MATING OF DISCRETE TREES AND WALKS IN THE QUARTER PLANE

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The joy of Integrable Combinatorics

Conference for Philippe Di Francesco 60

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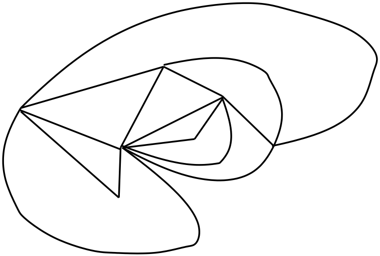
Planar Maps give cellular decompositions of the plane into polygonal regions



A triangulation of the plane

There is a huge literature on maps in relation to other topics like

Matrix integrals and quantum gravity

P. Di Francesco, P. Ginsparg, J. Zinn-Justin, 2D gravity and random matrices, Physics Reports Volume 254 March 1995

Bijections with tree-like structures

J. Bouttier, P. Di Francesco, E. Guitter, Planar Maps as Labeled Mobiles, Electronic Journal of Combinatorics 2004

The purpose of the talk is to give a general construction of

planar triangulations (or more generally planar maps)

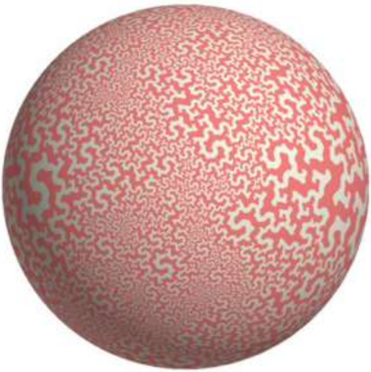
from

lattice paths in the plane

using

mating of trees

Mating of trees originates in complex dynamics (as special case of "mating of polynomials") where one mates Julia sets which can be dendrites i.e. continuous trees



Picture from Arnaud Chéritat's webpage

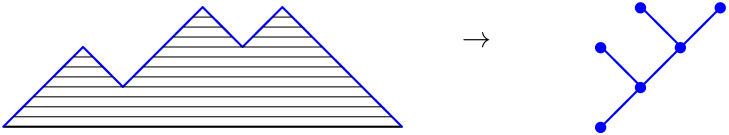
This has been used by Le Gall and Paulin (2009) then by Duplantier, Miller and Sheffield (2014) in the context of random maps.

There is also a discrete model for this construction, coming from the work of Mullin (1967).
See surveys by Gwynne, Holden and Sun (2020) or Sheffield (2022).

I will explain how it works and how to generalize it.

From paths to trees

There is a well known way to associate a rooted planar tree to a Dyck path by matching up and down steps:



Cut out the striped area under the Dyck path and sew the opposite up and down steps.

From paths to trees

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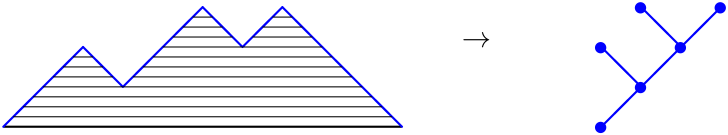
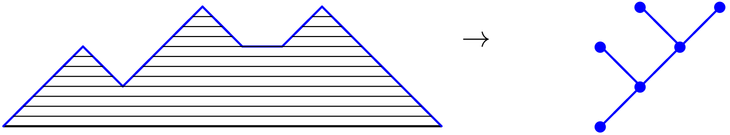


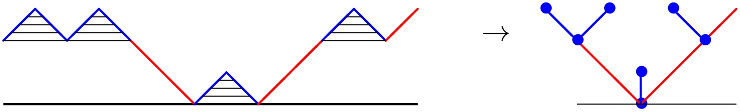
Cut out the striped area under the Dyck path and sew the opposite up and down steps.
The Dyck path is recovered by making a contour exploration of the tree.

Generalize this to Motzkin paths by shrinking each horizontal step to a point.



Some information has been lost, the horizontal steps of the path.

