Integrable Monopoles

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Based on:

- C.K., & K.Zarembo, JHEP08 (2023) 184
  ArXiv:2305.03649+work in progress

At the crossroads of physics and mathematics :
the joy of integrable combinatorics
IPhT, Saclay, Paris
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Motivation

- ”The existence of magnetic monopoles seems to be one of the safest bets that one can make about physics not yet seen” (Joe Polchinski, at the Dirac Centennial Symposium)
- A 4D QFT containing a Monopole and a Higgs particle
- A novel example of an integrable susy dCFT based on N=4 SYM
- One-point functions computable in closed form
- Novel insights on S-duality (‘t Hooft line dual to Wilson line)

Plan of the talk

I. Introducing the monopole-Higgs configuration

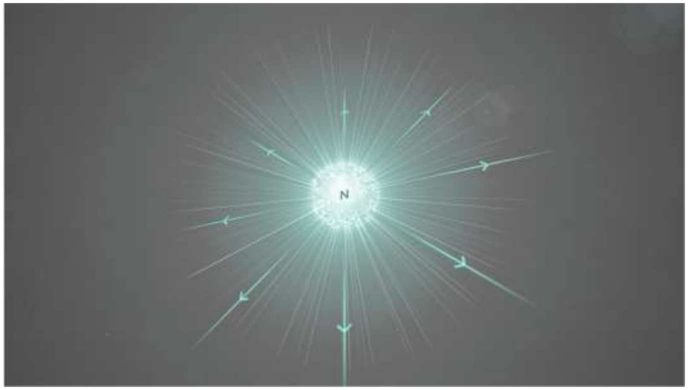
II. Quantization in the monopole background

III. Integrable One-point Functions

IV. Conclusion

Introducing the monopole

$$\vec{B} = \frac{B\vec{r}}{2r^3}$$



Dirac '31

$$\frac{q_e q_m}{2\pi\epsilon_0\hbar c^2} \in \mathbb{Z}$$

Dirac quantization condition: $B \in \mathbb{Z}$

't Hooft loop: world line of a monopole (static at the origin)

Disorder operator:
Prescribes certain singular behaviour of the gauge field

$$A_\phi = \frac{B}{2r}\frac{1-\cos\theta}{\sin\theta}, \qquad A_r = A_\theta = 0$$

A monopole in $\mathcal{N} = 4$ SYM

$$\mathbf{A}^{cl}_\phi = \mathbf{B}\,\frac{1-\cos\theta}{2r\sin\theta}, \qquad \mathbf{A}^{cl}_r = \mathbf{A}^{cl}_\theta = \mathbf{0},$$

Kapustin '05

$$\Phi^{cl}_I = \mathbf{B}\,\frac{n_I}{2r}, \qquad I = 1, 2, \ldots, 6, \qquad \sum_I n_I^2 = 1$$

Simplest case: $\mathbf{B} = \mathrm{Diag}(1, 0, \ldots)$ $\quad n_I = (1, 0, 0, 0, 0, 0)$

$$\mathbf{B} = \begin{array}{c} \begin{array}{cc} 1 & N-1 \end{array} \\ \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ \\ N-1 \end{array}$$

Supersymmetry conserved: 1/2 BPS configuration

Set-up constitutes a 1D dCFT (co-dimension $= 3$)

Quantizing around the monopole background

1. Expand around classical fields $(\Phi_1^{\text{cl}}, A_\mu^{\text{cl}})$

$$
\begin{array}{c} \\ A_\mu, \Phi_i, \Psi = \end{array}
\begin{array}{c} \begin{array}{cc} 1 & \quad N-1 \end{array} \\ \left[\begin{array}{c|ccc} \alpha & \beta & \beta & \beta \\ \hline \beta & \gamma & \gamma & \gamma \\ \beta & \gamma & \gamma & \gamma \\ \beta & \gamma & \gamma & \gamma \end{array}\right] \end{array}
\begin{array}{c} 1 \\ \\ N-1 \end{array}
$$

2. Gauge fix

3. Invert the quadratic part of the action (determine propagators)

Spectral decomposition

$$S = \Phi \, G^{-1} \, \Phi$$

$$G(x,y) = \sum_k \Psi_k(x) \, \frac{1}{\lambda_k} \, \Psi_k^\dagger(y), \qquad G^{-1}\Psi_k(x) = \lambda_k \Psi_k(x)$$

Quantum mechanical problem (field components β)

