Integrable dynamics on polygons and the dimer integrable system

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Joint works with:

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Philippe60, the joy of integrable combinatorics
IPhT, June 24 2024

- Several integrable dynamical systems on spaces of polygons have been studied in the last decades.

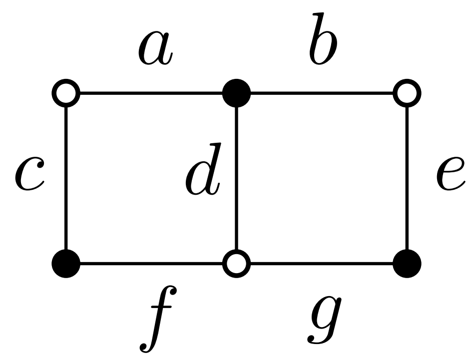
- Another class of integrable systems associated with bipartite dimer models on the torus was introduced by Goncharov and Kenyon in 2013.

- The setting of triple crossing diagram maps provides a common framework for both the geometric integrable systems and the dimer integrable system.

1 Integrable systems from bipartite dimer models on the torus

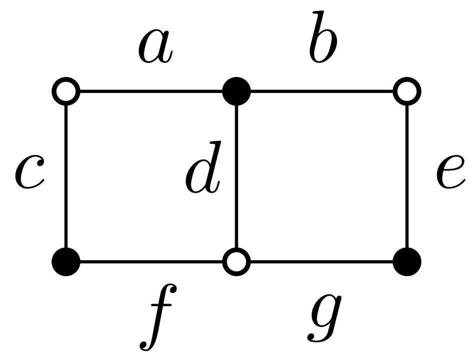
A model from statistical mechanics

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *edge weights*.



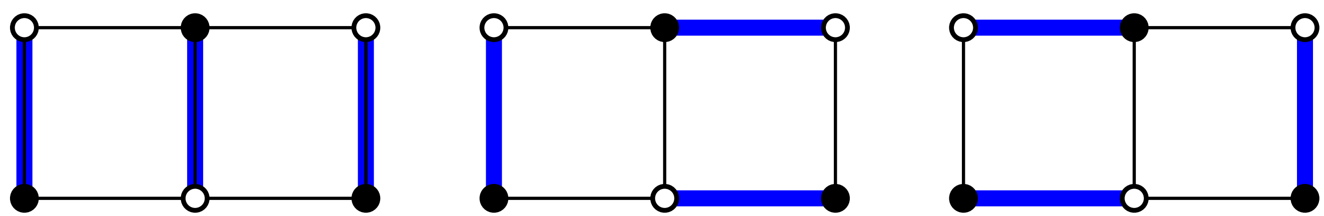
- In probability, edge weights are positive real numbers.

- For integrable systems and geometry purposes (this talk), edge weights are complex numbers.



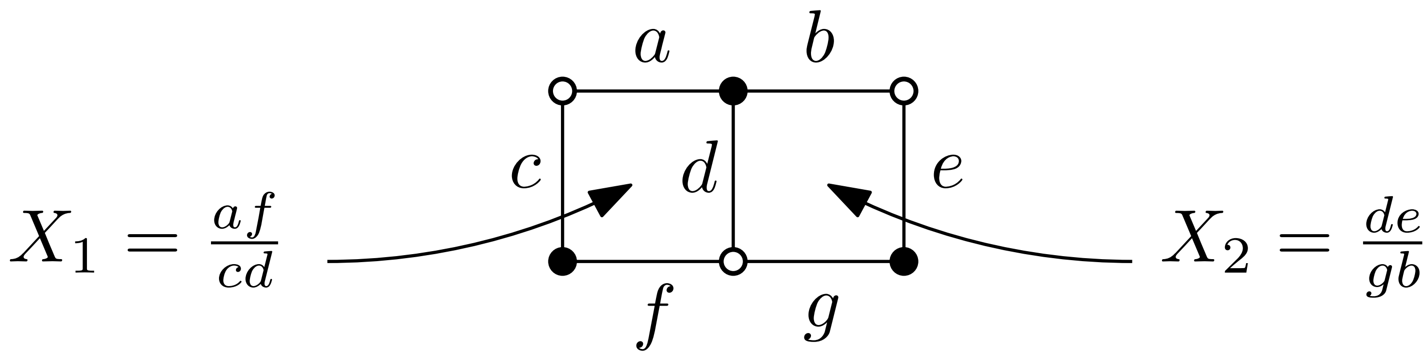
- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.

Dimer coverings:



Weights: cde bcg aef

- *Boltzmann probability measure*: draw a dimer covering at random with probability proportional to its weight.

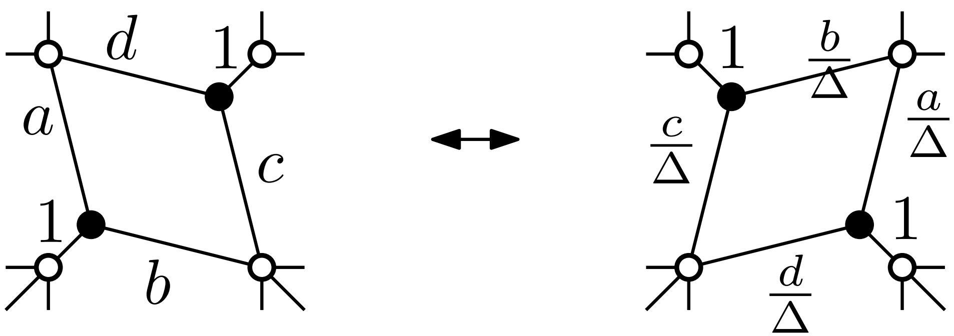


- Define *face weights* as alternating products of edge weights around faces.

- Two collections of edge weights induce the same Boltzmann probability measure if and only if they have the same face weights.

Two moves preserving correlations

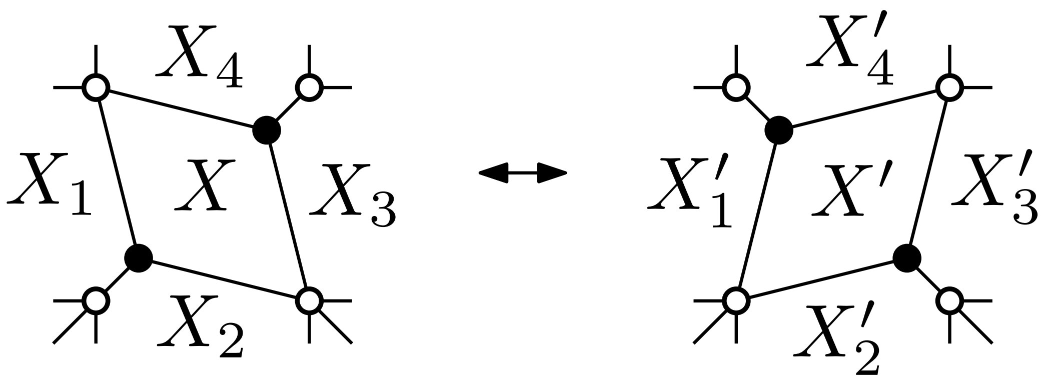
1. Spider move:



$$\Delta := ac + bd$$

Two moves preserving correlations

1. Spider move:

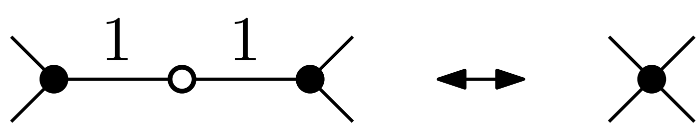


$$X' = X^{-1}$$
$$X'_1 = X_1(1+X)$$
$$X'_2 = \frac{X_2}{1+X^{-1}}$$
$$X'_3 = X_3(1+X)$$
$$X'_4 = \frac{X_4}{1+X^{-1}}$$

- The change in the face weights is a special instance of mutation of coefficient variables in *cluster algebras*.