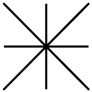
One can also consider paths which do not start or end at 0 and build a forest of trees, rooted on a V-shaped path:

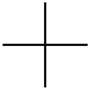


I will consider mainly two dimensional walks with *small steps*

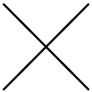
$$(\pm 1, 0) \quad (0, \pm 1) \qquad (\pm 1, \pm 1)$$

in the quarter plane $x, y \geq 0$

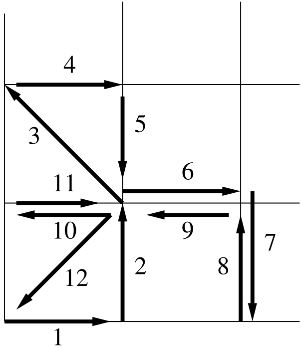




Straight steps



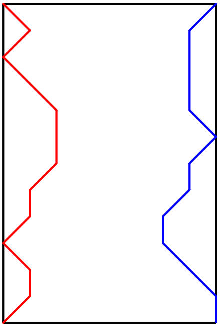
Oblique steps



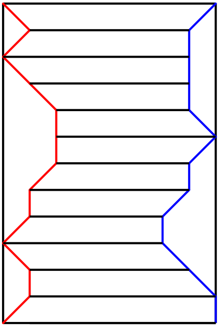
An excursion with small steps in the quarter plane

Constructiong a map from a walk

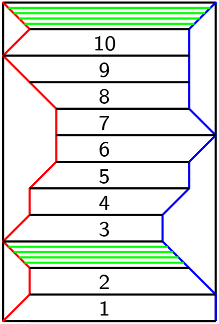
We draw two Motzkin paths vertically facing each other: the horizontal coordinate on the left and the vertical coordinate on the right, the paths running from bottom to top.

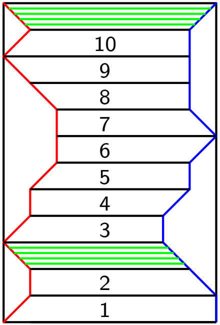


The mating consists in drawing horizontal lines between the vertices of the two paths, as below:

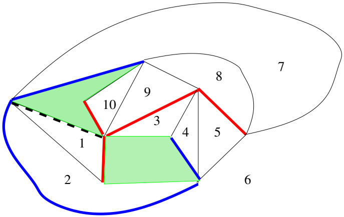


Between the two Motzkin paths we have quadrilaterals corresponding to straight or oblique steps (in green)

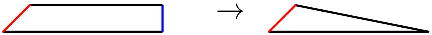




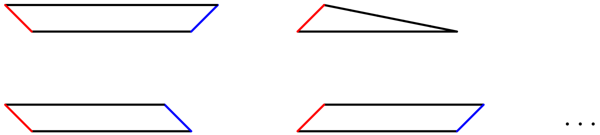
Contract the Motzkin paths into two trees
identify the upper and lower boundaries
to get the planar map below



In this contraction the quadrilaterals corresponding to straight steps become triangles as, for example:



The map consists of triangles and quadrilaterals



The main idea of this talk is to contract the quadrilaterals

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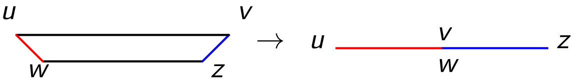
There are two ways to do so, like this



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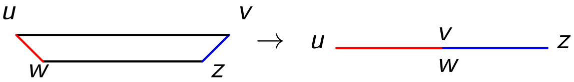
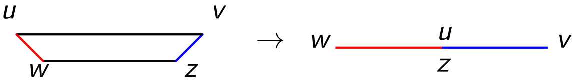
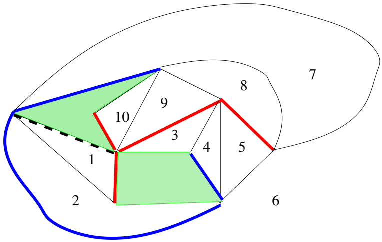


or this

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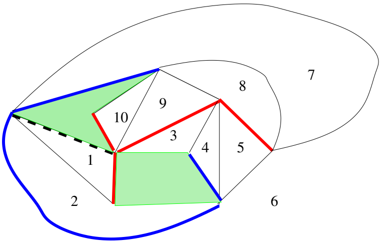
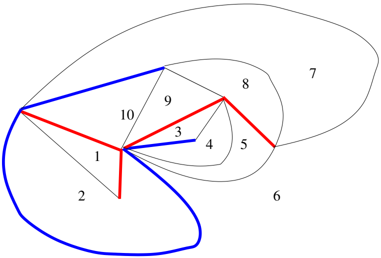


