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Quantum integrable systems on integrable classical background

Nicolai Reshetikhin

YMSC, Tsinghua University & BIMSA & UC Berkeley.

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Plan:

- Hybrid quantum systems
  (Quantum systems on a classical background)
- Hybrid integrable systems
- Relation to deformation quantization
- An example: spin Calogero-Moser system and dynamical Haldane-Shastry models.

(joint with A. Liashyk and I. Sechin)

1. Quantum systems on a classical background

Born-Oppenheimer approximation, Quantum computers: quantum-classical interface

- (M,ω) - symplectic manifold, the phase space of the system.

- We want to have a quantum system which is "driven" by a classical system:

The bundle of hybrid observables

$$\begin{array}{ll} E \leftarrow A_x = \pi^{-1}(x) \simeq \mathrm{End}(\mathbb{C}^N) \\ \pi \downarrow & \\ M \leftarrow & \end{array}$$

Assume for simplicity that quantum system is finite dimensional

A_x - associative, $*: A_x \to A_x$, $(ab)^* = b^* a^*$, $(\lambda a)^* = \bar{\lambda} a^*$

unital $1_x \in A_x$, simple

- The algebra of hybrid observables:

$A = \Gamma(M, E)$ the space of (smooth) sections

$s : M \to E$, $x \mapsto s(x)$,

pointwise multiplication

$$(s_1 s_2)(x) = s_1(x) s_2(x)$$

has a $*$-structure, $s(x) \mapsto s(x)^*$, i.e. A is a $*$-algebra.

- $Z(A) = C(M)_{\mathbb{C}} \cdot I \subset A$ (I is a section $x \mapsto 1_x$)

has a natural Poisson structure

$$\{z_1, z_2\} = \omega^{-1}(dz_1 \wedge dz_2)$$

- A is a module over the center $Z(A) \subset A$

. We want A to be a module over the Poisson algebra $Z(A)$:

$$\{z, s_1 s_2\} = \{z, s_1\} s_2 + s_1 \{z, s_2\}$$

$$\{z_1 \{z_2\, s\}\} = \{\{z_1, z_2\}, s\} + \{z_2, \{z_1, s\}\}$$

(R., Voronov, Weinstein)

$$\{z, s\} = (\omega^{-1})^{ij}\, \partial_i z\, \nabla_j^{\alpha} s$$

where $\nabla_i^{\alpha} s = \partial_i s + [\alpha_i, s]$ the covariant derivative with respect to a connection $\alpha = \sum_i \alpha_i dx^i$

<u>Def</u>. A <u>hybrid quantum system</u> is a bundle of algebras over (M, ω) with a $*$ and a hermitian connection on it.

(A quantum system on a classical background)   (Poisson Azumaya algebras)

More algebraically, algebra of observables of a hybrid integrable system is

- An associative algebra A, (finite dimensional) and simple over $Z(A)$
- Poisson algebra structure on $Z(A)$ with trivial Poisson center (M, ω)
- Poisson algebra $Z(A)$ acts by derivations on A.

Representations of A

Representation Bundle:

$$\begin{array}{ccc} V & \leftarrow & V_x \\ \downarrow & & \\ \mathcal{M} & & \end{array}$$

V_x - finite dimensional Hermitian

- V_x is an irr module over A_x, $\rho_x : A_x \to \mathrm{End}(V_x)$

  $(V_x \simeq \mathbb{C}^N, \quad A_x \simeq \mathrm{End}(\mathbb{C}^N))$.

$$\rho_x(a_x)^\dagger = \rho_x(a_x^*)$$

- with a connection β

The associated representation space of A is $\mathcal{H} = \Gamma(\mathcal{M}, V)$. The pointwise action $a_x : f_x \mapsto a_x f_x$

Connections α (on E) and β (on V) agree:

$$\nabla^\beta(af) = \nabla^\alpha(a)f + a\nabla^\beta f$$

Note that the Poisson algebra $Z(A) \simeq C(\mathcal{M})$ is also represented:

$$\{z, f\} = (\omega^{-1})^{ij}\, \partial_i z\, \nabla^\beta_j f$$

$$\{z, af\} = \{z, a\}f + a\{z, f\}$$

$$\{z_1, \{z_2, f\}\} = \{\{z_1, z_2\}, f\} + \{z_2, \{z_1, f\}\}$$

Lagrangian modules

Fix a Lagrangian fibration $\begin{array}{c}\mathcal{M}_{2n}\\ p\downarrow\\ B_n\end{array}$ (generic fiber $p^{-1}(b) \subset \mathcal{M}$ is Lagrangian)