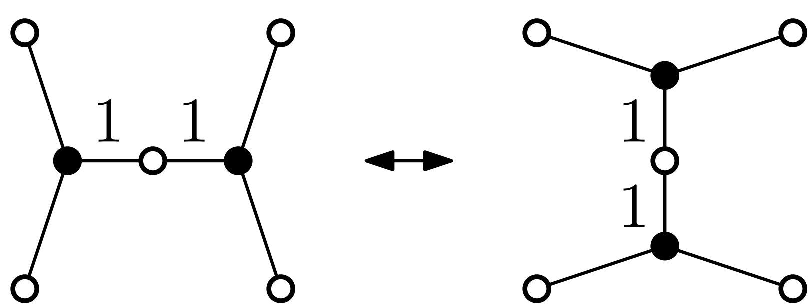
Two moves preserving correlations

2. <u>Contraction/expansion of degree-two vertex:</u>



- Face weights don't change.

- May be recombined into a *resplit* move:

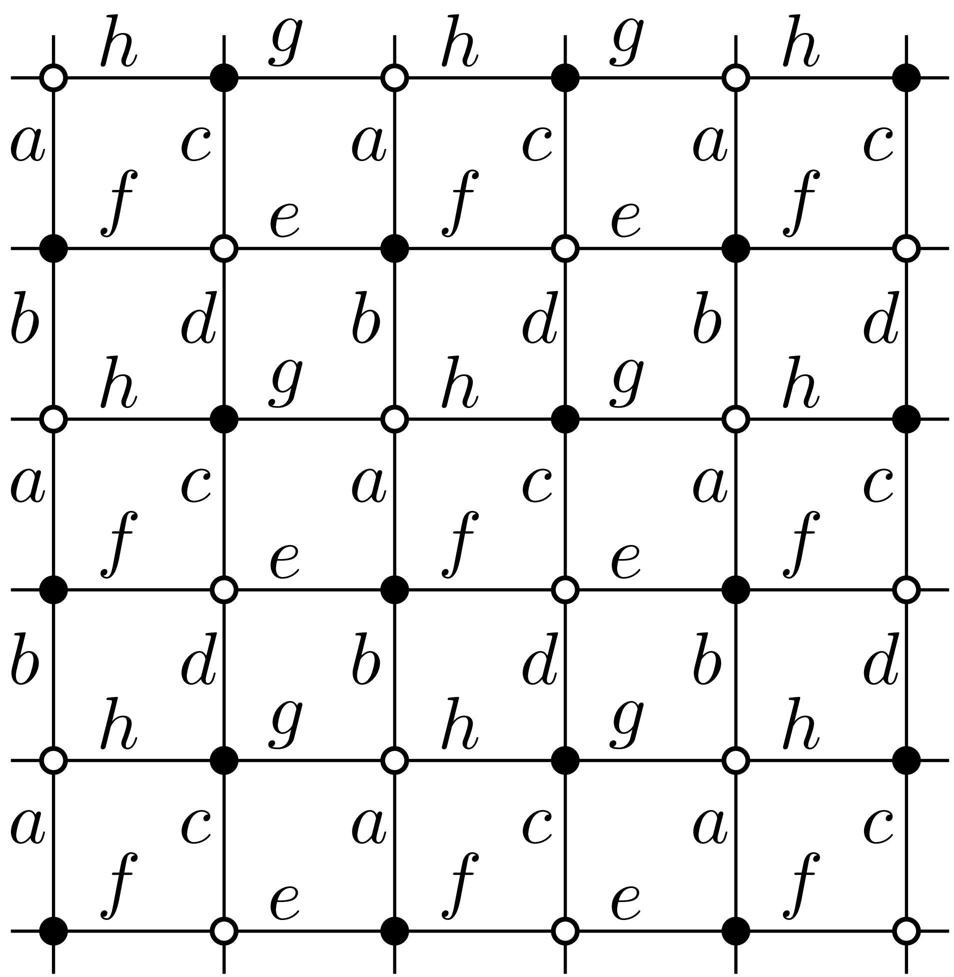


Discrete-time integrable dynamics

- We use these moves to define discrete-time dynamics on face-weighted bipartite graphs on the torus.

- One step of the dynamics will bring us back to the same combinatorial graph, but the face weights will potentially have changed.

- An equivalent way of working with torus graphs is to consider infinite planar graphs that are periodic in two directions, with weights also periodic in two directions.



