Last passage percolation in a strip

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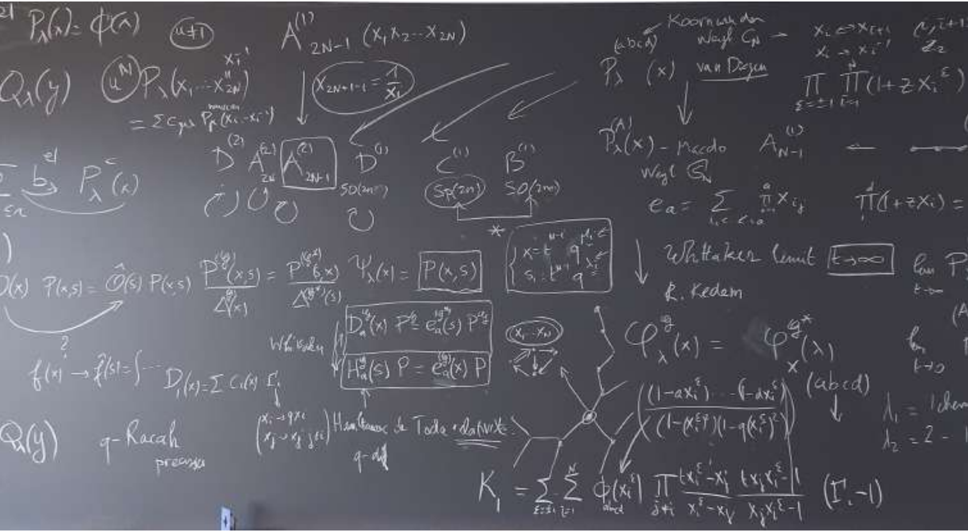
At the crossroads of physics and mathematics:
The joy of integrable combinatorics
in honor of Philippe Di Francesco's 60th birthday


Koornwinder
van Diejen
Whittaker limit
R. Kedem
q-Racah

Symmetric functions

The Schur functions

$$s_\lambda(x) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\det\left(x_i^{n-j}\right)_{i,j=1}^{n}}$$

where $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0)$ and $x = \{x_1, \ldots, x_n\}$ are symmetric polynomials satisfying the Cauchy summation identity

$$\sum_\lambda s_\lambda(a_1, \ldots, a_n) s_\lambda(b_1, \ldots, b_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - a_i b_j} =: \Pi(a; b).$$

Schur functions satisfy a branching rule that allows to expand them in monomials

$$s_\lambda(x) = \sum_{\emptyset \prec \lambda^1 \prec \cdots \prec \lambda^n = \lambda} \prod_{i=1}^{n} x_i^{|\lambda^i| - |\lambda^{i-1}|}$$

where $\emptyset \prec \lambda^1 \prec \cdots \prec \lambda^n = \lambda$ is a sequence of interlaced partitions. We write $\mu \prec \lambda$ for $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \ldots$

RSK correspondence

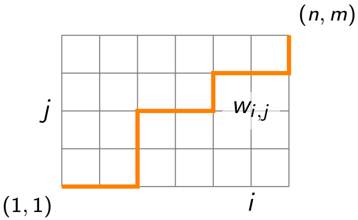
Expanding in monomials each side of the Cauchy identity

$$\sum_{\lambda} s_\lambda(a_1, \dots, a_n) s_\lambda(b_1, \dots, b_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - a_i b_j},$$

the two sides acan be matched using Robinson-Schensted-Knuth correspondence, a bijection

$$\begin{Bmatrix} \emptyset \prec \lambda^1 \prec \cdots \prec \lambda^n = \lambda \\ \emptyset \prec \mu^1 \prec \cdots \prec \mu^m = \lambda \end{Bmatrix} \quad \longleftrightarrow \quad W = (w_{i,j}) \in \mathbb{N}^{n \times m}$$

Greene's theorem implies that $\lambda_1 = G(n, m)$ where



$$G(n, m) = \max_{\text{paths } (1,1) \to (n,m)} \left\{ \sum_{(i,j) \in \text{ path}} w_{i,j} \right\}.$$

Schur measure

Assume that when $w_{i,j} \sim \mathrm{Geom}(a_i b_j)$ are independent (we say that $w \sim \mathrm{Geom}(q)$ if $\mathbb{P}(w = k) = (1-q)q^k$). Then,

$$\mathbb{P}\left(G(n,m) \leqslant r\right) = \frac{1}{\Pi(a;b)} \sum_{\lambda:\lambda_1 \leqslant r} s_\lambda(a) s_\lambda(b)$$

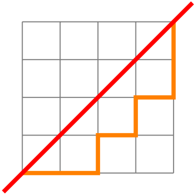
In other terms, $G(n,m)$ has the same law as λ_1 when λ is a random partition sampled according to the Schur measure

$$\mathbb{P}(\lambda) = \frac{1}{\Pi(a;b)} s_\lambda(a) s_\lambda(b).$$

Asymptotics

[Johansson, 2001] proved that after appropriate rescaling, $G(n,m)$ fluctuates according to the Tracy-Widom GUE distribution (governing fluctuations of the largest eigenvalue of Hermitian random matrices of large size).