Assume $L \subset \mathcal{M}$ is a <u>Lagrangian section</u>, i.e.

$$L \cap p^{-1}(B) = \{x_{L,B}\} \text{ for generic } B.$$

Define the vector Bundle $V^{B,L} \xrightarrow{p_L} B$

$$p_L^{-1}(B) = V_{x_{L,B}}$$

Define the space $\mathcal{H}_{L,B} = \Gamma(B, V^L)$

- $C(\mathcal{M})$-module structure, $\exists\, v \in \Gamma(B, V^L)$

$$f : v(B) \mapsto f(x_{L,B})\, v(B) \quad , \quad x_{L,B} = L \cap p^{-1}(B)$$

- A-module structure

$$a : v(B) \mapsto a_{x_{L,B}}\, v(B)$$

Poisson structure is not represented in $H_{L,B}$

Dynamics

The Hamiltonian dynamics generated by $H^{(0)} \in Z(A)$ lifts to A as

$$\frac{\partial a_t}{\partial t} = \{H^{(0)}, a_t\}, \quad a_0 = a$$

A hybrid quantum Heisenberg dynamics on A with a background Hamiltonian dynamics generated by H is

Heisenberg $$\frac{\partial a_t}{\partial t} = \{H^{(0)}, a_t\}_\alpha + i[H^{(1)}, a_t], \quad H^{(1)} \in A$$

Here $H^{(0)}$ is the classical Hamiltonian, $H^{(1)}$ is the quantum Hamiltonian.

Note: $H^{(1)}$ and $H^{(1)} + f \cdot 1$ define the same H. dynamic.

The <u>hybrid Schrödinger picture</u>: the evolution of a section $s \in \Gamma(M, V) = \mathcal{H}$

$$\frac{\partial s_t}{\partial t} = -\{H^{(0)}, s_t\}_{\beta} - i H^{(1)} s_t, \quad s_0 = s$$

Consider $\Psi_t(x) = s_t(x(t))$, $x(0) = x$,

$$i \frac{\partial \Psi_t}{\partial t}(x) = H^{(1)}(x(t))\Psi_t(x) \qquad (*)$$

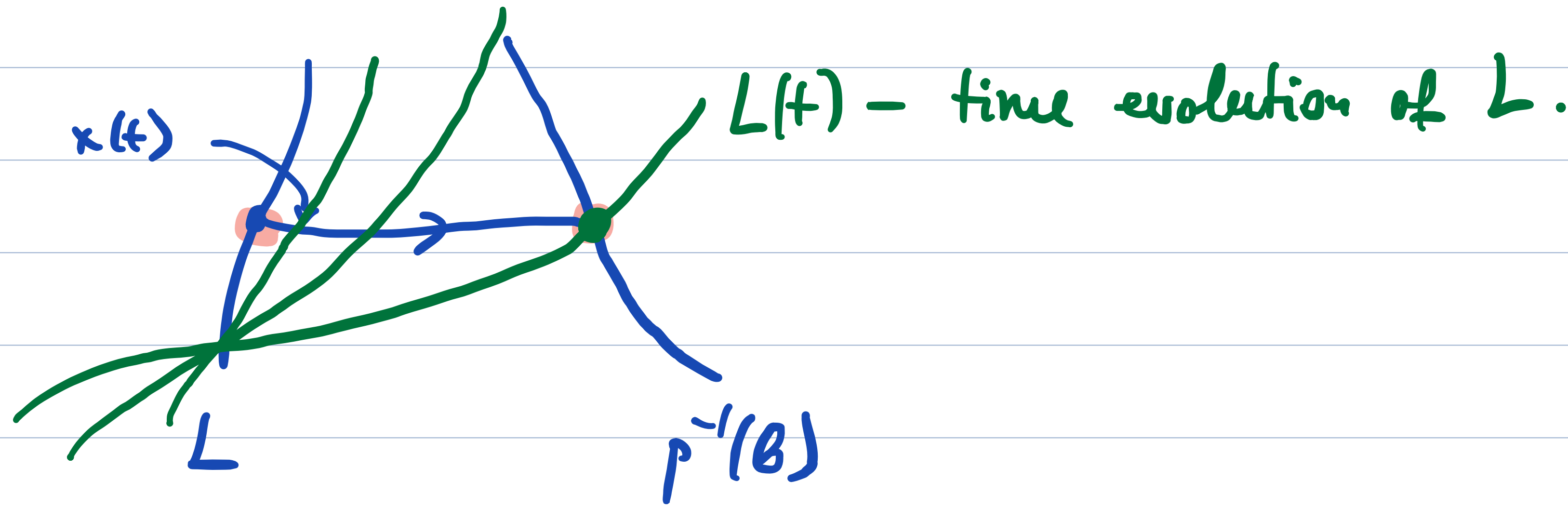
where $x(t)$ is a flow line of $v_H = \omega^{-1}(dH^{(0)})$ with $x(0) = x$.