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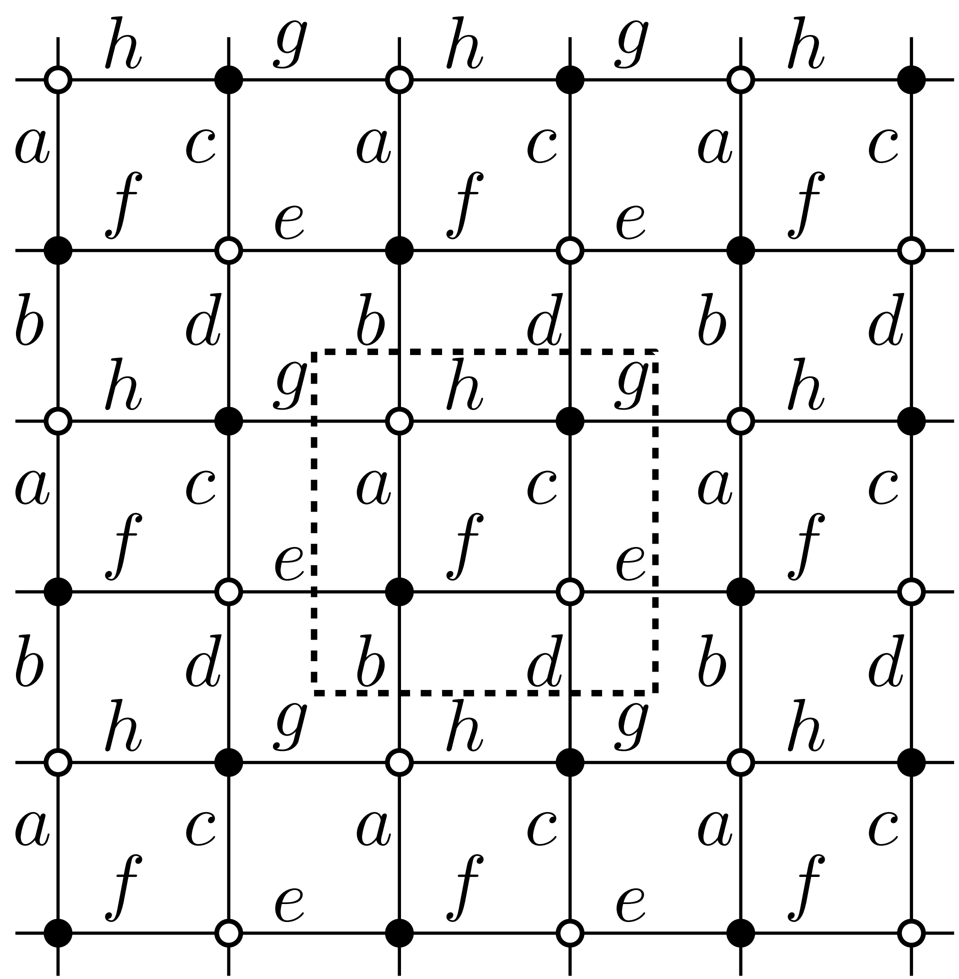
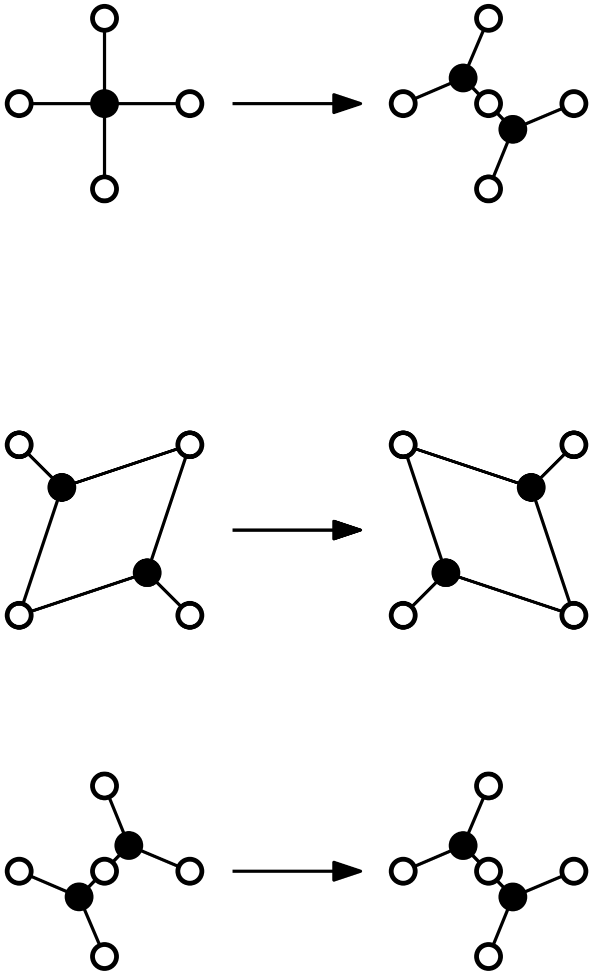
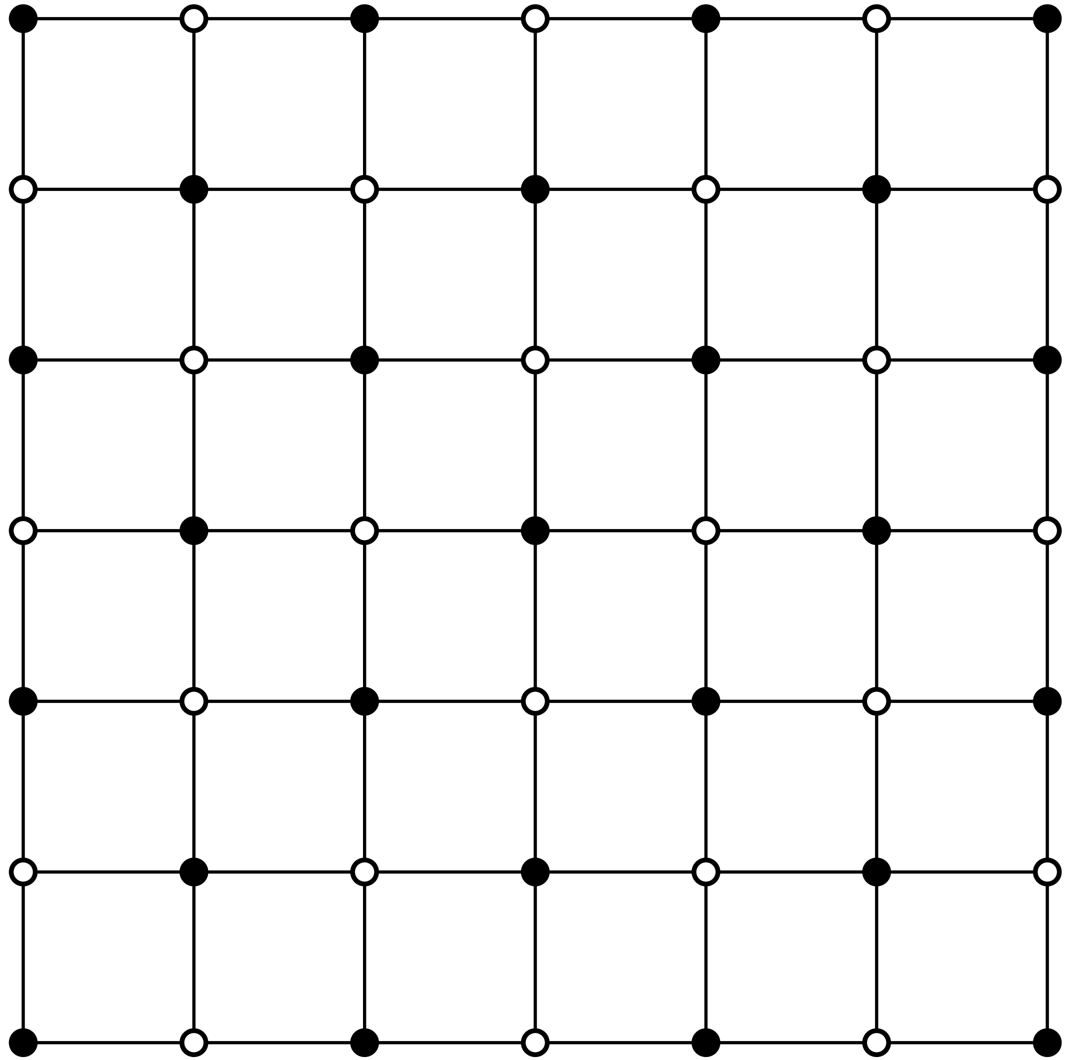
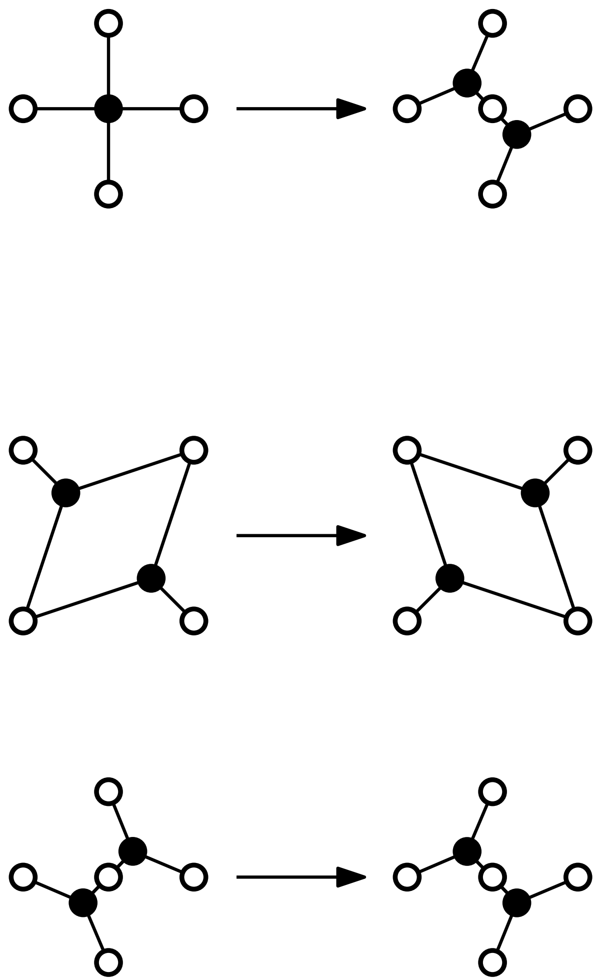