A variant



Assume that the weight matrix is symmetric:

$$w_{i,j} = w_{j,i}$$

$$w_{i,j} \sim \mathrm{Geom}(a_i a_j) \text{ for } i > j$$

$$w_{i,i} \sim \mathrm{Geom}(c a_i)$$

$$G^{\boxslash}(n,m) = \max_{\text{paths } (1,1)\to(n,m)} \sum_{(i,j)\in \text{ path}} w_{i,j}.$$

[Baik-Rains 2003] proved that

$$\mathbb{P}\left(G^{\boxslash}(n,n) \leqslant r\right) = \frac{1}{\Pi(a,c)\Phi(a)} \sum_{\lambda:\lambda_1\leqslant r} c^{\lambda_1-\lambda_2+\lambda_3-\lambda_4+\dots} s_\lambda(a)$$

where

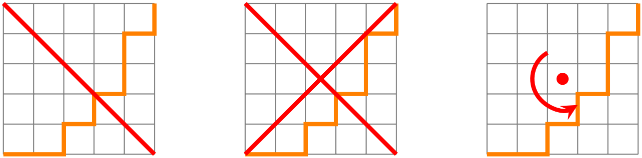
$$\Phi(x) = \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_{\lambda' \text{ even}} s_\lambda(x).$$

Asymptotics

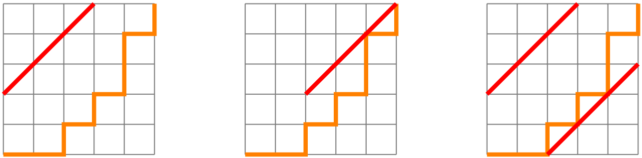
As $n\to\infty$, $G^{\boxslash}(n,n)$ fluctuates according to the Tracy-Widom GSE or GOE distributions (depending if $c=1$ or $c<1$).

Other variants

[Baik-Rains 2003] also computed $\mathbb{P}\left(G^{\bullet}(n,n) \leqslant r\right)$ in terms of Schur functions for other symmetry types $\bullet$:



The problem is however open for symmetries such as

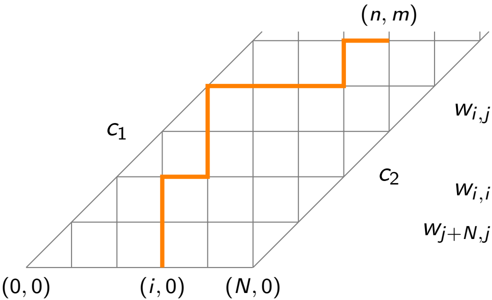


Last passage percolation with walls

Imposing two symmetry axis in the diagonal direction is equivalent to assuming that paths are constrained to remain between two walls.

Last passage percolation in a strip

Let $a_1, \ldots, a_N \in (0,1)$, $c_1, c_2 > 0$.



$w_{i,j} \sim \mathrm{Geom}(a_i a_j)$ for $j < i < j + N$ (indices modulo N)

$w_{i,i} \sim \mathrm{Geom}(c_1 a_i)$

$w_{j+N,j} \sim \mathrm{Geom}(c_2 a_j)$

We fix an initial condition $G(i,0) = G_0(i)$ for some function G_0.

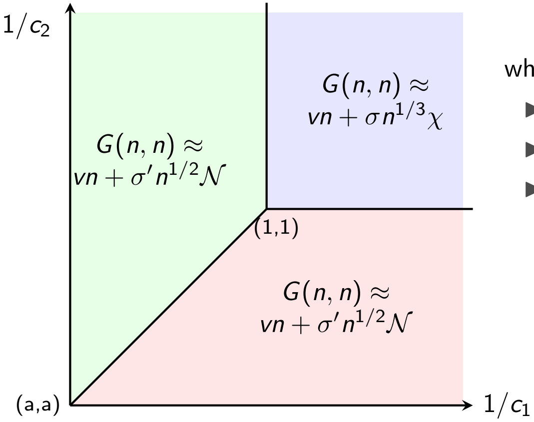
$$G(n,m) = \max_{\text{paths } (i,0)\to(n,m)} \left\{ G_0(i) + \sum_{(i,j)\in \text{ path}} w_{i,j} \right\}$$

Open Problem

Find the asymptotic distribution of $G(n,m)$ as $n, m \to \infty$, depending on the width N and the boundary parameters c_1, c_2.