<u>Schrödinger picture</u>: $\psi \in \mathcal{H}^{L,B}$

$$-i \frac{\partial v_t}{\partial t}(b) = H^{(1)}(x(t))\, v_t(b)$$

where $x(t)$ is the flow line of $v_H = \omega^{-1}(dH^{(0)})$

connecting L and $p^{-1}(B)$ in time t.



2. Hybrid integrable systems

(i) A classical integrable system:

$$\begin{array}{l} M_{2n} \leftarrow \text{generic fiber is Lagrangian} \\ \pi \downarrow \\ B_n \end{array}$$

The usual class of examples: $H_1^{(0)}, \dots, H_n^{(0)} \in C^\infty(M_{2n})$ s.t.

- $H_1^{(0)}, \dots, H_n^{(0)}$ — independent
- $\{H_i^{(0)}, H_j^{(0)}\} = 0$

$\pi(x) = (H_1^{(0)}(x), \dots, H_n^{(0)}(x))$. $B_n \subset \mathbb{R}^n$

Let $x(t_1, \dots, t_n)$ be multitime trajectory

$$\frac{\partial x(\underline{t})}{\partial t_j} = \omega^{-1}\left(d H_j^{(0)}(x(\underline{t}))\right), \quad \underline{t} = (t_1, \dots, t_n)$$

The multitime evolution of classical observables

$$\frac{\partial f_{\underline{t}}(x)}{\partial t_j} = \{ I_j, f_{\underline{t}} \}(x), \quad f_{\underline{0}}(x) = f(x) \in C(M)$$

(ii) A hybrid integrable system is a lift of this classical system to the hybrid algebra of observables $A = \Gamma(M, E)$ with the "twist" by quantum Hamiltonians $H_1^{(1)}, \dots, H_n^{(1)} \in A$.

Hybrid multitime Heisenberg dynamics of observables:

$$s(x) \longmapsto s_{\underline{t}}(x), \qquad s \in A$$

$$\frac{\partial s_{\underline{t}}}{\partial t_j}(x) = \{H^{(0)}_j, s_{\underline{t}}\}_\alpha(x) + i\,[H^{(1)}_j(x), s_{\underline{t}}(x)], \quad s_0(x) = s(x)$$

$j = 1, \dots, n$

The compatibility condition:

$$d\alpha + [\alpha,\alpha]\tfrac{1}{2}$$

$$\{H^{(0)}_k, H^{(1)}_j\} + \{H^{(1)}_k, H^{(0)}_j\} + i\,[H^{(1)}_k, H^{(1)}_j] - i\left(\iota_{v(f)} \wedge \iota_{v(g)}\right)(F_\alpha) \in Z(A)$$

$f = H^{(1)}_k$

$g = H^{(1)}_j$

Multitime Hybrid Schrödinger dynamics

$$v(x) \longmapsto v_{\underline{t}}(x), \qquad v \in \mathcal{H}$$

$$\frac{\partial v_{\underline{t}}}{\partial t_j}(x) = \{H^{(0)}_j, v_{\underline{t}}\}(x) + i\, H^{(1)}_j(x)\, v_{\underline{t}}(x), \quad v_{\underline{0}}(x) = \quad (x)$$

$$\{H^{(0)}_k, H^{(1)}_j\} + \{H^{(1)}_k, H^{(0)}_j\} + i\,[H^{(1)}_k, H^{(1)}_j] - i\,(\imath_{v(f)} \wedge \imath_{v(g)})(F_\alpha) = 0$$

Substitute $v_{\underline{t}}(x(\underline{t})) = \psi(\underline{t})$, $x(\underline{t})$ is a multitime integrable trajectory generated by vector fields $v_j = \omega^{-1}(dH^{(0)}_j)$, then

$$i\frac{\partial \psi_{\underline{t}}}{\partial t_k} = H^{(1)}_k(x(\underline{t}))\,\psi_{\underline{t}}$$

Differential operators $i\frac{\partial}{\partial t_k} + H^{(1)}_k(x(\underline{t}))$ commute

$$\left[i\frac{\partial}{\partial t_k} - H^{(1)}_k(x(\underline{t})),\ i\frac{\partial}{\partial t_j} - H^{(1)}_j(x(\underline{t}))\right] = 0$$

They also generate a multitime dynamics in $\mathcal{H}_{L,B}$

$$i\,\frac{\partial \Psi_{\underline{t}}}{\partial t_k}(b) = H_k^{(1)}(x(\underline{t}))\,\Psi_{\underline{t}}(b)$$

$x(\underline{t})$ - multitime trajectory connecting L & $\pi^{-1}(b)$

Note: $H_1^{(1)}, \dots, H_n^{(1)}$ define a flat connection on generic level surfaces of Poisson commuting integrals $H_1^{(0)}, \dots, H_n^{(0)}$

The relation to deformation quantization

(a) A_0 - associative algebra, $Z(A_0) \subset A_0$ be its center.