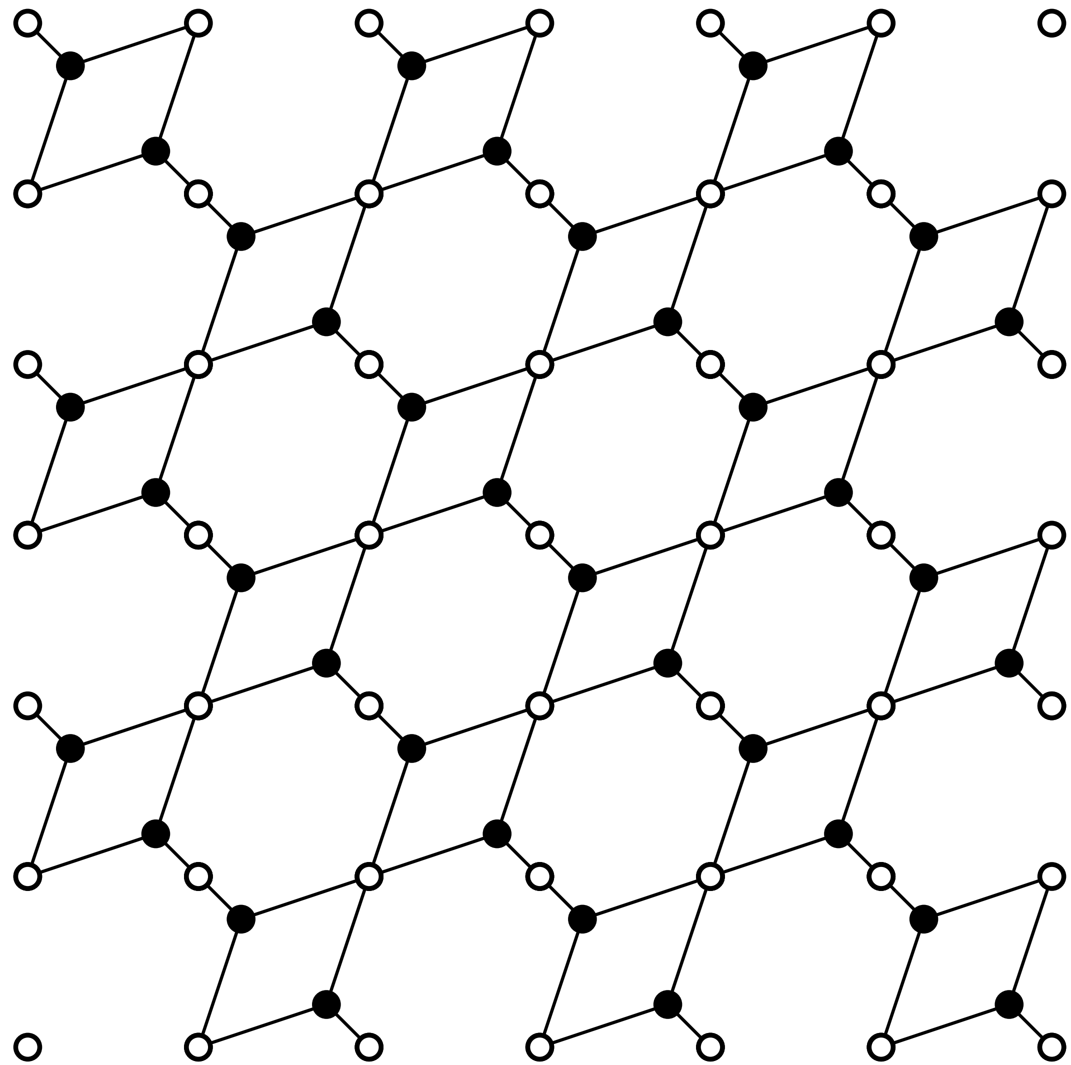
Square grid example



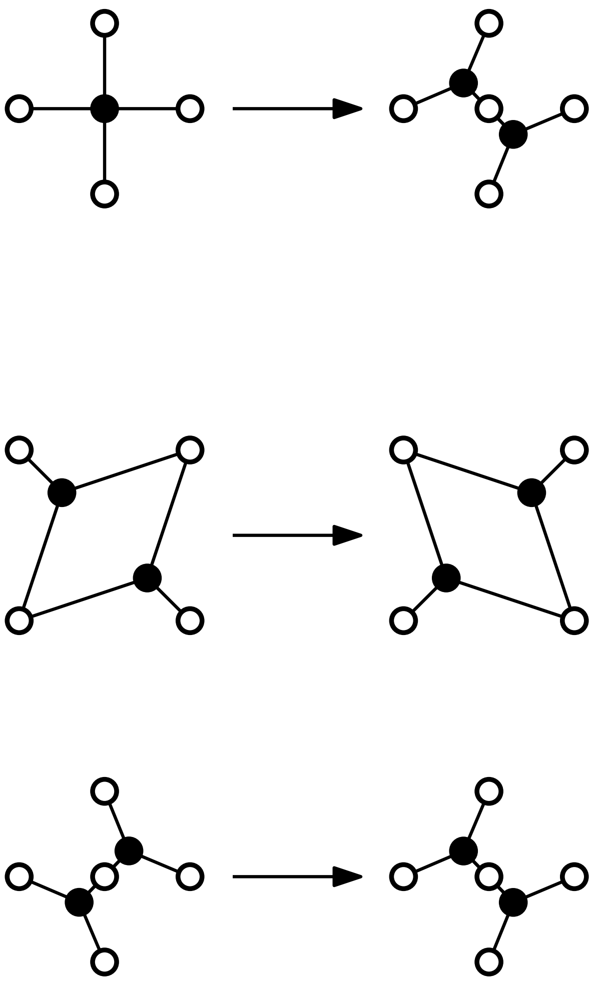


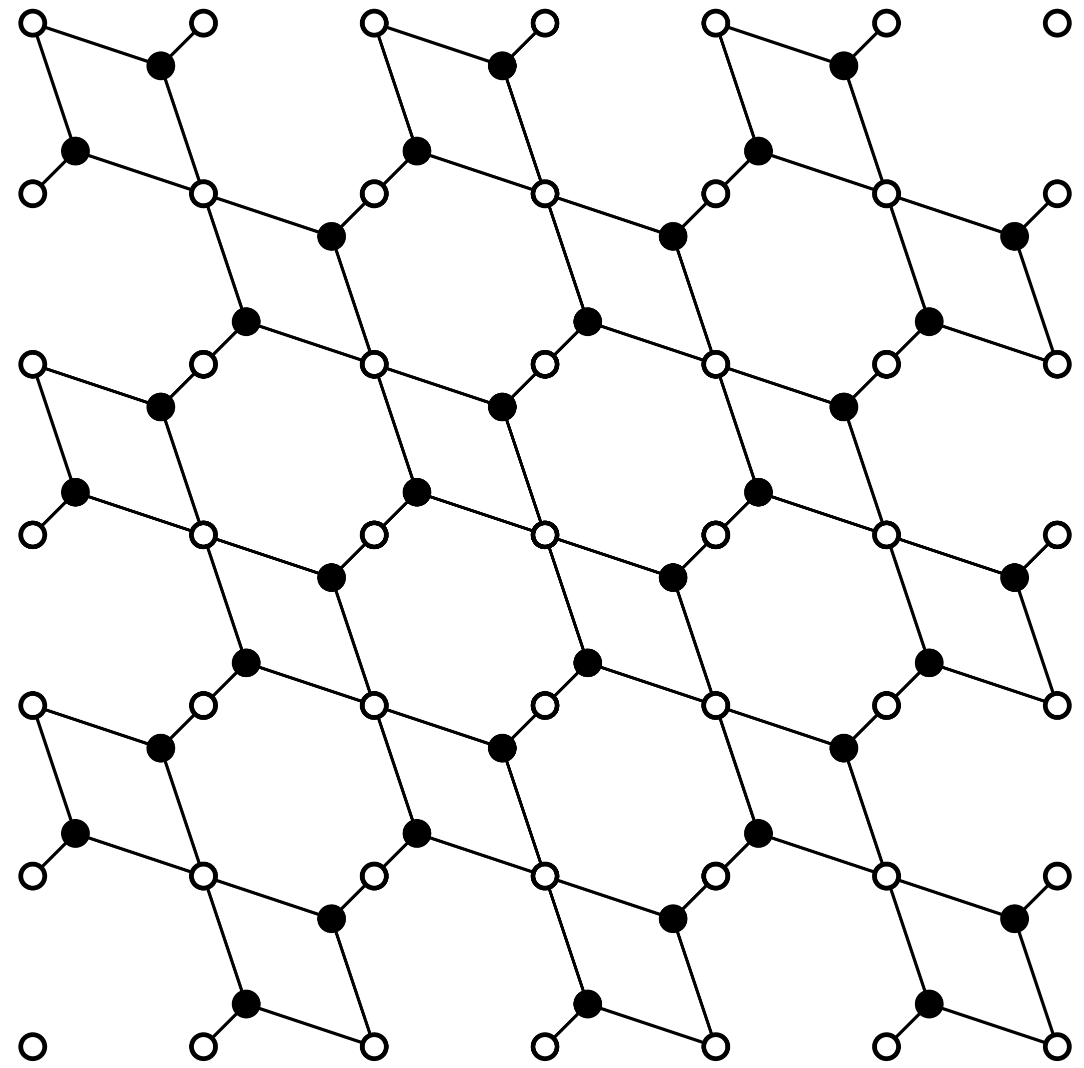
Square grid example





Square grid example