One of the four possible rooted triangulations of the plane:

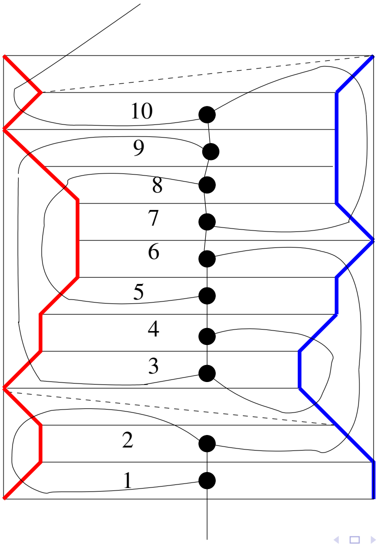


A walk with small steps gives several triangulations of the sphere and each triangulation can be obtained in several ways.

In the sequel I show how to restrict the class of paths and equip the map with some supplementary structure in order to obtain bijections.

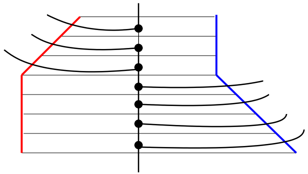
In particular we obtain new bijections and recover several known bijections.

In order to visualize the maps we depict also the dual map putting a dashed line between the opposite vertices which have been identified.

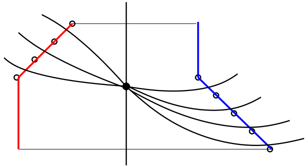


A variant: more general steps

One can also use other types of steps, e.g. with a $(3,4)$ step:



Erasing the internal edges we get a face with nine edges

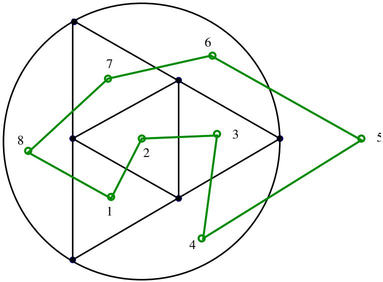


More variants

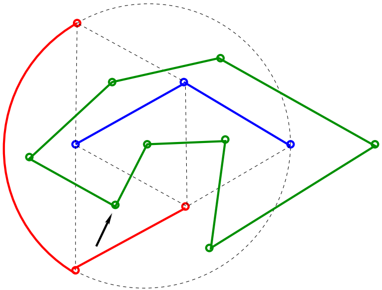
One can use paths not starting and ending at zero, not staying in the quarter plane, one can also identify opposite sides of quadrangles (or larger polygons) and get higher genus maps etc.

Triangulations with a Hamiltonian cycle and Mullin's construction

A triangulation of the plane with a Hamiltonian cycle going through its faces

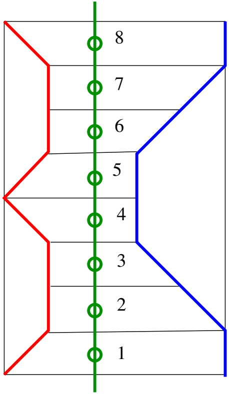


Keep edges not crossed by the cycle:



There remains two trees and the path performs a depth first search of each tree.

We go through the path and record the successive triangles:

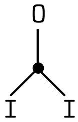


This gives an excursion in the quarter plane with straight steps
This is (a variant of) Mullin's bijection.

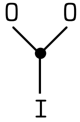
Prographs and Kenyon-Miller-Sheffield-Wilson bijection

Prographs are formed with vertices of two types:

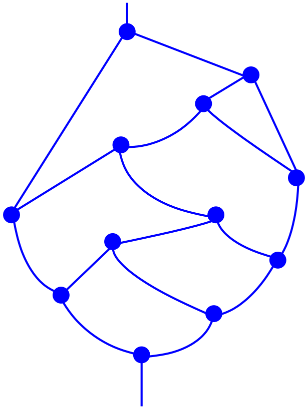
Products with two inputs and one output:



Coproducts with one input and two outputs:

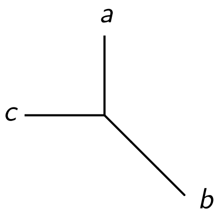


Each output can be used as input in another operator to build a prograph: an oriented (planar) trivalent graph with one input and one output e.g.:

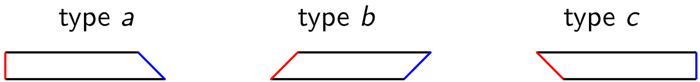


Prographs are counted as standard Young tableaux of rectangular shape $3 \times n$, in bijection with *tandem excursions* with steps

$$a = (0,1) \quad b = (1,-1) \quad c = (-1,0)$$



There are three types of cells:



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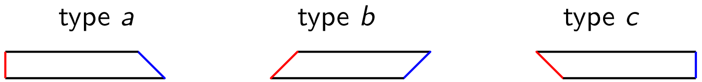
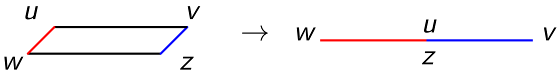
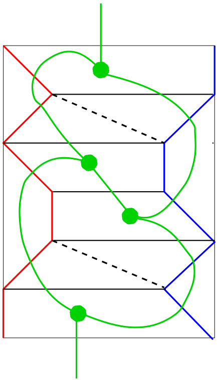


Contraction Rules



From a tandem excursion we get a prograph dual to the triangulation.
Each *a* step corresponds to a coproduct while a *c*-step corresponds to a product.
Here is an example with the word *abacbc*:



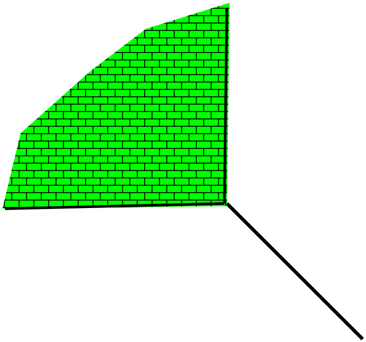
This correspondance between tandem walks and prographs is a bijection.

Maps with a bipolar orientation

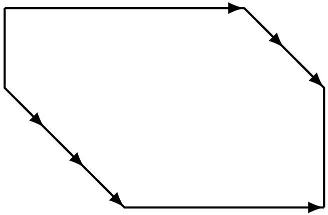
(Kenyon, Miller, Sheffield, Wilson)

The preceding construction can be generalized to walks with steps

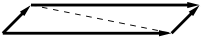
$$(1,-1) \qquad (-i,j) \quad i,j \geq 0, \quad i+j>0$$



Maps with a bipolar orientation:
Orient the sides of a face from west to east, e.g. :



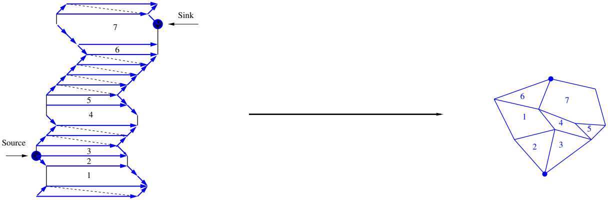
For a quadrilateral corresponding to a step $(1, -1)$ make the usual contraction, consistent with the orientation:



Recover in this way a bijection of Kenyon, Miller, Sheffield and Wilson between paths and maps with a bipolar orientation.

Figure: A walk with steps (1,−1);(0,2);(−1,0);(0,1);(1,−1);(1,−1);(−1,1);(0,1);(1,−1);(1,−1);(1,−1);(1,−1);(−1,0);(−2,1);(1,−1).

The final map

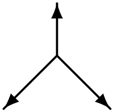


A new example: reversed Y-walks

Walks with steps:

$$(0,1) \qquad (-1,-1) \qquad (1,-1)$$

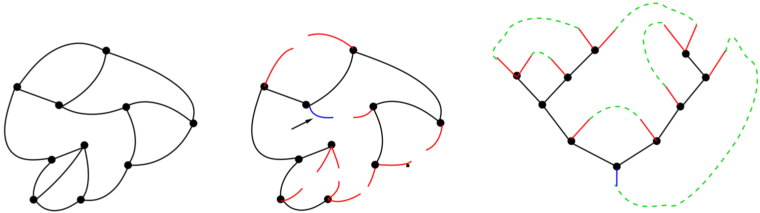
.