$A_\hbar$ — flat deformation family of A_0

- $A_\hbar$ - associative algebra for each $\hbar$
- $\varphi_\hbar : A_\hbar \xrightarrow{\sim} A_0$ an isomorphism of top. spaces

"Equivalently", a family of associative mult. on A

$$a *_\hbar b = \varphi_\hbar\left(\varphi_\hbar^{-1}(a)\, \varphi_\hbar^{-1}(b)\right) = ab - i\hbar\, m_1(a,b) + \hbar^2 m_2(a,b) + \cdots$$

Ex: $A_0[[\hbar]]$, $a *_\hbar b = ab + \sum_{n\geq 1}^{\infty} (-i\hbar)^n m_n(a,b)$,

<u>Well known fact.</u> Such deformation family induces the following structures on A_0:

(i) $\{z, z'\} = m_1(z,z') - m_1(z',z)$ is a <u>Poisson structure</u> on $Z(A_0)$

(ii) $\{z, s\} = m_1(z,s) - m_1(s,z)$ is the action of $Z(A_0)$ by derivations on A_0

(b) Assume we have an integrable system on a deformation family $\varphi_\hbar : A_\hbar \to A_0$ i.e. $H_1^\hbar, \dots, H_n^\hbar \in A_0$ such that

$$(*) \qquad H_k^\hbar *_\hbar H_\ell^\hbar - H_\ell^\hbar *_\hbar H_k^\hbar = 0 \quad \text{in } A_0$$

Assume that as $\hbar \to 0$ $H_k^\hbar = H_k^{(0)} 1_{A_0} - i\hbar H_k^{(1)} + O(\hbar^2)$, with $\{H_k^{(0)}\} \in Z(A_0)$

- terms of order $\hbar$ in $(*)$ give $\{H_k^{(0)}, H_\ell^{(0)}\} = 0$
- terms of order $\hbar^2$ give

$$\{H_k^{(0)}, H_\ell^{(1)}\}_\alpha - \{H_k^{(1)}, H_\ell^{(0)}\}_\alpha + i[H_k^{(1)}, H_\ell^{(1)}] - \{H_k^{(0)}, H_\ell^{(0)}\}_2 = 0$$

Here $\{a, b\}_2 = m_2(a,b) - m_2(b,a)$

Because $\{a,b\}_2 \in Z(A)$ for $a, b \in Z(A)$ we have a hybrid integrable Heisenberg system.

Quantum spin Calogero-Moser system.

Cherednik - Dunkle operators act on $\mathbb{C}(z_1,..,z_n)$

$$d_j = \hbar z_j \frac{\partial}{\partial z_j} + \sum_{i>j} \frac{z_i}{z_i - z_j} K_{ij} - \sum_{i<j} \frac{z_j}{z_j - z_i} K_{ij}$$

where $K_{ij} : (z_1, \dots, z_i, \dots, z_j \cdots z_n) \mapsto (z_1, \dots, z_j, \dots, z_i \cdots z_n)$

They give a representation of the degenerate affine Hecke algebra :

$$[d_i, d_j] = 0 \,, \quad K_{i\,i+1} d_i = d_{i+1} K_{i\,i+1} + 1 \,,$$

$$[d_i, K_{kl}] = 0 \quad \text{if} \quad i \neq k, l$$

K_{ij} generate the group algebra of S_n.

The elements

$$H_k = \frac{1}{k} \sum_{i=1}^{n} d_i^{\,k}$$

acting on

$$\mathcal{H} = \left(\mathbb{C}(z_1, .., z_n) \otimes (\mathbb{C}^N)^{\otimes n}\right)^{S_n}$$

define commutative differential operators $\hat{H}_k$.