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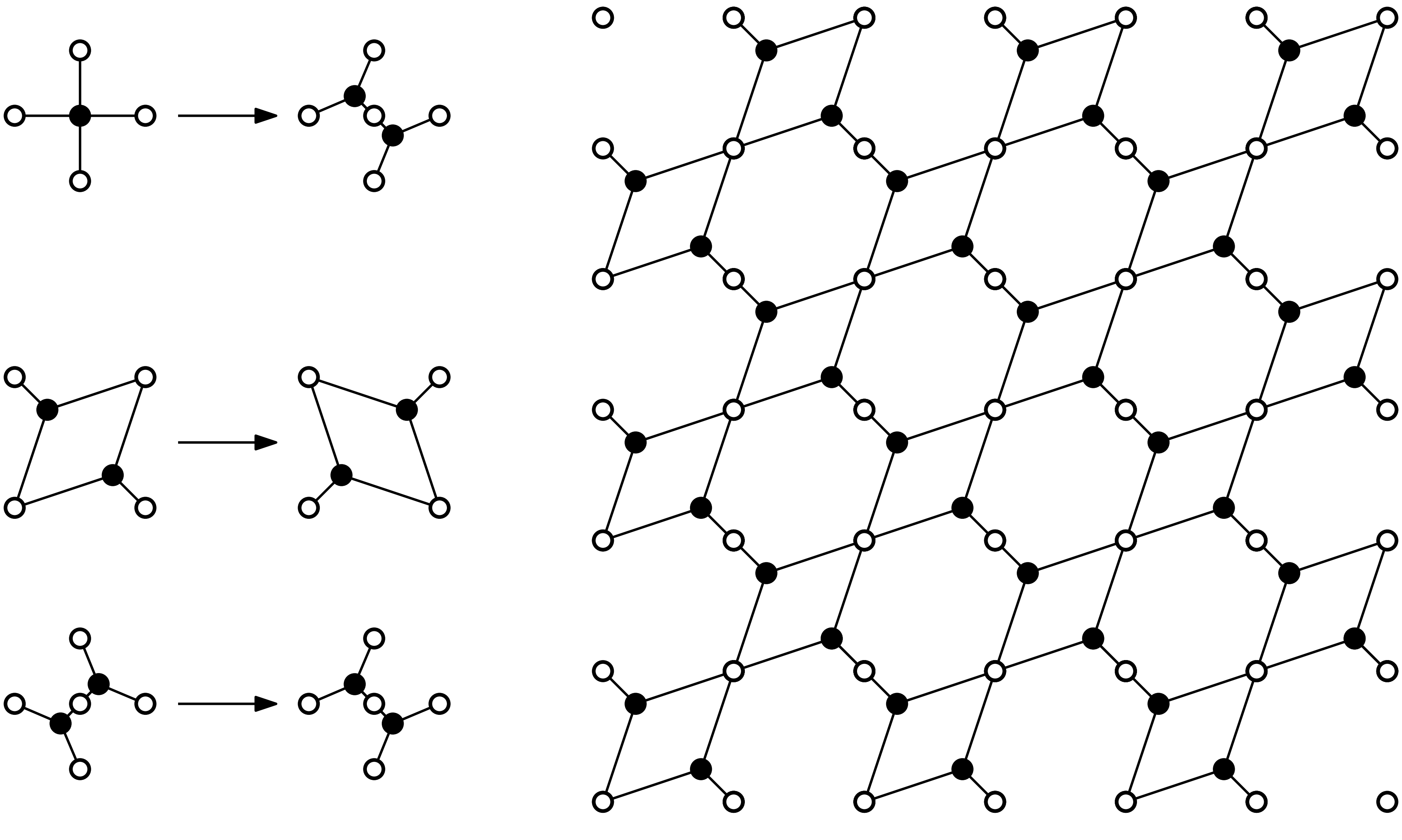

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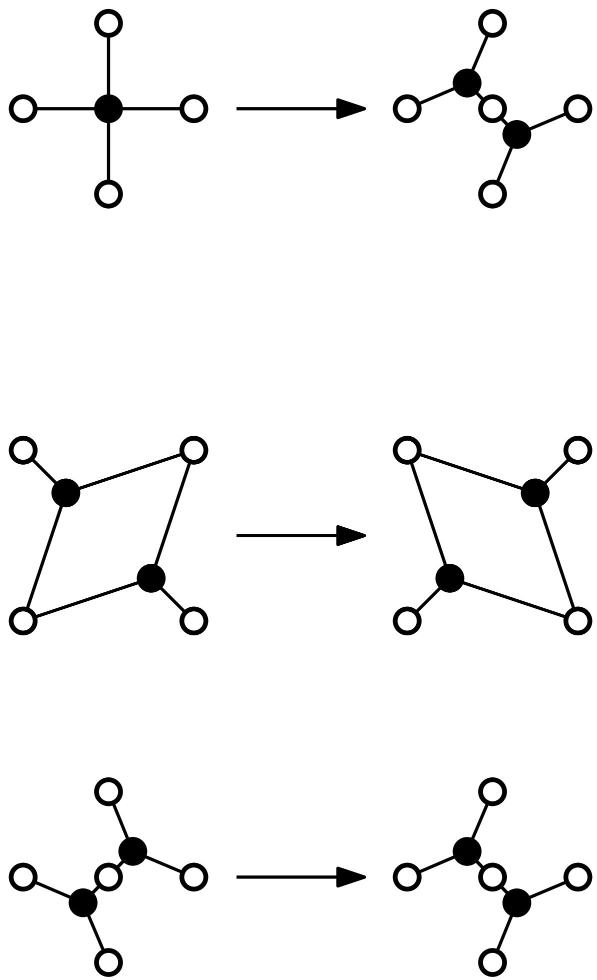

