PDF, ASM, DPP and TSSCPP

June 23, 2024

At the crossroads of physics and mathematics:
the joy of integrable combinatorics – A conference in honour of
Philippe Di Francesco's 60th birthday

Some pictures. . . ESI '12

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Some pictures. . . Giens (Stroganov memorial) '14

Some pictures. . . Giens (Stroganov memorial) '14 cont'd

Some pictures. . . Giens (Stroganov memorial) ’14 cont’d

Papers

2003

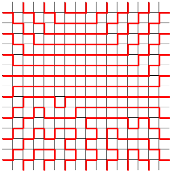
- PDF, PZJ, JBZ, *A bijection between classes of Fully Packed Loops and Plane Partitions.*
- PDF, JBZ, *On FPL configurations with four sets of nested arches.*
- PDF, *A refined Razumov–Stroganov conjecture.*
- PDF, *A refined Razumov–Stroganov conjecture II.*
- PDF, PZJ, JBZ, *Determinant Formulae for some Tiling Problems and Application to Fully Packed Loops.*
- PDF, PZJ, *Around the Razumov–Stroganov conjecture: proof of a multi-parameter sum rule.*
- PDF, PZJ, *Inhomogenous model of crossing loops and multidegrees of some algebraic varieties.*
- PDF, *Inhomogeneous loop models with open boundaries.*
- PDF, PZJ, *Quantum Knizhnik–Zamolodchikov equation, generalized Razumov–Stroganov sum rules and extended Joseph polynomials.*
- PDF, *Boundary qKZ equation and generalized Razumov–Stroganov sum rules for open IRF models.*
- PDF, PZJ, *From Orbital Varieties to Alternating Sign Matrices.*
- PDF, PZJ, JBZ, *Sum rules for the ground states of the O(1) loop model on a cylinder and the XXZ spin chain.*
- PDF, *Totally Symmetric Self-Complementary Plane Partitions and Quantum Knizhnik–Zamolodchikov equation: a conjecture.*
- PDF, *Open boundary Quantum Knizhnik–Zamolodchikov equation and the weighted enumeration of Plane Partitions with symmetries.*
- PDF, PZJ, *Quantum Knizhnik–Zamolodchikov Equation, Totally Symmetric Self-Complementary Plane Partitions and Alternating Sign Matrices.*

2007

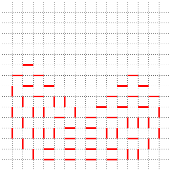
- PDF, PZJ, *Quantum Knizhnik–Zamolodchikov equation: reflecting boundary conditions and combinatorics.*
- RB, PDF, PZJ, *On the weighted enumeration of alternating sign matrices and descending plane partitions.*
- RB, PDF, PZJ, *A doubly-refined enumeration of alternating sign matrices and descending plane partitions.*

2012

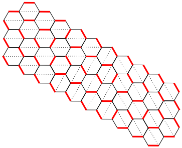
Result 1: Bijection FPLs / plane partitions



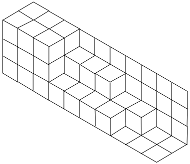
FPL



active part of FPL



dimers



tiling

Theorem (PDF, PZJ, JBZ '03)

The process above gives a bijection between FPLs with three sets of a, b, c nested arches and lozenge tilings of a $a \times b \times c$ hexagon (a.k.a. boxed plane partitions).

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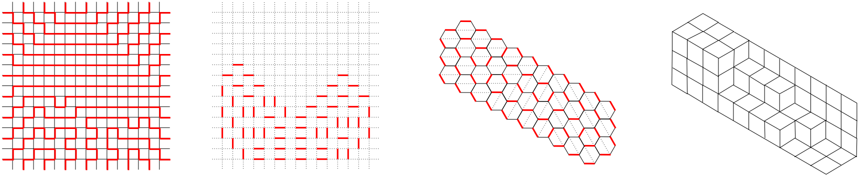

FPL   active part of FPL   dimers   tiling

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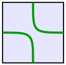
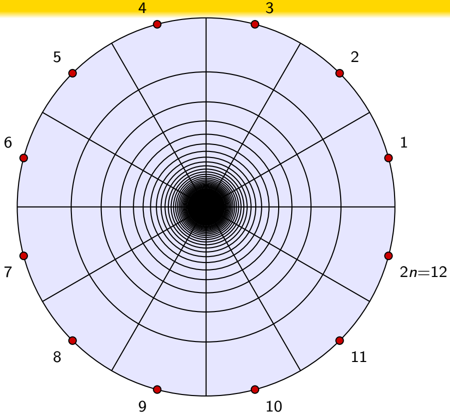
Result 2: Noncrossing loop model and ASMs

Decorate a semi-infinite cylinder with randomly rotated :



Probability P_n that $k \leftrightarrow 2n+1-k$ for all k, i.e.,

 ?