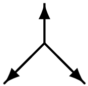
The vertical coordinate performs a Dyck path, the horizontal a Motzkin path with horizontal steps when the vertical coordinate moves up.

Maps with a complete spanning tree

Take a trivalent map (a) and cut edges (b) until you get a *complete rooted spanning tree (c)* (or *blossoming tree* where each leaf has *buds*) and a *root*:

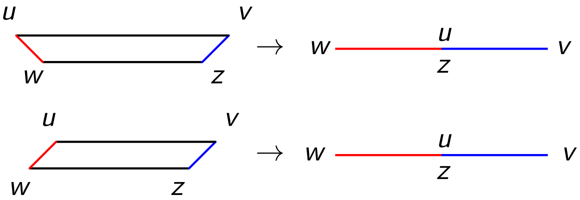


(a) (b) (c)



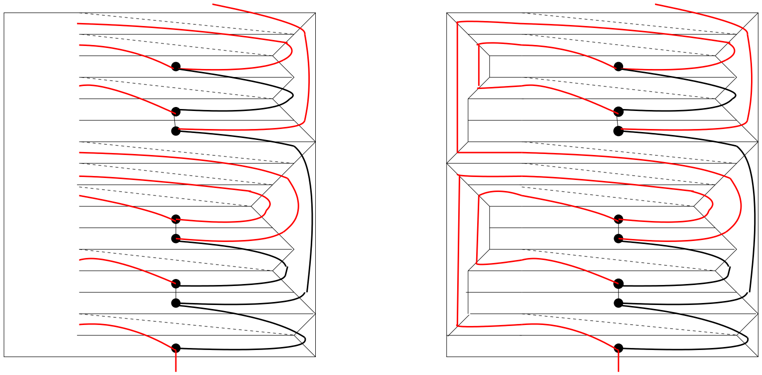
The rY-walks have two types of oblique steps, so we must give the rules for contracting the associated quadrilaterals. These rules are simple and shown below.

Contraction Rules



in both cases identify the NW and SE corners.

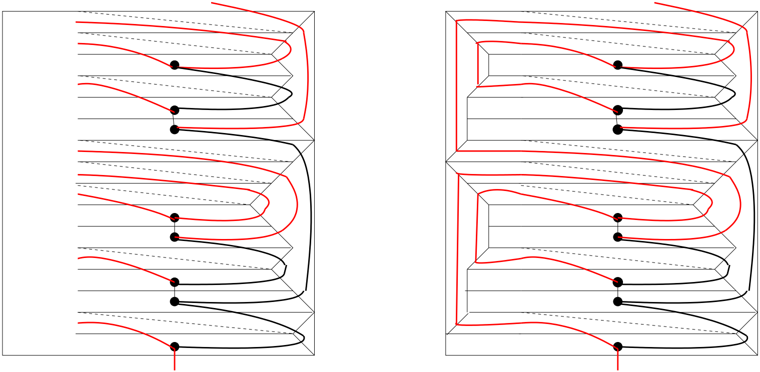
rY-walks and their associated maps.



(a) (b)

(a) Construction of the spanning tree using the vertical coordinates.

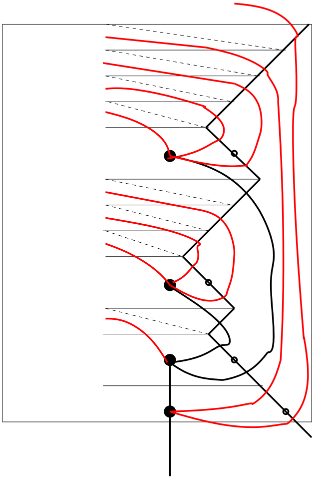
rY-walks and their associated maps.

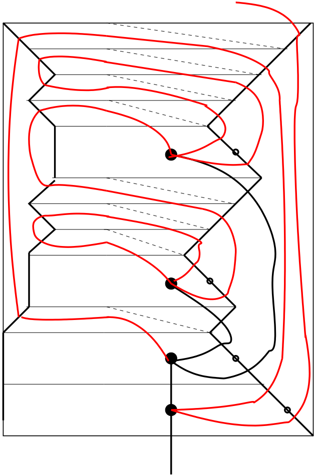


(a) (b)

(a) Construction of the spanning tree using the vertical coordinates.
(b) Matching the leaves using the horizontal coordinates.