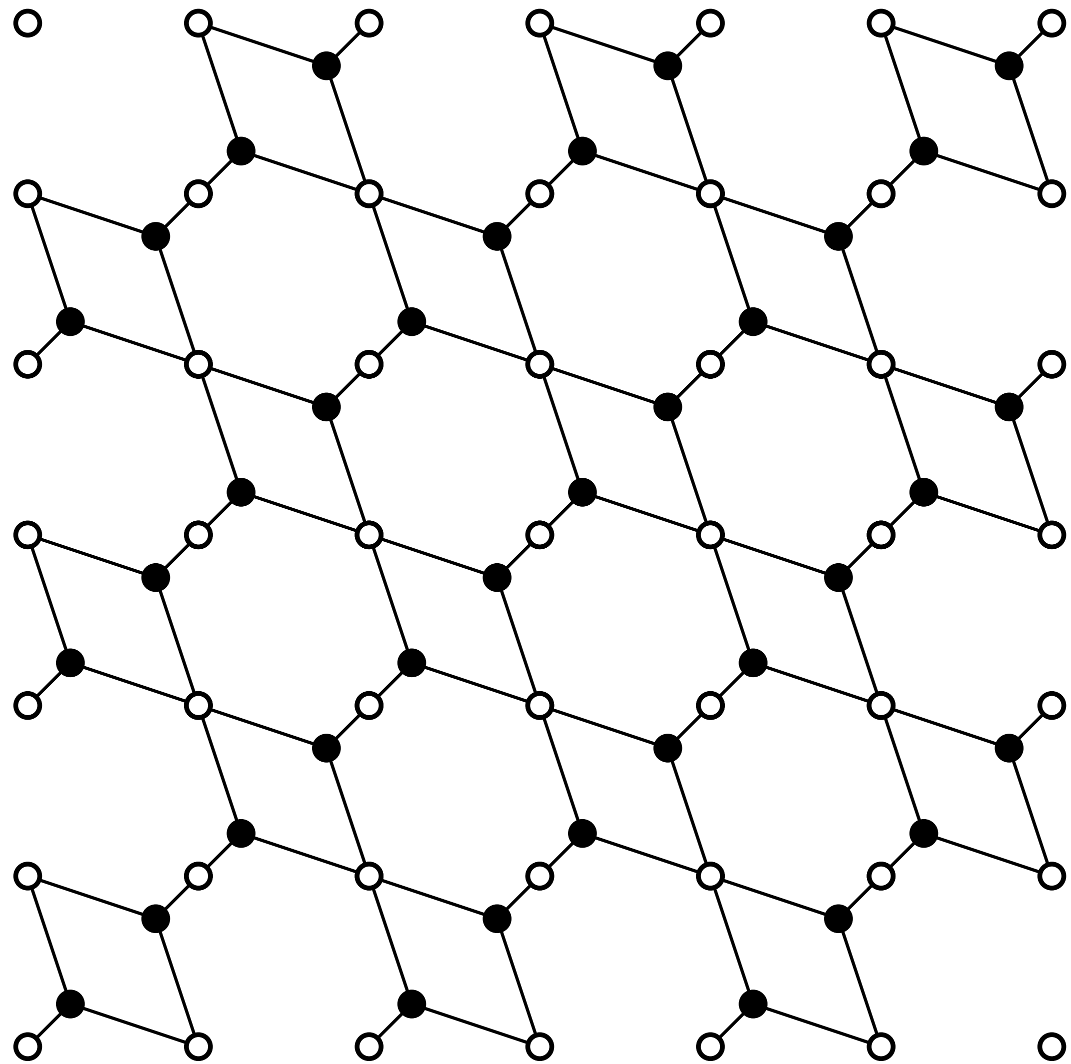

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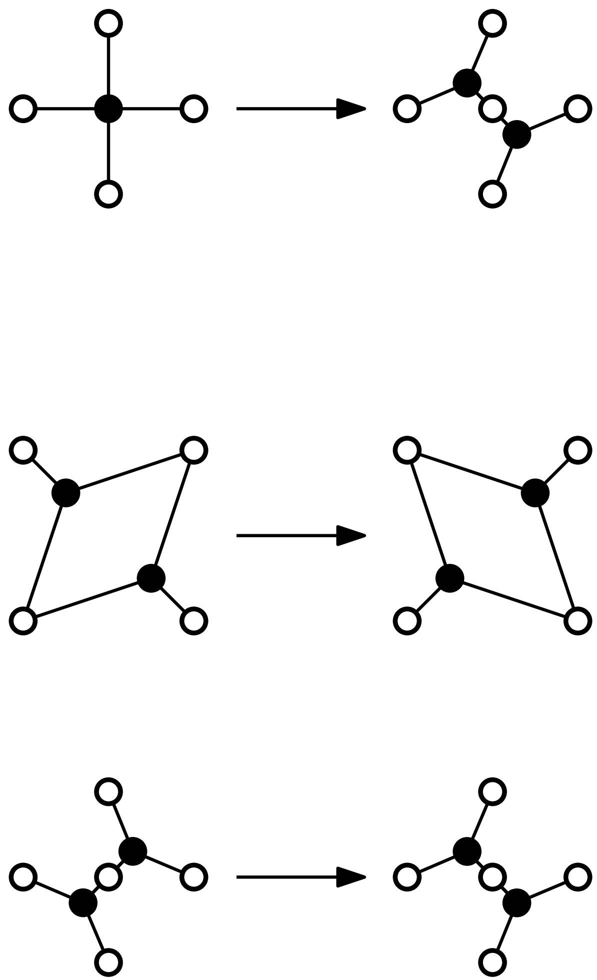
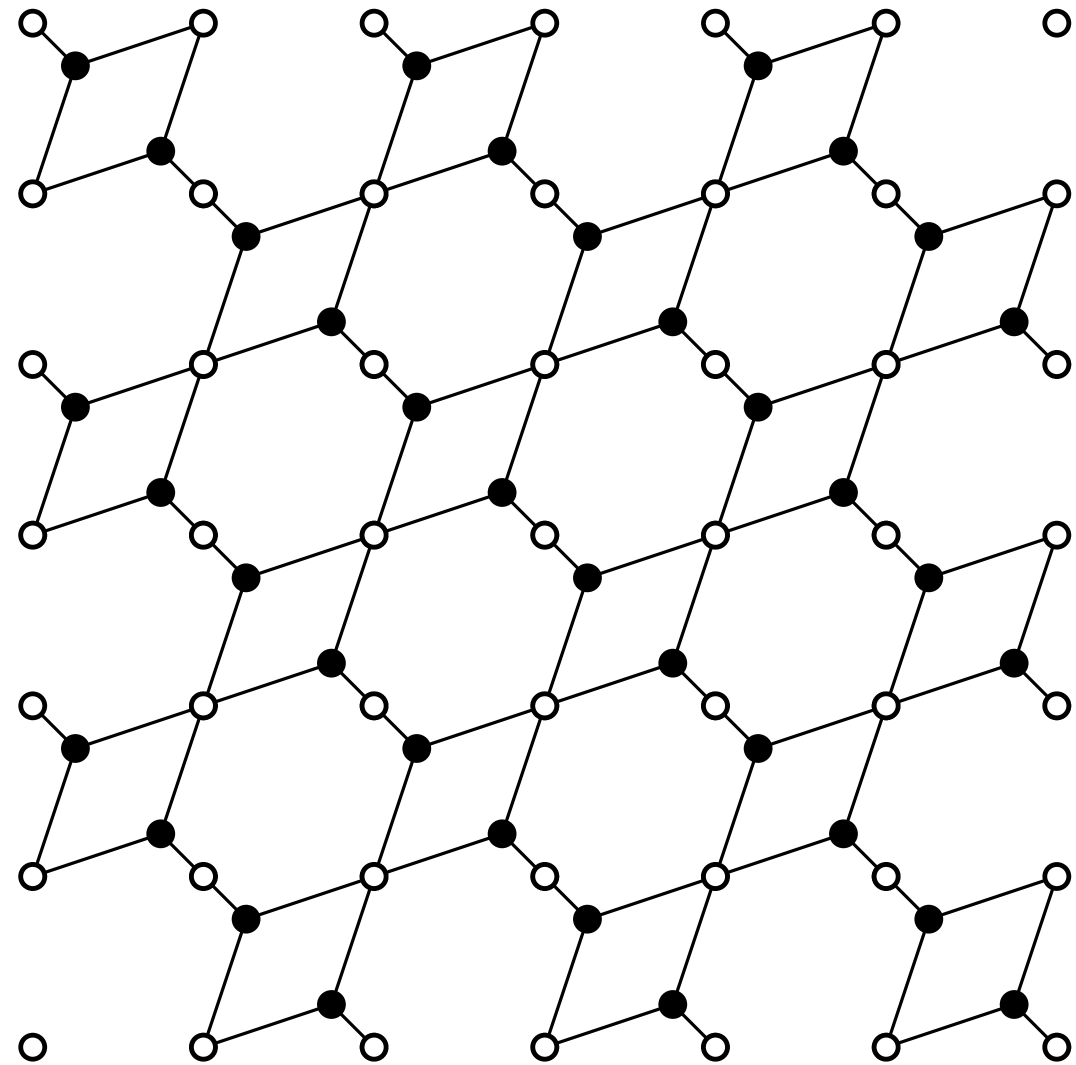




- All these discrete-time dynamics are integrable, in the sense that they have “enough” conserved quantities.

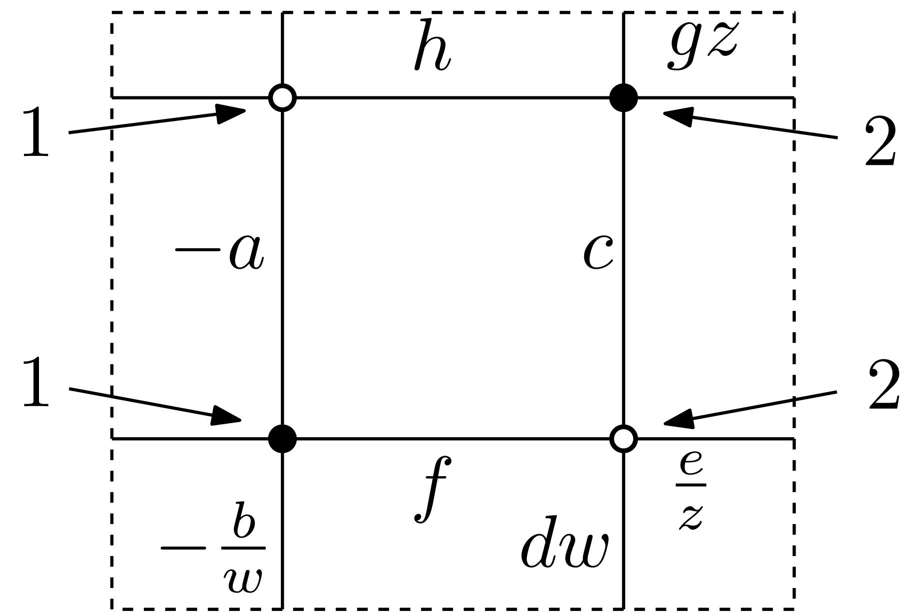
- These conserved quantities have a particular structure with respect to a Poisson bracket. In the right coordinates, the motion is translation on some high-dimensional torus (Goncharov-Kenyon, Fock-Marshakov, Vichitkunakorn, George-Inchiostro).

