Philippe60: Paths to positivity

Rinat Kedem

Philippe60 IPhT Saclay 2024

A few favorite objects:

- Chebyshev polynomials

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- Chebyshev polynomials
- Partial fraction decomposition
- Lindström Gessel-Viennot
- Heaps, dimers and hard particles
- Generating functions
- Continued fractions

$\vdots$

Essential facts

- Hobby: Calculating

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- Superpower: Calculating

Essential facts

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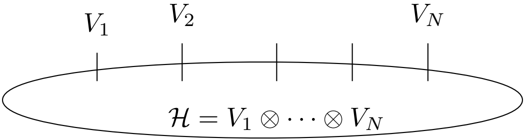
- Superpower: Calculating
- Motto: “Formulas speak to me”

Role model: Alexei Stakhanov

A formula: "Kirillov-Reshetikhin conjecture"

Counting weighted Bethe states of generalized Heisenberg spin chains: "Fermionic character formulas"

$$Z_{\mathcal{H},\lambda}(t) = \sum_{\substack{m_i \geq 0 \\ \text{selection rules}}} \prod_i \begin{bmatrix} p_i(\mathbf{m}) + m_i \\ m_i \end{bmatrix}_t t^{E(\mathbf{p},\mathbf{m})}$$



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- Grading=Power of $t \sim$ sum of Bethe integers

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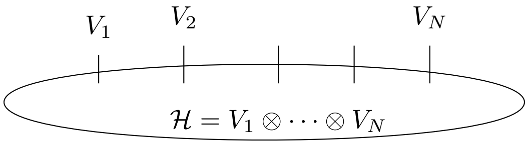

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Problem: Prove a positivity condition for "selection rules" on $\{m_i\}$

Solution: Chebyshev polynomials! ($SU(2)$)

Relax selection rules: Partition function $\rightsquigarrow$ generating function

$$Z_{\mathcal{H},\lambda}(t; Q_0, Q_1) = \sum_{m_i \geq 0} Q_0^{p_1 - p} Q_1^{p} \prod_i \begin{bmatrix} p_i - p + m_i \\ m_i \end{bmatrix}_t t^{\widetilde{E}(\mathbf{p},\mathbf{m})}$$

with $Q_0 Q_1 = t^{\frac{1}{2}} Q_1 Q_0$. **No section rules**.

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- Constant term in Q_1 of $Z_{\mathcal{H},\lambda}(t;1,Q_1)$ is the character formula.
- Can sum over $m_1, m_2, \ldots$: result is factorized:

$$Z_{\mathcal{H},\lambda}(q)=q^{\sharp}\, Q_1Q_0^{-1}\left(\prod_k^{\rightarrow} Q_k^{n_k}\right)\left(\lim_{n\to\infty} Q_nQ_{n+1}^{-1}\right)^{\lambda+1}, \quad (\{n_j\}\leftrightarrow\mathcal{H})$$

if Q_k are solutions of the quantum Q-system

$$t^{\frac{1}{2}}Q_{k+1}Q_{k-1}=Q_k^2-1$$

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5. Non-commutative $\mathbf{R}_n$: [Kontsevich] $\mathbf{R}_{n+1}C\mathbf{R}_{n-1} = \mathbf{R}_n^2 + 1$ with $C = \mathbf{R}_{n+1}^{-1}\mathbf{R}_n\mathbf{R}_{n+1}\mathbf{R}_n^{-1}$:
   $\Rightarrow \mathbf{R}_n$ Laurent in $\mathbf{R}_0, \mathbf{R}_1$ with coefficients in {0,1}

Path solution to Kontsevich recursion

1. $\mathbf{R}_{n+1} C \mathbf{R}_{n-1} = \mathbf{R}_n^2 + 1$ is integrable discrete evolution $n \mapsto n+1$,
2. Conserved quantities $C = \mathbf{R}_{n+1}^{-1} \mathbf{R}_n \mathbf{R}_{n+1} \mathbf{R}_n^{-1}$ and

$$H = \underbrace{\mathbf{R}_{n+1} \mathbf{R}_n^{-1}}_{y_1} + \underbrace{\mathbf{R}_{n+1}^{-1} \mathbf{R}_n^{-1}}_{y_2} + \underbrace{\mathbf{R}_{n+1}^{-1} \mathbf{R}_n}_{y_3}$$

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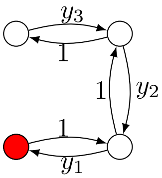
3. Linear recursion relations with constant coefficients $\Rightarrow R_n R_0^{-1} =$ partition function of weighted paths of length $2n$ on Graph:

Graph



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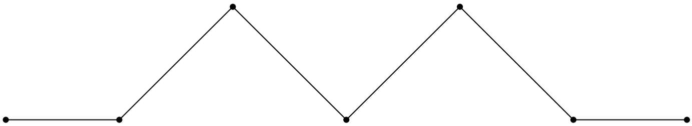

weight$=y_2^2 y_1$

Higher rank Chebyshev

$SU(N)$ renormalized Q-system

$$R_{a,n+1}R_{a,n-1} = R_{a,n}^2 + R_{a+1,n}R_{a-1,n}, \; 1 \leq a < N$$

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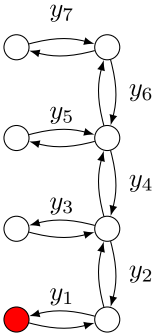
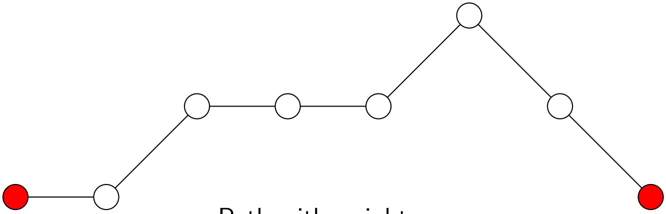
- $R_{1,n}R_{1,0}^{-1}$ = partition function of paths ● to ● of length $2n$ on:



Graph for $N = 4$



Path with weight $y_3y_4y_2y_1$

Lindström-Gessel-Viennot

The Q-system $R_{a,n+1}R_{a,n-1} = R_{a,n}^2 + R_{a+1,n}R_{n-1,n}$ is a Desnanot-Jacobi relation for Wronskian determinants

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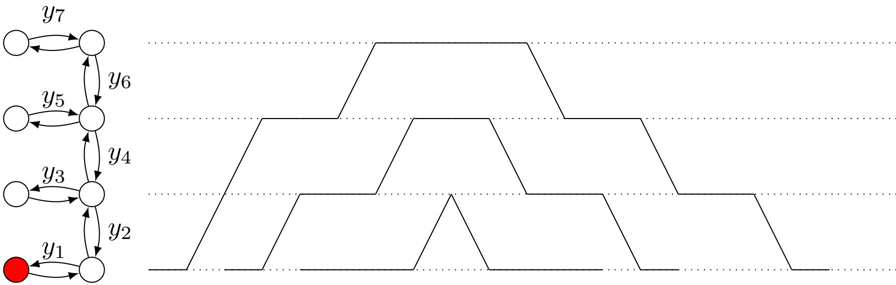
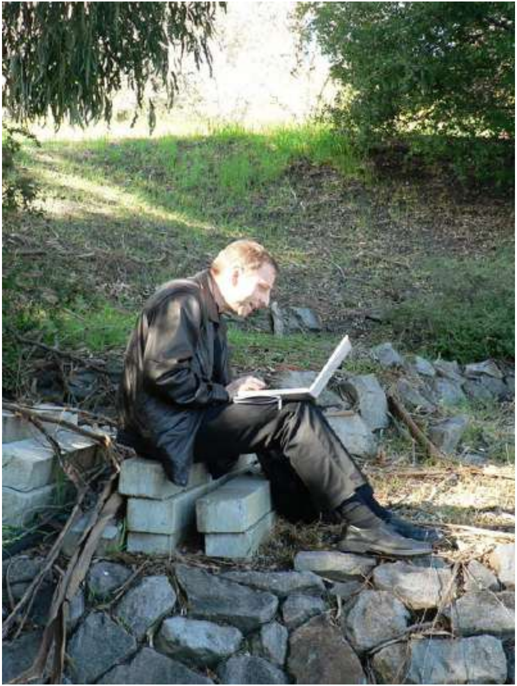
Lindström-Gessel-Viennot: $R_{a,n}$ = partition function of a non-intersecting paths on the same graph.



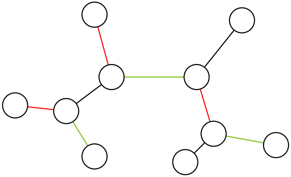
$\Rightarrow$ $R_{a,n}$ positive polynomials in the weights $\{y_k\}$, Laurent monomials in initial data.



“Formulas speak to me”

Cluster algebras

- Introduced by Fomin-Zelevinsky [2000], quantization by Berenstein-Zelevinsky, Gekhtman-Shapiro-Vainshtein, Fock-Goncharov.
- r Generators (cluster variables) at each node in a regular r-tree



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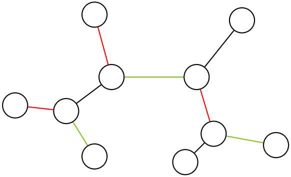

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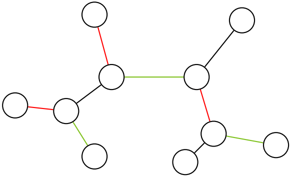

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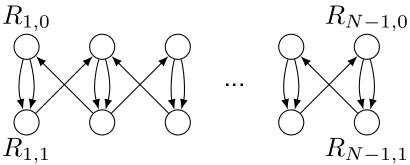
- Basic theorem: **Laurent property** [FZ]: Any cluster variable is a Laurent polynomial in the cluster variables of any other fixed node.
- **Positivity** (conjecture/theorem): The Laurent polynomial has positive integer coefficients.