Quantum mechanical problems involved

I. For $\Phi_2, \Phi_3, \ldots \Phi_6, A_0$ and ghost c

Scalar particle in monopole potential — Dirac '31

II. For Φ_1, $\vec{A}$:

Scalar coupled to spin-1 particle in monopole potential

Spin-1 particle & monopole
Olsen, Osland & Wu '90

III. For Ψ^I_α $(\alpha, I \in \{1,2,3,4\})$

Fermions in (non-standard) monopole potential

Standard case
Kazama, Yang &
Goldhaber '77

All are beautiful, exactly solvable quantum mechanical systems

Scalar in monopole potential

Dirac 31, Tamm '31
Fierz '44

$$G^{-1} = \partial_t^2 + \hat{H}, \qquad \hat{H} = -(\partial^k + iA^k_{cl})(\partial_k + iA^{cl}_k) + \frac{B}{4r^2},$$

$$\Psi(\vec{x},t) = e^{i\omega t}\,\Phi(r,\theta,\phi), \qquad \hat{H}\,\Phi(r,\theta,\phi) = k^2\,\Phi(r,\theta,\phi)$$

Define $L_\pm, L_z$ with standard SU(2) commutation relations and $[\vec{L},\hat{H}]=0$

Eigenfunctions in terms of monopole spherical harmonics $(q = B/2)$

$$\Psi(r,\theta,\phi) = \frac{1}{r}(kr)^{1/2} J_{j+1/2}(kr)\, Y^{(q)}_{jm}(\theta,\phi), \qquad k > 0$$

$$Y^{(q)}_{jm}(\theta,\phi) = e^{i(m-\frac{B}{2})\phi}\, U_{jm}(\theta)$$

Involves Jacobi polynomial

SU(2) representation theory $\Longrightarrow\ m \in \frac{\mathbb{Z}}{2}$

Single valuedness of wavefunction $\Longrightarrow\ B \in \mathbb{Z}$ (Dirac quantization condition)

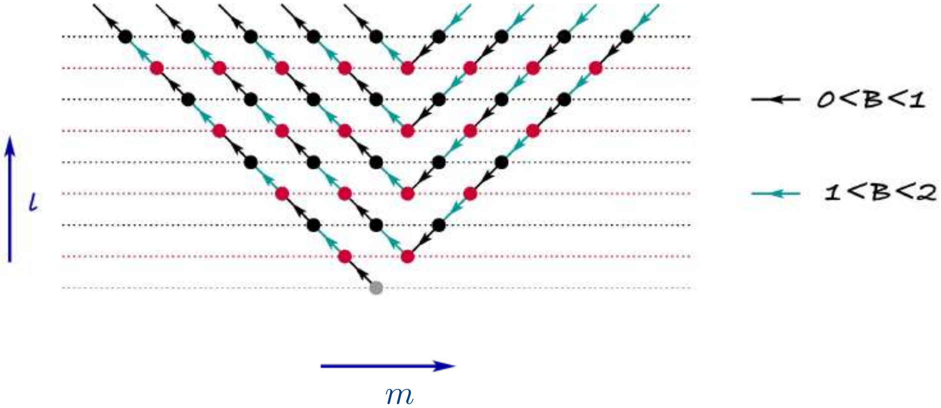
OBS: Spectral flow: $j = \frac{B}{2}, \frac{B}{2}+1, \ldots$

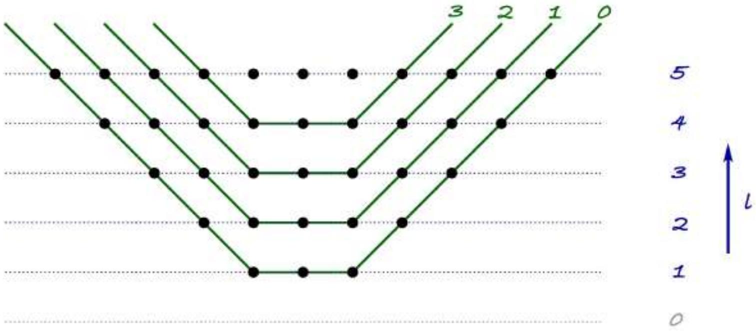
Spectral flow

$$B = 0: \; l = 0, 1, 2, \ldots, \; m = -l, \ldots, l$$

$$B = 1: \; l = \frac{1}{2}, \frac{3}{2}, \ldots, \; m = -l, \ldots, l$$

Wilczek '82





In general: $l = \frac{B}{2}, \frac{B}{2} + 1, \ldots$

The scalar propagator

$$G(x,x') = \frac{1}{rr'}\sum_{jm} Y^{(q)*}_{jm}(\mathbf{n})Y^{(q)}_{jm}(\mathbf{n}')$$
$$\times \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_0^\infty dk\, \frac{\mathrm{e}^{\,i\omega(t-t')}\sqrt{kr}\,J_{j+\frac{1}{2}}(kr)\sqrt{kr'}\,J_{j+\frac{1}{2}}(kr')}{\omega^2+k^2}\,.$$