- The conserved quantities are partition functions for dimer covers on the torus with prescribed homology.

Kasteleyn matrix determinant

- The Kasteleyn matrix $K(z,w)$ is the signed twisted adjacency matrix of a bipartite graph on the torus.

$$\begin{array}{c|cc} & \bullet_1 & \bullet_2 \\ \hline \circ_1 & -a - \frac{b}{w} & h + gz \\ \circ_2 & f + \frac{e}{z} & c + dw \end{array}$$



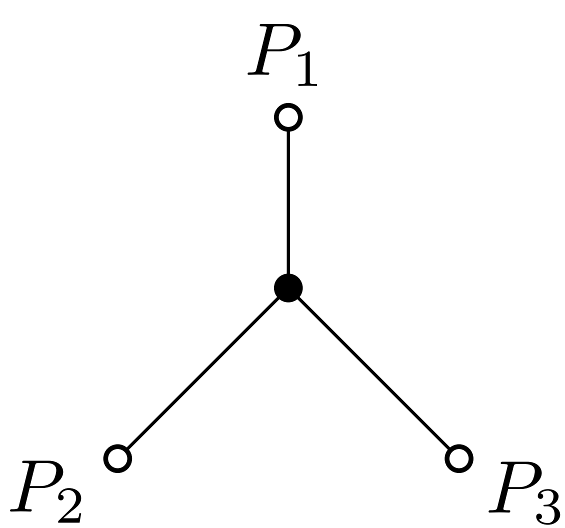
- The Laurent polynomial $\det K(z,w)$ is the generating function of the conserved quantities.

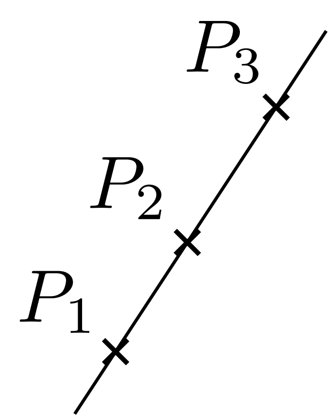
2 Triple crossing diagram maps

(following Affolter-Glick-Pylyavskyy-R. '24 and Affolter-George-R. '21)

- A *triple crossing diagram* (TCD) is a bipartite graph such that all the black vertices have degree 3.

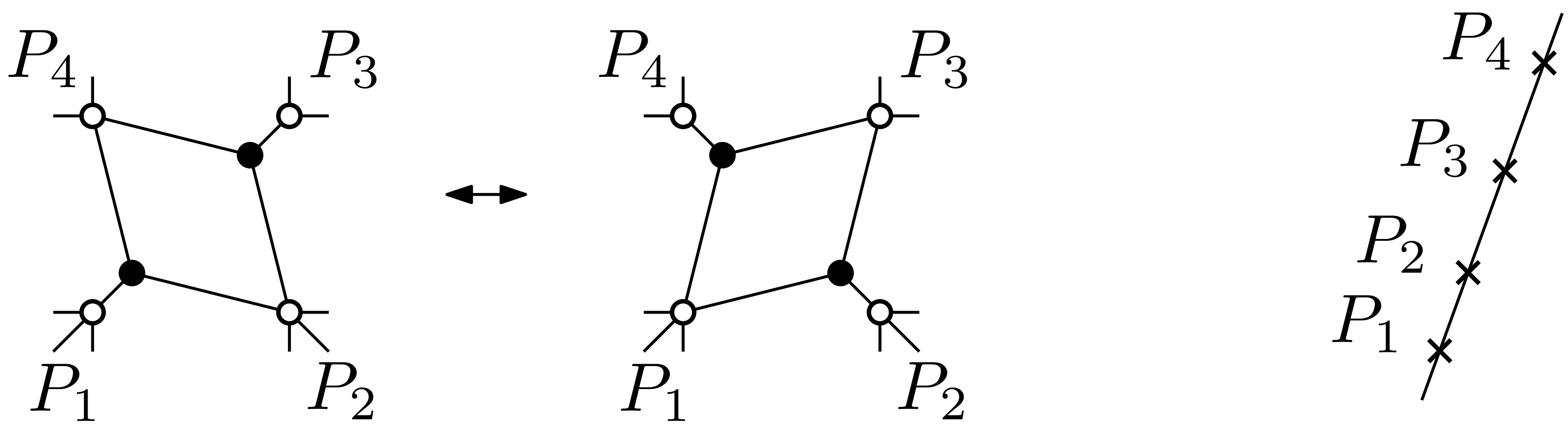
- Let $m \geq 1$ and Γ be a TCD with white vertex set W and black vertex set B. A *TCD map* is a map from W to $\mathbb{C}P^m$ such that for any $b \in B$, the three vertices around b are mapped to three collinear points.





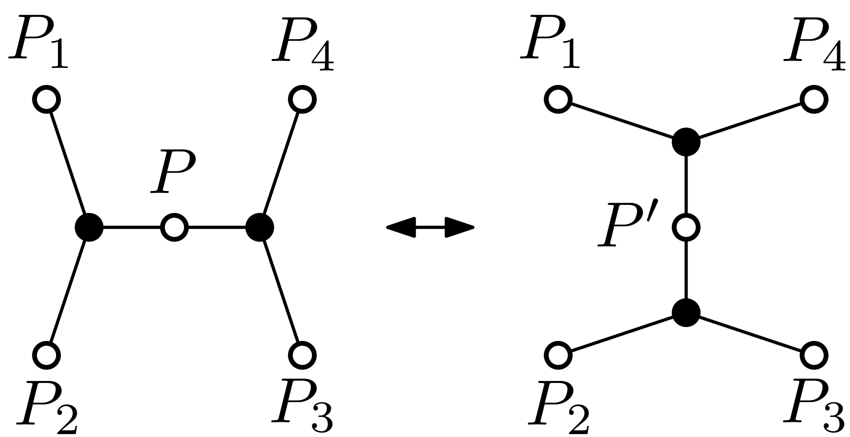
$P_1, P_2, P_3 \in \mathbb{C}P^m$

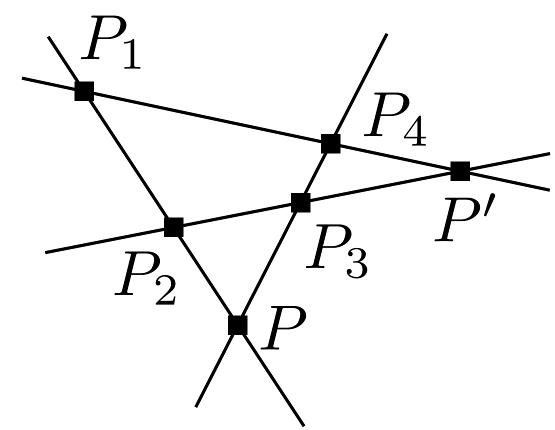
Spider move for TCD maps



- The points of the TCD maps do not change, this is a reparametrization.