Instead of the step $(0, 1)$ we could consider $(0, 2)$ and get a quartic map





One can generalize further to the step set

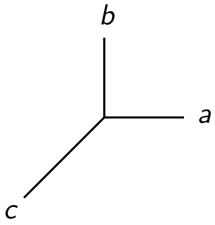
$$(1,-1) \quad (-1,-1) \qquad (0,k),\; k \geq 1$$

and get a bijection between rooted maps with a complete spanning tree and generalized rY walks.

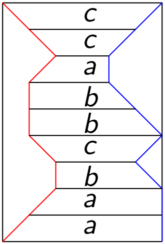
Kreweras walks and Bernardi's bijection (2009)

Kreweras steps:

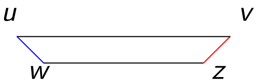
$$a = (1,0) \quad b = (0,1) \quad c = (-1,-1)$$



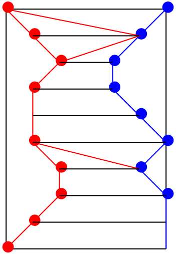
Here is an example *aabcbbacc*:



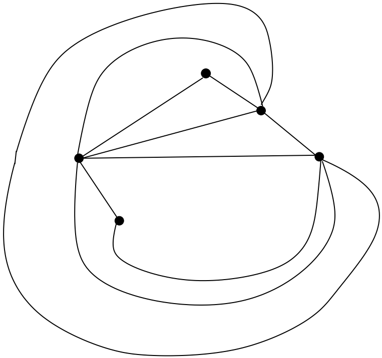
Rules for the contractions:



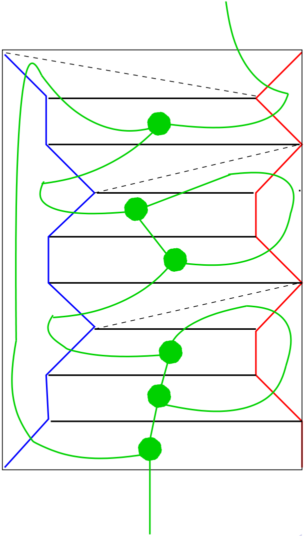
Consider red and blue segments (uw and vz above). They are paired with two segments red and blue whose heights are i and j. If $i < j$ (red is lower) identify u and z, if not identify v and w .



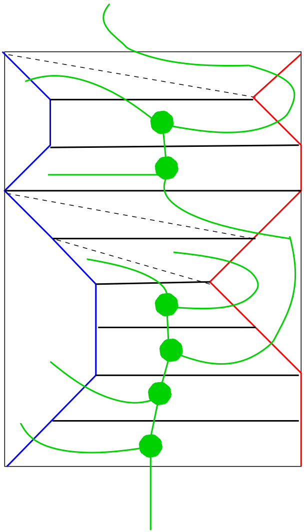
The final triangulation:



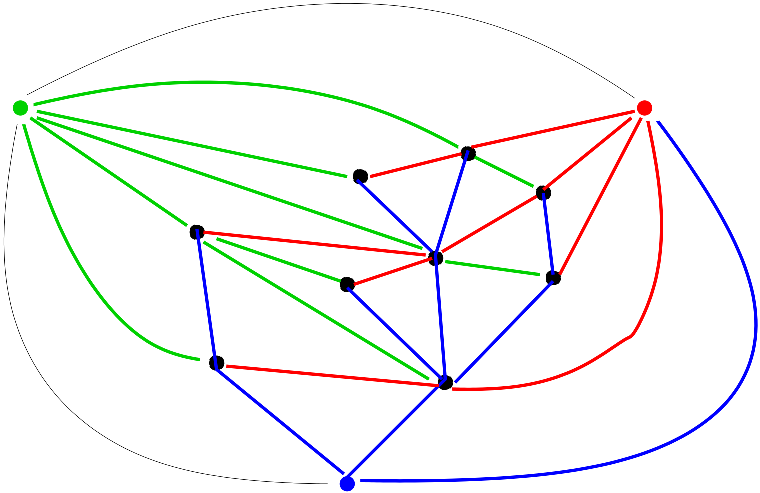
The preceding construction is not a bijection, we need more information to recover the walk.
Consider the dual map with trivalent vertices