Conjectural phase diagram

The richest behaviour is when $N = Ln^{2/3}$. For $a_1 = \cdots = a_N = a$, it is expected that



where

- $v = v(c_1, c_2, L)$,
- $\mathcal{N} \sim$ Gaussian
- $\chi \sim$ unknown distribution depending on c_1, c_2, L. This is an open problem for any model in the same universality class.

Stationary measure (=non-equilibrium steady-state)

However, we can find the asymptotic distribution of

$$G_t(i) = G(t+i,t) - G(t,t)$$

as t goes to infinity. G_t is a Markov process on $\mathbb{Z}^N$.

Problem

Find the law of the initial condition $(G_0(i))_{1\leqslant i\leqslant N}$ such that for all t, $(G_t(i))_{1\leqslant i\leqslant N} \overset{(d)}{=} (G_0(i))_{1\leqslant i\leqslant N}$

- For models such as Asymmetric Simple Exclusion Process (ASEP), the standard method is the **matrix product ansatz** [Derrida-Evans-Hakim-Pasquier 1993].
- We will illustrate another approach based on symmetric functions, taking the example of Last Passage Percolation and Schur functions.
- The method works as well for other models: Log-gamma polymer and KPZ equation (Whittaker functions), stochastic six vertex model (Hall-Littlewood polynomials), other examples.

Stationary measure

Assume for simplicity that $a_1 = \cdots = a_N = a$.

For $R = (R(j))_{1 \leqslant j \leqslant N}$, let

$$\mathbb{P}_q^{\mathrm{RW}}(R) = \prod_{j=1}^{N} (1-q) q^{R(j)-R(j-1)}$$

be the probability that R is a random walk with $\mathrm{Geom}(q)$ increments.

Define a Pitman-type operation

$$R_1 \star R_2(k) = \min_{1 \leqslant j \leqslant k} \{ R_1(j-1) + R_2(k) - R_2(j) \} .$$

Consider the probability measure

$$\mathbb{P}_{a,c_1,c_2}(R_1, R_2) = \frac{1}{Z} (c_1 c_2)^{-R_1 \star R_2(N)} \times \mathbb{P}^{\mathrm{RW}}_{ac_1}(R_1) \times \mathbb{P}^{\mathrm{RW}}_{ac_2}(R_2).$$

Theorem ([B.-Corwin-Yang 2023])

For any parameters a, c_1, c_2, the marginal law of R_1 under $\mathbb{P}_{a,c_1,c_2}$ is the unique stationary measure of the Markov process G_t.

A variant of the Schur process

The random walks R_1, R_2 are related to a sequence of ~~partitions~~ signatures:

$$R_1(j) = \lambda_1^{(j)} - \lambda_1^{(0)}, \quad R_2(j) = \lambda_2^{(j)} - \lambda_2^{(0)}$$

where $\boldsymbol{\lambda} = \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(N)}$ is a sequence of interlaced signatures $\lambda^{(j)} = (\lambda_1^{(j)} \geqslant \lambda_2^{(j)}) \in \mathbb{Z}^2$ distributed as

$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{Z_{a,c_1,c_2}(N)} c_1^{\lambda_1^{(0)} - \lambda_2^{(0)}} c_2^{\lambda_1^{(N)} - \lambda_2^{(N)}} \prod_{j=1}^{N} s_{\lambda^{(j)}/\lambda^{(j-1)}}(a_i)$$

and $s_{\lambda/\mu}$ denote skew Schur functions

$$s_{\lambda/\mu}(x) = \mathbb{1}_{\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2} x^{\lambda_1 + \lambda_2 - \mu_1 - \mu_2}.$$

The construction is similar to the free boundary Schur process [Betea-Bouttier-Nejjar-Vuletic 2017] except that the $\lambda^{(j)}$ are no longer integer partitions:

- parts can be negative,
- all signatures have length 2.
- the measure $\mathbb{P}$ is infinite (but becomes a probability measure if we fix $\lambda_2^{(0)} = 0$).

Properties of Schur functions yield explicit formulas:

$$\mathbb{E}\left[t^{2R_1(N)}\right] = \frac{1}{Z_{a,c_1,c_2}(N)} \oint \frac{dz}{2\mathbf{i}\pi z} \left| \frac{1-z^2}{(1-taz)^N(1-zc_1/t)(1-zc_2 t)} \right|^2 .$$

More generally, for $0 = x_0 < \cdots < x_k = N$, there is a simple formula for

$$\mathbb{E}\left[\prod_{i=1}^{k} t_i^{2(R_1(x_i)-R_1(x_{i-1}))}\right].$$

In particular, one can deduce that starting from the stationary initial data, i.e. $G_0 \overset{(d)}{=} R_1$, we have

$$\mathbb{E}[G(n,n)] = n \times v(c_1, c_2, N)$$

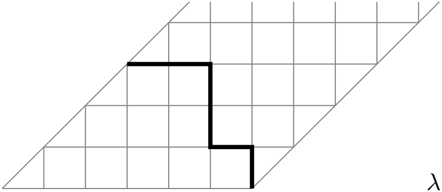
where

$$v(c_1, c_2, N) = \frac{(1-a^2)Z_{a,c_1,c_2}(N+1) - Z_{a,c_1,c_2}(N)}{Z_{a,c_1,c_2}(N)}$$