They are Hamiltonians of quantum spin Calogero-Moser model (Hikami-Wadati, ...)

$$\hat{p}_j = -i\hbar \frac{\partial}{\partial q_j}, \quad z_j = e^{i q_j}$$

$$\hat{H}_1 = \sum_{j=1}^{n} \hat{p}_j$$

$$\hat{H}_2 = \frac{1}{2}\sum_{j=1}^{n} \hat{p}_j^{\,2} + \sum_{i \neq j} \frac{(1+\hbar P_{ij})}{8 \sin^2\left(\frac{q_i - q_j}{2}\right)},$$

$$\hat{H}_3 = \frac{1}{3}\sum_{j=1}^{n} \hat{p}_j^2 - \sum_{i\neq j} \frac{z_i z_j (1+\hbar P_{ij})}{(z_i - z_j)^2} -$$

$$- \frac{\hbar}{3} \sum_{k\neq j\neq i} \frac{z_i z_j z_k \quad P_{jk} P_{ij}}{(z_i - z_j)(z_j - z_k)(z_k - z_i)}$$

$$\ldots$$

As $\hbar \to 0$

$$\hat{H}_k = H_k^{(0)}(p,q) + \hbar H_k^{(1)}(p,q) + O(\hbar^2)$$

$$H_k^{(0)} \in C(T^* \mathbb{R}^N), \quad H_k^{(1)} \in C(T^* \mathbb{R}^N \to (\mathbb{C}^N)^{\otimes n})$$

$$H_1^c = \sum_{i=1}^{n} p_i \quad , \quad H_1^{(1)} = 0$$

$$H_2^c = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} \frac{1}{4\sin\left(\frac{q_i - q_j}{2}\right)^2} \quad , \quad H_2^{(1)} = -\frac{1}{2}\sum_{j\neq i} \frac{z_i z_j P_{ij}}{(z_i - z_j)^2}$$

$$H_3^c = \frac{1}{3}\sum_{i=1}^{n} p_i^3 - \sum_{i\neq j} \frac{z_i z_j p_i}{(z_i - z_j)^2} \quad , \quad H_3^{(1)} = -\frac{1}{2}\sum_{i\neq j} \frac{z_i z_j p_i P_{ij}}{(z_i - z_j)^2} - $$

$$H_k^c = \frac{1}{k} \operatorname{Tr}\left(\begin{pmatrix} p_1 & & \frac{z_i z_j}{z_i - z_j} \\ & \ddots & \\ \frac{z_j z_i}{z_j - z_i} & & p_n \end{pmatrix}^k\right) \quad , \quad -\frac{1}{3}\sum_{k\neq i\neq j} \frac{z_i z_j z_k P_{jk} P_{ij}}{(z_i - z_j)(z_j - z_k)(z_k - z_i)}$$

$$\cdots\cdots$$

The classical multitime evolution: $\quad x = (p, q)$

$$(M) \qquad \frac{\partial x(t)}{\partial t_k} = \omega^{-1}(dH_k)(t)$$

- On a generic (n-dimensional) level surface

$$\mathcal{M}(E) = \{ H_k^{(0)}(x) = E_k \}$$

the multitime evolution gives a hybrid integrable system

$$\left[i\frac{\partial}{\partial t_j} - H_j^{(1)}(x(t)),\ i\frac{\partial}{\partial t_k} - H_k^{(1)}(x(t)) \right] = 0$$

- <u>Thm</u> (A. Liashyk, N.R., I. Sechin) The "freezing point"

$q_j^* = \frac{2\pi j}{N}$, $p_j^* = 0$ is the only fixed point

of the multitime evolution (M)

Corollary $\quad [H^{(1)}_k(x_*), H^{(1)}_\ell(x_*)] = 0$

The operator

$$H^{(1)}_2(X_*) = \sum_{i\neq j} \frac{P_{ij}}{\sin^2(q_i^* - q_j^*)} = \sum_{i\neq j} \frac{P_{ij}}{\sin^2\left(\frac{\pi(i-j)}{N}\right)}.$$

is a well known Haldaine - Shastry model for long range spin interactions.

Thm (A. Liashyk, G. Ma, N.R., I. Sechin) For each $1 \leq k \leq n-1$ there are degenerate Liouville tori of dimension k. ($\binom{n}{k}$ types).

The next goal is to construct covariantly constant sections for $i\frac{\partial}{\partial t_k} - H_k^{(1)}(x(t))$ on Liouville tori of all dimensions, study this nonstationary dynamics and the limit $n \to \infty$.

Related upcoming work:

<u>Integrable systems on stratified symplectic spaces</u>

(CHEN Zhuo, JIANG Kai, N.R., XIAO Husileng)

Examples: G - simple, compact Lie group

T^*G/G - stratified Poisson space

$\cup$

$S(G) = \mu^{-1}(0)/G$ stratified symplectic spaces

Happy Birthday

Philippe !!!