- Integral over ω can be carried out
- Sum over m can be carried out

Jacobi pol.

$$G(x,x') = \frac{1}{4\pi rr'}\sum_j (2j+1)\left(\frac{1+\eta}{2}\right)^q D_j(\xi)P^{(0,2q)}_{j-q}(\eta),$$

AdS$_2$ propagator

$m^2 = j(j+1)$

$$\eta = \mathbf{n}\cdot\mathbf{n}', \qquad \xi = \frac{(t-t')^2+r^2+r'^2}{2rr'}\,,$$

Scalar and vector in monopole potential

$$\hat{H}\begin{pmatrix}\Phi_1\\ \vec{A}\end{pmatrix} = \frac{1}{r^2}\begin{pmatrix} r^2p_r^2+\mathbf{L}^2 & -iB\,\hat{\mathbf{r}}^{\mathbf{T}} \\ iB\hat{\mathbf{r}} & r^2p_r^2+\mathbf{L}^2-iB\,\hat{\mathbf{r}}\times \end{pmatrix}\begin{pmatrix}\Phi_1\\ \vec{A}\end{pmatrix} = E\begin{pmatrix}\Phi_1\\ \vec{A}\end{pmatrix}$$

$$\mathbf{J}=\mathbf{L}+\mathbf{S}$$

For B=1: $\ell=\frac{1}{2},\frac{3}{2},\ldots$ For $\ell \geq \dfrac{3}{2}$: $J=\ell-1,\ell,\ell+1,$

$s=1$ For $\ell = \dfrac{1}{2}$: $J=\ell,\ell+1.$

Mode expansion

$$(\Phi_1)_{JM}(x) = C\,F_J(r)Y^{(q)}_{JM}(\theta,\phi)$$

$$\vec{A}_{JM}(x) = F_J(r)\left[C_-\mathbf{Y}^{(\mathbf{q})}_{\mathbf{JJ-1M}}(\theta,\phi)+C_0\mathbf{Y}^{(\mathbf{q})}_{\mathbf{JJM}}(\theta,\phi)+C_+\mathbf{Y}^{(\mathbf{q})}_{\mathbf{JJ+1M}}(\theta,\phi)\right]$$

$F_J(r)=(kr)^{-1/2}J_\nu(kr)$, $\quad J_\nu$ Bessel function, $\quad \nu=\nu(J)$

$\nu(J)$ and (C,C_-,C_0,C_+) eigenvalues and eigenvectors of 4×4 matrix

Spectrum

Solution for ν:

$$\nu = \left\{ J - \frac{1}{2}, J + \frac{1}{2}, J + \frac{1}{2}, J + \frac{3}{2} \right\}, \qquad J \geq 3/2.$$

Interesting contrast to a gauge field with no scalar coupling

Olsen, Osland & Wu, '90

$$\nu = \left\{ \left[\frac{1}{4} + \left(\sqrt{J^2 + J} - 1 \right)^2 \right]^{1/2}, J + \frac{1}{2}, \left[\frac{1}{4} + \left(\sqrt{J^2 + J} + 1 \right)^2 \right]^{1/2} \right\}$$

Coupling to scalar is dictated by susy of $\mathcal{N} = 4$ SYM

The simple spectrum is a manifestation of underlying integrability

Propagators can be found by spectral re-summation

Fermions in a (non-standard) monopole potential

$$S_2^{ferm} = \frac{1}{2}\mathrm{tr}\left(i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + \bar{\Psi}G^1\left[\Phi_1^{cl},\Psi\right] + \bar{\Psi}\gamma^\mu\left[A_\mu^{cl},\Psi\right]\right)$$

Need to solve eigenvalue problem

$$\begin{pmatrix} E & i\vec{\sigma}\cdot(\vec{\partial}-i\vec{A}^{\mathrm{cl}})+i\frac{q}{r} \\ i\vec{\sigma}\cdot(\vec{\partial}-i\vec{A}^{\mathrm{cl}})+i\frac{q}{r} & -E \end{pmatrix}\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix} = \lambda\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}$$

Mode expansion in spinor spherical harmonics

$$\Psi^A_{JM} = f_+(r)\,\xi^+_{\mathbf{JM}}(\theta,\phi) + f_-(r)\,\xi^-_{\mathbf{JM}}(\theta,\phi),$$
$$\Psi^B_{JM} = g_+(r)\,\xi^+_{\mathbf{JM}}(\theta,\phi) + g_-(r)\,\xi^-_{\mathbf{JM}}(\theta,\phi), \quad J = 0,1,2,\ldots \quad (B=1)$$