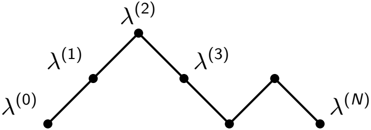
with

$$Z_{a,c_1,c_2}(N) = \oint \frac{dz}{2\mathbf{i}\pi z} \left| \frac{1-z^2}{(1-az)^N(1-zc_1)(1-zc_2)} \right|^2 .$$

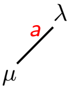
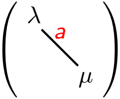
More general two-layer Schur process



$$\boldsymbol{\lambda} = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(N)}\right)$$



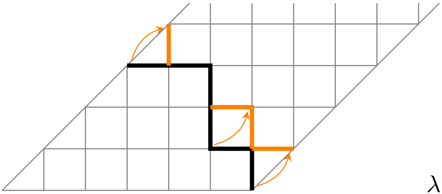
Vertices on the path are decorated by signatures $\lambda = (\lambda_1 \geqslant \lambda_2) \in \mathbb{Z}^2$.
We define a probability measure on $\boldsymbol{\lambda}$ by taking the product of Boltzmann weights

wt  = wt  $= s_{\lambda/\mu}(a)$

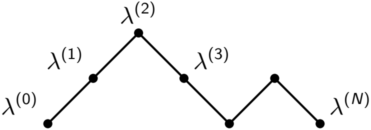
$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{Z_{a,c_1,c_2}(N)} c_1^{\lambda_1^{(0)} - \lambda_2^{(0)}} c_2^{\lambda_1^{(N)} - \lambda_2^{(N)}} \prod_{\text{edges } e} \mathrm{wt}(e)$$

When $c_1 c_2 < 1$, this is a well-defined probability measure.

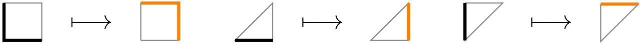
Dynamics on the two-layer Schur process



$$\boldsymbol{\lambda} = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(N)}\right)$$



We construct dynamics on $\boldsymbol{\lambda}$, inspired by [Borodin-Ferrari 2008], such that when the path evolves by the elementary moves



1. The two-layer Schur process is mapped to a two layer Schur process;
2. the λ_1 marginal of the dynamics corresponds to the recurrence of geometric LPP.

After averaging over $\lambda_1^{(0)}, \lambda_2^{(0)}$, the law of $R_1(j) = \lambda_1^{(j)} - \lambda_1^{(0)}$ simplifies and can be analytically continued to all c_1, c_2.

Connection to the Matrix Product Ansatz

The law of the stationary measure R_1 can be written, in terms of increments $\Delta(j) = R_1(j) - R_1(j-1) \in \mathbb{N}$ as

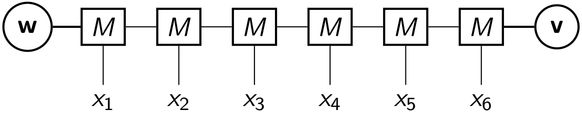
$$\mathbb{P}\left(\bigcap_{j=1}^{N} \{\Delta(j) = x_j\}\right) = \frac{1}{Z(N)} \mathbf{w}^t \left(\prod_{j=1}^{N} M(x_j; \cdot, \cdot)\right) \mathbf{v}$$

where

$$M(x; n, n') = s_{\lambda'/\lambda}(a) \text{ where } \begin{cases} n & = \lambda_1 - \lambda_2, \\ n' & = \lambda'_1 - \lambda'_2, \\ x & = \lambda'_1 - \lambda'_1, \end{cases}$$

with $\mathbf{w}^t = (1, c_1, c_1^2, \dots)$ and $\mathbf{v}^t = (1, c_2, c_2^2, \dots)$.

In other terms, the stationary measure is the Matrix Product State



Hence, another way to interpret the talk is that **skew Schur functions provide a representation of the MPA relations for Last Passage Percolation.**

Conclusion

Summary

The two-layer Schur processes allows to describe the stationary measure of LPP in a strip in terms of reweighted random walks.

The stationary measure is not a priori a Gibbs measure, but becomes so on some enlarged state space.

The method becomes more interesting when applied to more complicated models (log-gamma polymer, KPZ equation) related to Whittaker functions.

Outlook

- The method applies to other families of symmetric functions [B.-Corwin, in progress]
- Go beyond the stationary measure.

Thank you for your attention!