Philippe and the joyful integrable combinatorics of 2D quantum gravity

Jérémie Bouttier

Philippe60, 24 June 2024

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It quickly became clear to me that I wanted to do my PhD with him and Emmanuel.
They agreed.

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Ann. Inst. Henri Poincaré,

Vol. 59, n° 2, 1993, p. 117-139. *Physique théorique*

A generating function for fatgraphs

by

P. DI FRANCESCO and C. ITZYKSON

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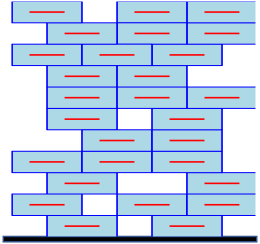
ELSEVIER

Physics Reports

Volume 254, Issues 1–2, March 1995, Pages 1-133

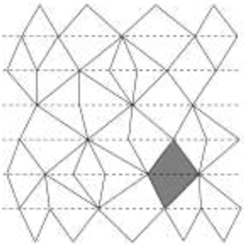
2D gravity and random matrices

P. Di Francesco [a], P. Ginsparg [b], J. Zinn-Justin [a]



heaps

X. Viennot (1986)



Integrable 2D Lorentzian gravity and random walks

P. Di Francesco [a,1], E. Guitter [a], C. Kristjansen [b,2]

[a] CEA-Saclay, Service de Physique Théorique, F-91191, Gif sur Yvette Cedex, France
[b] The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Received 14 July 1999; accepted 11 October 1999

Our first common paper

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J. Phys. A: Math. Gen. **35** (2002) 3821–3854 — PII: S0305-4470(02)33016-6

Critical and tricritical hard objects on bicolourable random lattices: exact solutions

J Bouttier, P Di Francesco and E Guitter

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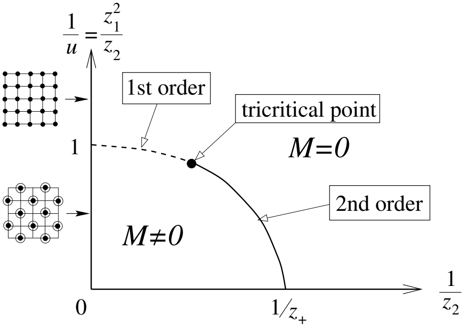

We studied a random lattice analogue of the hard-square model, and a dilute version thereof. We computed exact generating functions via matrix integral techniques, identified the phase diagram, critical exponents, etc. I loved this mixture of combinatorics and physics.

The bijective revolution

The bijective revolution

arXiv > math > arXiv:math/0205226

Mathematics > Combinatorics

[Submitted on 22 May 2002]

Random Planar Lattices and Integrated SuperBrownian Excursion

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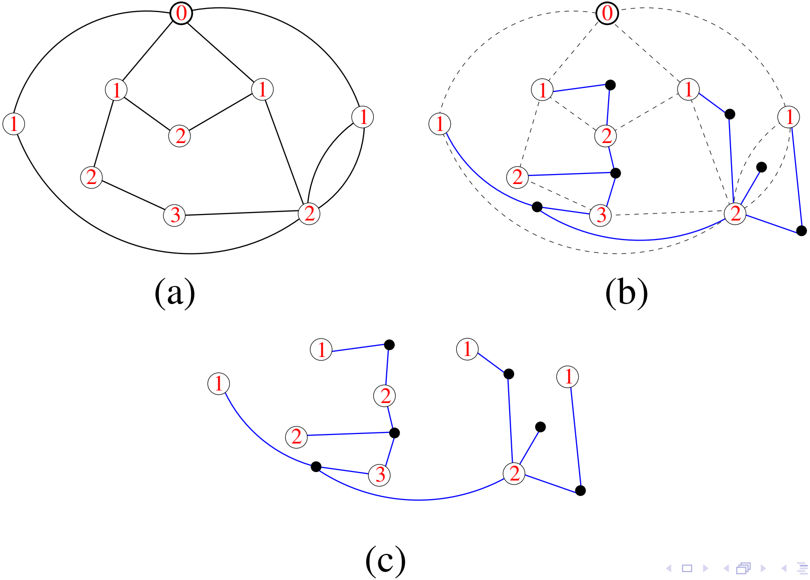
Mireille and Gilles replied with:

- The degree distribution in bipartite planar maps: applications to the Ising model

[Submitted on 4 Nov 2002]

Our most successful paper in this vein has been Planar maps as labeled mobiles (2004) where we introduce the BDG bijection, extending the Cori-Vauquelin-Schaeffer bijection between planar quadrangulations and well-labeled trees to maps with arbitrary (controlled) face degrees.



Integrability strikes back

One day, we noticed that the tree bijections led naturally to recurrence equations such as

$$R_n = 1 + gR_n(R_{n-1} + R_n + R_{n+1}), \quad n \geq 1, \qquad R_0 = 0.$$

Here, R_n is essentially the generating function of planar quadrangulations with two points at distance at most n, i.e. the discrete two-point function.

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Shortly after, Philippe came up with the explicit solution:

$$R_n = R\frac{(1-x^n)(1-x^{n+3})}{(1-x^{n+1})(1-x^{n+2})}$$

with $R = 1 + 3gR^2$, $x + x^{-1} = \frac{1-4gR}{gR}$.

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with $R = 1 + 3gR^2$, $x + x^{-1} = \frac{1-4gR}{gR}$. By taking an appropriate scaling limit, we were able to prove the Ambjørn-Watabiki prediction for the (continuum) two-point function of pure 2D quantum gravity. See the paper Geodesic distances in planar graphs (2003).

Beyond quadrangulations: Philippe's "prediction"

Upon redefining them like in previous section as $\alpha_i = x_i(1-x_i)(1-x_i^2)\lambda_i$, and resumming the expression for R_n, we arrive at

$$R_n = R\frac{u_n^{(m)} u_{n+3}^{(m)}}{u_{n+1}^{(m)} u_{n+2}^{(m)}},$$

$$u_n^{(m)} = \sum_{l=0}^{m} (-1)^l \sum_{1 \leqslant m_1 < \cdots < m_l \leqslant m} \prod_{i=1}^{l} \lambda_{m_i} x_{m_i}^{n+m} \prod_{1 \leqslant i < j \leqslant l} c_{m_i, m_j},$$

$$c_{a,b} \equiv \frac{(x_a - x_b)^2}{(1 - x_a x_b)^2}, \tag{5.4}$$

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$$c_{a,b} \equiv \frac{(x_a - x_b)^2}{(1 - x_a x_b)^2}, \qquad (5.4)$$

Almost ten years later, after many detours, Emmanuel and I were able to prove this formula in the paper Planar maps and continued fractions (2012).

Integrability and distance statistics of random planar maps

Many other instances of this integrability phenomenon have been discovered since:

- in various models of maps and trees studied by Bousquet-Mélou, Kuba, Ambjørn-Budd, Fusy-Guitter...
- other distance-related observables, such as the three-point function, are determined by integrable recurrence equations (B.-Guitter 2008-2010).

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Conclusion: what Philippe really taught me

Thank you!