Solution of eigenvalue problem

Standard case

$$\Psi_- \sim \begin{bmatrix} J_{\mu-\frac{1}{2}}(kr)\,\xi^-_{\mathbf{JM}}(\theta,\phi) \\ J_{\mu+\frac{1}{2}}(kr)\,\xi^+_{\mathbf{JM}}(\theta,\phi) \end{bmatrix}, \qquad \Psi_+ \sim \begin{bmatrix} J_{\mu-\frac{1}{2}}(kr)\,\xi^+_{\mathbf{JM}}(\theta,\phi) \\ J_{\mu+\frac{1}{2}}(kr)\,\xi^-_{\mathbf{JM}}(\theta,\phi) \end{bmatrix}, \qquad k = \sqrt{\lambda^2 - E^2}$$

$$\mu = \left[\left(J + \frac{1}{2}\right)^2 - q^2\right]^{1/2},$$ OBS. Not integer or half-integer

Non-standard case

Relevant Bessel functions J_{ν_+}, J_{ν_-}

$$\nu_- = J, \qquad \nu_+ = J + 1$$

Coupling to scalar is dictated by susy of $\mathcal{N} = 4$ SYM

The simple spectrum is a manifestation of underlying integrability

Integrable One-point Functions

Generic operators built from scalars

$$\mathcal{O} = \Psi^{I_1 \ldots I_L} \operatorname{tr} \Phi_{I_1} \ldots \Phi_{I_L}, \quad i_1, \ldots, i_L \in \{1, 2, \ldots, 6\}$$

Good conformal operators are eigenstates of integrable $SO(6)$ spin chain

$$\hat{H} = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{L} \left(2 - 2P_{\ell\ell+1} + K_{\ell\ell+1}\right),$$

Minahan & Zarembo '02

Eigenstates characterized by three sets of rapidities

$$|u_{1i}, u_{2j}, u_{3k}\rangle$$

Fullfil a set of algebraic Bethe equations

$$\left(\frac{u_{aj} - \frac{iq_a}{2}}{u_{aj} + \frac{iq_a}{2}}\right)^L \prod_{bk} \frac{u_{aj} - u_{bk} + \frac{iM_{ab}}{2}}{u_{aj} - u_{bk} - \frac{iM_{ab}}{2}} = -1,$$

One-point functions

Can be expressed as overlap with boundary state. At leading order

de Leeuw, C.K.
Zarembo '15

$$\langle \text{Bst}| = \text{Bst}_{I_1 \ldots I_L} \text{Tr}\, \Phi_{I_1} \ldots \Phi_{I_L}\,, \qquad \text{Bst}_{I_1 \ldots I_L} = n_{I_1} \ldots n_{I_L}$$

Overlap formula at leading order

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{2\pi^2}{\lambda r^2} \right)^{\frac{L}{2}} L^{-\frac{1}{2}} \frac{\langle \text{Bst} \,|\Psi\rangle}{\langle \Psi \,|\Psi\rangle^{\frac{1}{2}}}\,.$$

Integrable boundary state $|\text{Bst}\rangle$

Piroli, Pozsgay,
Vernier '17

de Leeuw, C.K,
Zarembo '15

$$Q_{2n+1} |\text{Bst}\rangle = 0, \;\; n = 1, 2, \ldots$$

Expect closed overlap formula to exist

Expressible entirely in terms of Bethe roots,
and including the Gaudin determinant

Result

For $n_I = \delta_{I,1}$, general scalar operator

de Leeuw, Gombor, C.K., Linardopoulos, Pozsgay '19

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{\pi}{\sqrt{\lambda}\, r} \right)^L \sqrt{ \frac{1}{L} \, \frac{\prod_j u_{2j}^2 \left(u_{2j}^2 + \frac{1}{4}\right)}{\prod_j u_{1j}^2 \left(u_{1j}^2 + \frac{1}{4}\right) \prod_j u_{3j}^2 \left(u_{3j}^2 + \frac{1}{4}\right)} \, \frac{\det G^+}{\det G^-} } \,,$$

State needs to have paired roots, $G =$ Gaudin matrix

Higher loop orders: Use the perturbative framework set up above

Should be possible to integrability bootstrap the full result the entire theory, all loops

Komatsu & Wang '20

Bajnok & Gombor '20

Conclusions

- Magnetic monopoles fascinating --- here in $\mathcal{N} = 4$ SYM
- Quantization completed
- Novel example of an integrable dCFT (co-dimension 3)
- Should be possible to integrability-bootstrap to get the all loop result for 1-pt fcts.

Thank you