Q systems and cluster algebras

- Q-system variables are cluster ($\mathcal{A}$-) variables for a cluster algebra with initial quiver/exchange matrix



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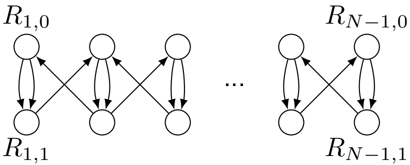

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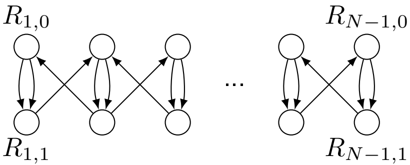

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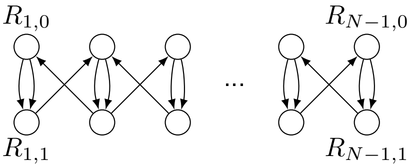

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- Path solutions $\Rightarrow$ (partial) positivity proof.
- Q-systems for each (affine) root system associated with quiver coded by root system.

Toda Hamiltonians and Macdonald theory

Quantum Q-system for $SU(N)$

$$\begin{aligned} q^a Q_{a,n+1} Q_{a,n-1} &= Q_{a,n}^2 - Q_{a+1,n} Q_{a-1,n},\ 1 \leq a \leq N \\ Q_{a,0} Q_{b,1} &= q^{\min(a,b)} Q_{b,1} Q_{a,0} \end{aligned}$$

- Integrable evolution in discrete time n, Conserved quantities $H_a =$ quantum Toda Hamiltonians
- $\{Q_{a,k}, H_a\}_a$ generate $t \to \infty$ limit of spherical double affine Hecke algebra
- Functional representation of $Q_{a,k}$: q-difference operators, limit of Macdonald operators and their time-translation.

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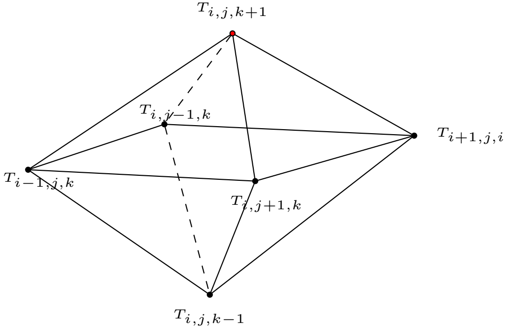
c.f. Talk by Alexander Shapiro, Wednesday afternoon.

Further generalization: T-systems

Q-systems with spectral parameter = T-systems = Octahedron recursion

$$T_{a,j,k+1}T_{a,j,k-1} = T_{a,j+1,k}T_{a,j-1,k} + T_{a+1,j,k}T_{a-1,j,k}$$



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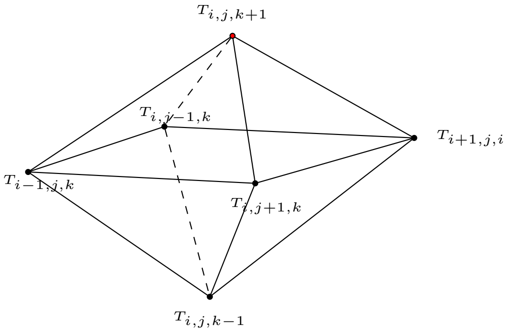
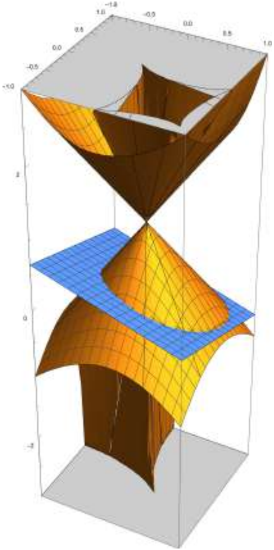


- Infinite rank cluster algebra/Discrete evolution in k
- Initial data surface $\mathcal{S}_0 = \{a, j, k(a, j)\}_{a,j}$
- $T_{i,j,k}$ = dimer partition function in region in finite region in $\mathcal{S}_0$



Large k asymptotics gives arctic curves
[work with: Soto Garrido, Trung Vu]

c.f. talks later this afternoon...

Happy 60th Philippe!

To a hundred and twenty!