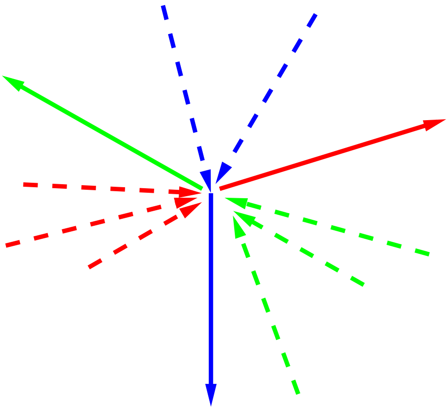
orient the edges from bottom to top an cut edges with the wrong orientation to get a spanning tree. This allows to recover the path from which we started.



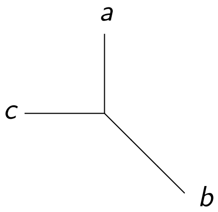
Schnyder woods (Bernardi, Bonichon 2009, Li, Sun, Watson 2017)



A triangulation with three trees rooted on three vertices of the external face.
In each internal vertex we have *Schnyder condition* :



Consider tandem walks with steps



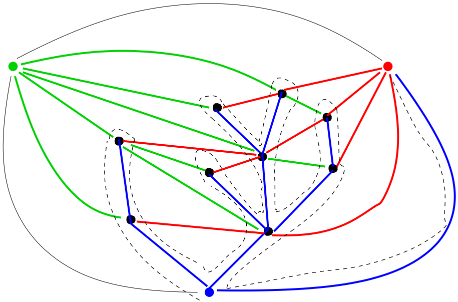
type a

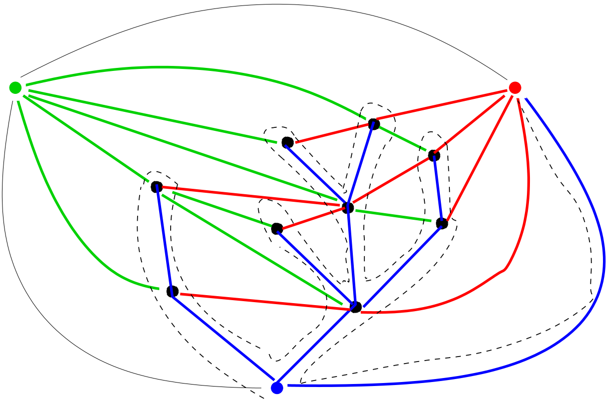
type b

type c

Make a contour walk of the blue tree

1. when you go up a blue edge make an *a* step
2. when you go down a *b* step
3. when you cross a red edge for the second time make a *c* step
4. do not go down the last blue step and do not make the last *b* step.



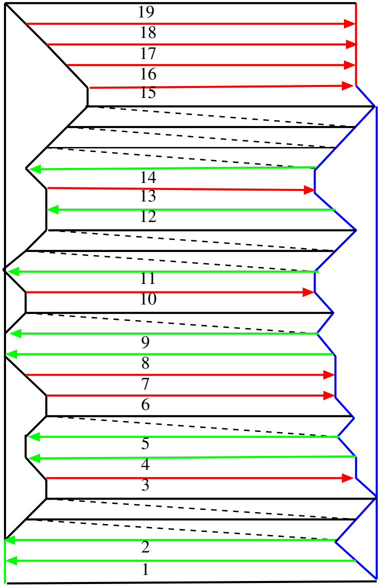


Here is the word you get in the example

$$aabbacabaccabacbbaacbbbacccc(b)$$

There is never a step *c* following immediately a step *b*. The upper and lower edges of the rectangle, together with the *a* step not paired with the last *b* step, form the external triangle.

This construction recovers the Li Sun Watson bijection between such walks and Schnyder woods.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CONCLUSION

I have shown how to produce triangulations, or more genral maps, from walks in the plane.

The construction has a very simple principle and recovers several known bijections.

Many variants are possible and could hopefully be used to produce more bijections.

HAPPY BIRTHDAY PHILIPPE!