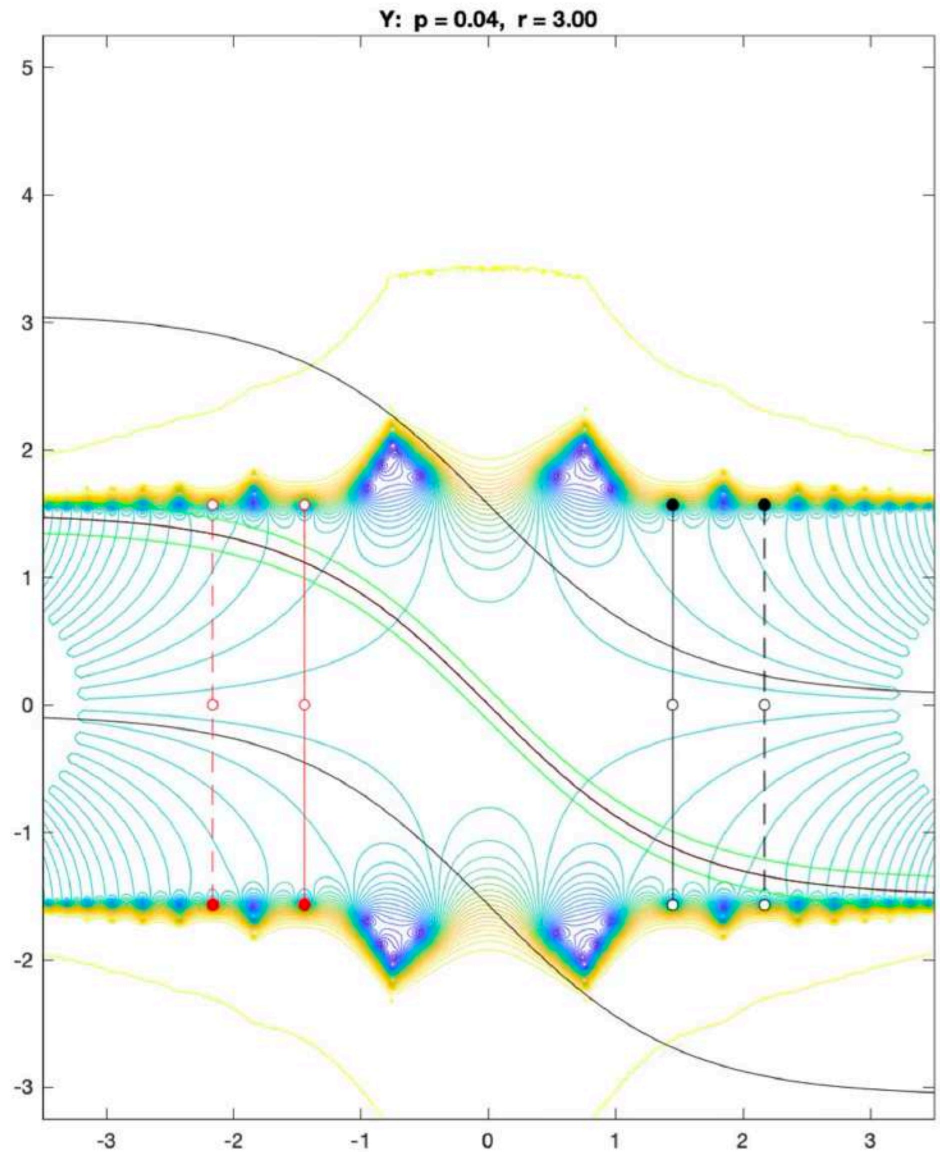
$|1+Y|$: 2 particles, $N=2$



$|1+Y|$: 2 particles, $N=3$



$||+Y||$: 2 particles, $N = 4$



$|1+Y|$: 4 particles, $N_1, N_2 = 1, 2$

As r leaves the real axis the zeros of $|1+Y|$ wander about and induce transitions between the different TBAs.

<see movies>

Conclusions:

- The $p \to 0$ limit shows regularities of structure which make us expect that a more-complete picture of $c(r)$ in the complex plane will be possible.

- The precise working of this limit is subtle, and much remains to be done both analytically and numerically.

- It would be interesting to study other models in a similar way, & also other limits of sinh-Gordon such as the staircase models...

Happy 60th!

↖ Philippe@40

Happy

60th !

Philippe@40