Theorem (PDF, PZJ '05)

$$P_n = \frac{1}{\#\mathrm{ASM}(n)} = \prod_{i=0}^{n-1} \frac{(n+i)!}{(3i+1)!} = 1, \frac{1}{2}, \frac{1}{7}, \frac{1}{42} \ldots$$

where $\mathrm{ASM}(n)$ is the set of Alternating Sign Matrices of size n.

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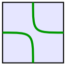
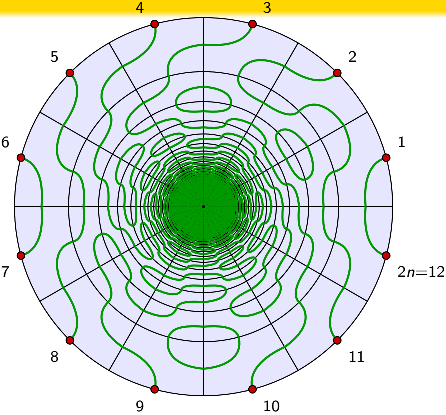
Decorate a semi-infinite cylinder with randomly rotated :



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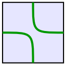
Decorate a semi-infinite cylinder with randomly rotated :

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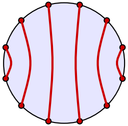
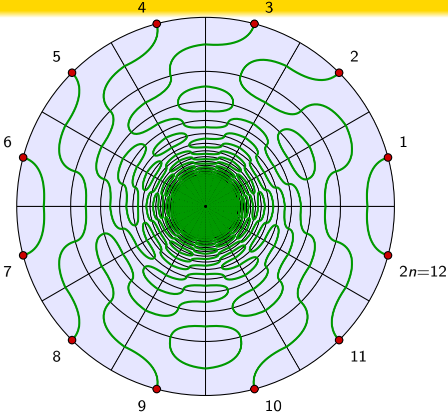
 ?



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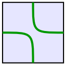
Decorate a semi-infinite cylinder with randomly rotated :

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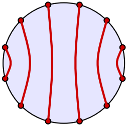
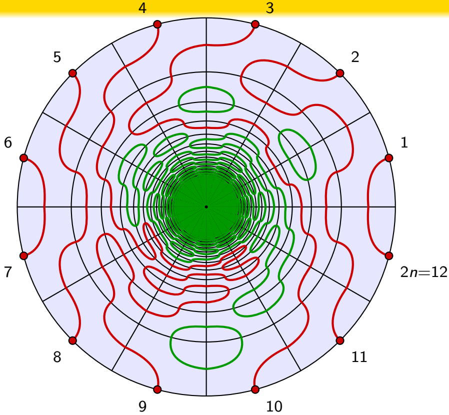
 ?



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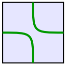
Decorate a semi-infinite cylinder with randomly rotated :

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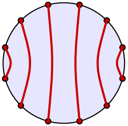
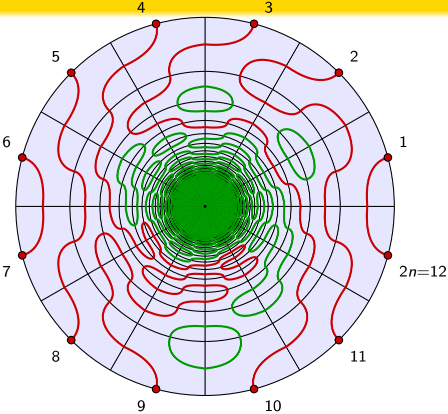
 ?



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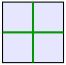
Result 3: Crossing loop model and geometry

Add to the loop model a crossing  with probability $1/9$:

Probability P'_n that $k \leftrightarrow 2n+1-k$ for all k, i.e.,

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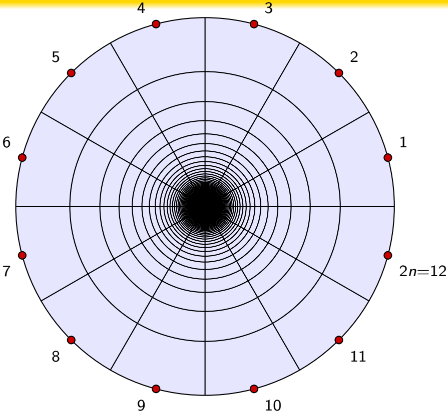

Theorem (PDF, PZJ '06)

$$P'_n = \frac{\deg C_n}{\det_{i,j=0,\dots,n-1} \binom{2i+2j+1}{2i}} = 1, \frac{3}{7}, \frac{31}{307}, \frac{1145}{82977} \dots$$

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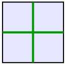
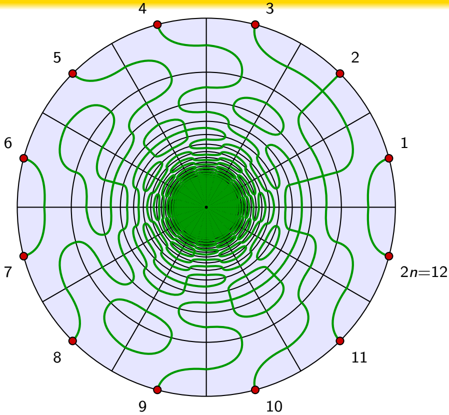
Add to the loop model a crossing  with probability 1/9:



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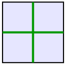
Add to the loop model a crossing  with probability 1/9:

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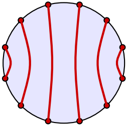
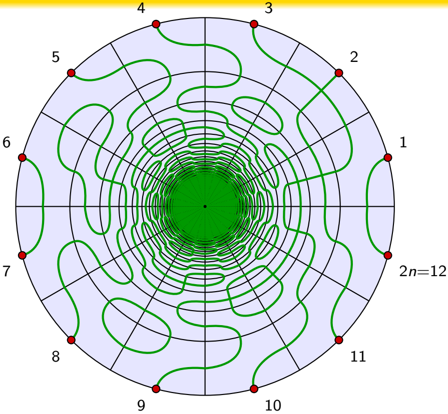
 ?



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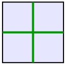

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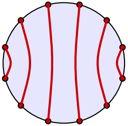

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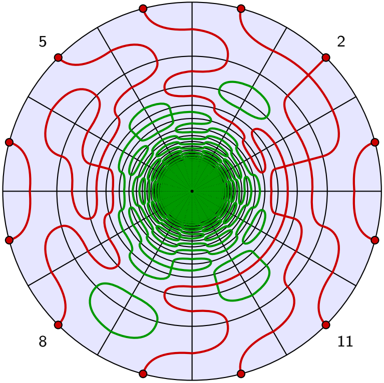

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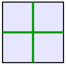

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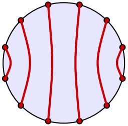

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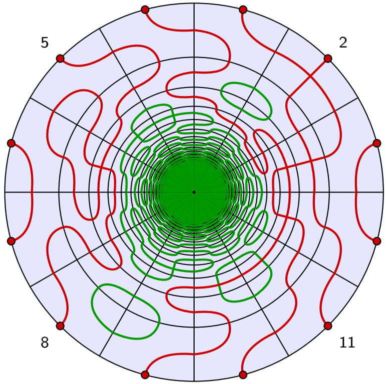

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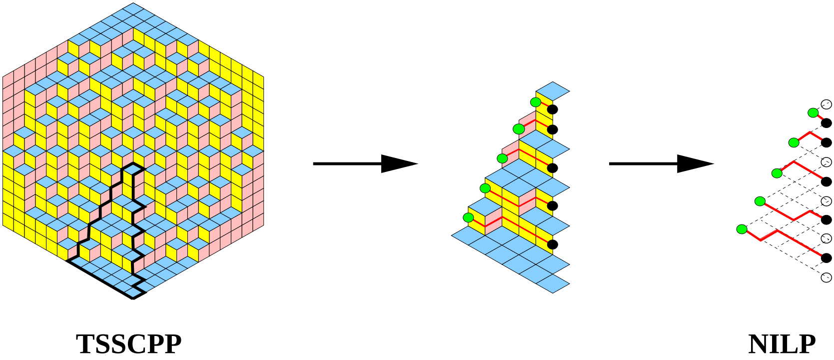
Result 4: qKZ and TSSCPPs

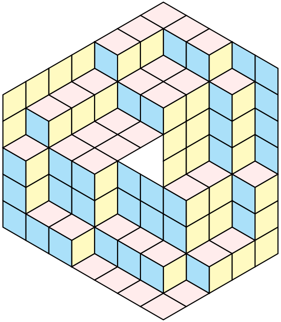


Theorem (PDF '06; PDF, PZJ '08)

$$\sum_{\pi} \Psi_{\pi}|_{homogeneous} = \sum_{\text{TSSCPPs}} \tau^{\#\ pink\ lozenges}$$

Remark: #TSSCPPs = #ASMs. (but no known bijection!)

Result 5: DPPs and ASMs



$\longleftrightarrow$

0	1	0	0
0	0	1	0
1	-1	0	1
0	1	0	0

Theorem (RB, PDF, PZJ '12)

The weighted enumeration of ASMs and DPPs coincide, with two bulk statistics and two boundary statistics.

Generalises a conjecture of [Mills, Robbins, Rumsey '83].
On the ASM side, the statistics are: number of -1s, inversion number, and locations of top/bottom 1s.

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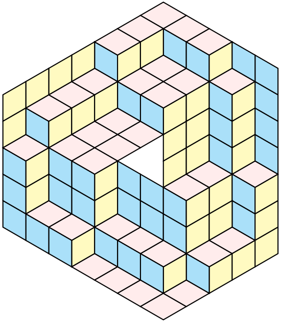

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$$\sum_{m=0}^{\infty} q^m \prod_{n=1}^{m} \frac{1}{1-q^n} \prod_{n=1}^{m+3} \frac{1}{1-q^n}$$
$$= 1 + 2q + 5q^2 + 10q^3 + 19q^4 + 34q^5 + 60q^6 + 100q^7 + \cdots$$

Congratulations!

Wish you heaps and heaps of new adventures!