Resplit for TCD maps





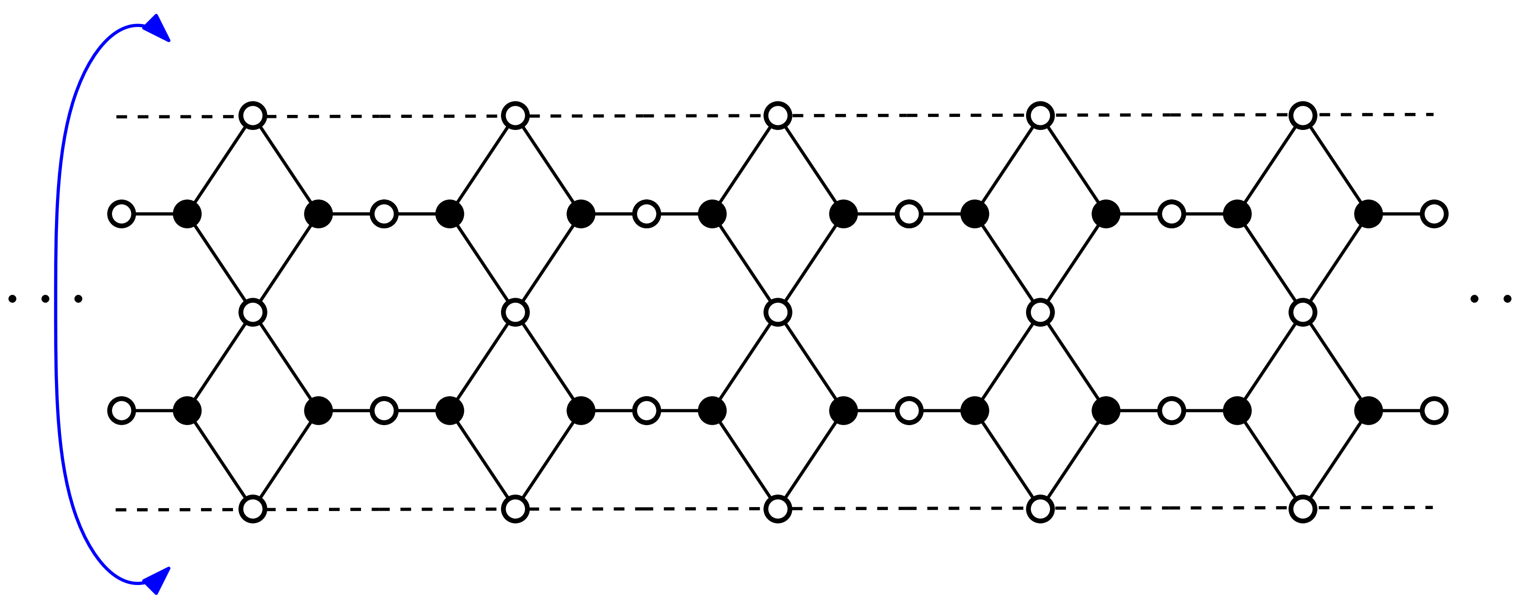
- If the points are in $\mathbb{C}P^m$ with $m \geq 2$, the new point P' is determined by Menelaus theorem for complete quadrilaterals:

$$\mathrm{mr}(P_1, P, P_2, P_3, P', P_4) = -1.$$

- In $\mathbb{C}P^1$, we use the multi-ratio formula to define P'.

Twisted TCD maps

- It is a TCD map on an infinite cylinder such that the graph is periodic and the points are quasi-periodic, with some monodromy $M \in PGL_{m+1}(\mathbb{C})$.



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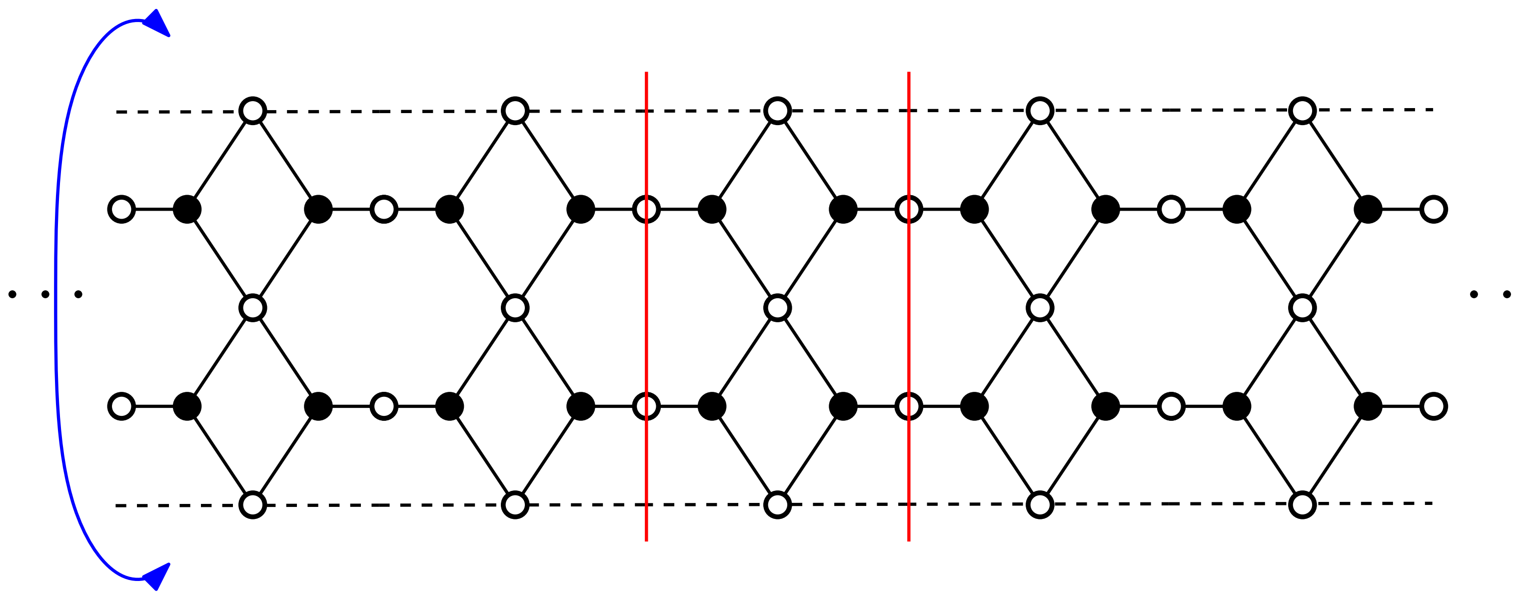

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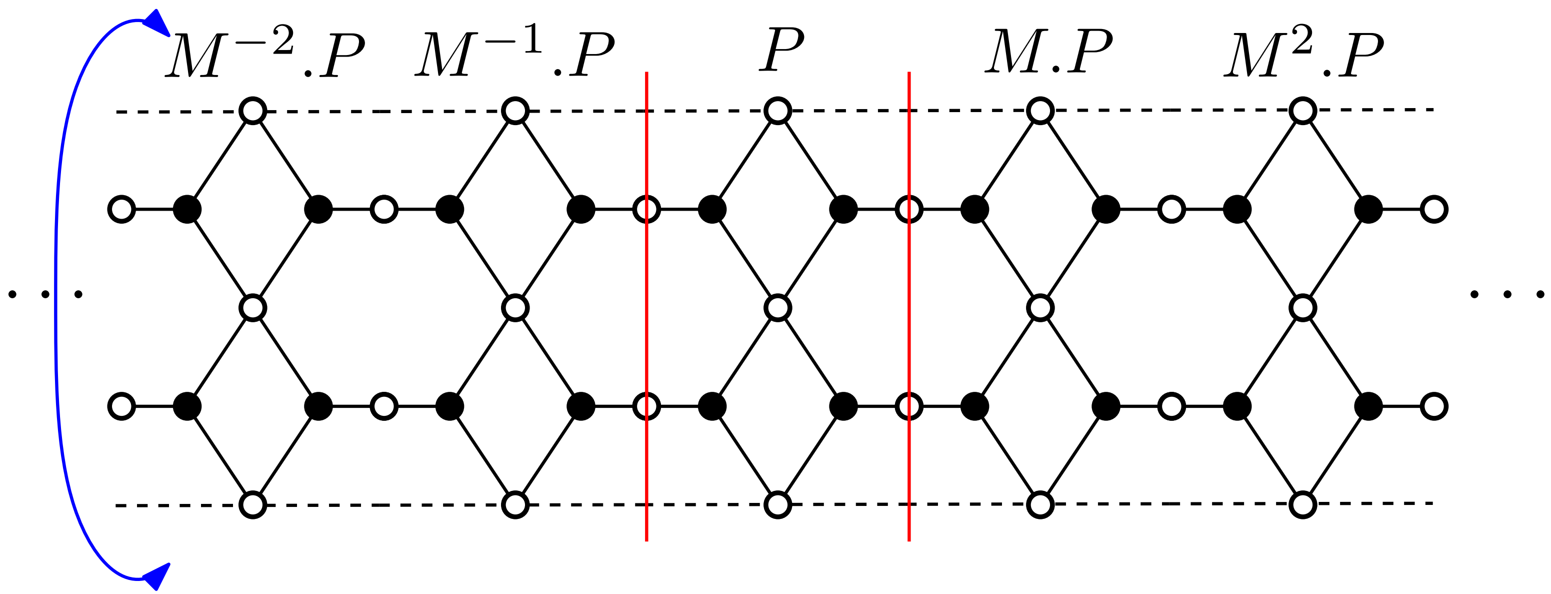
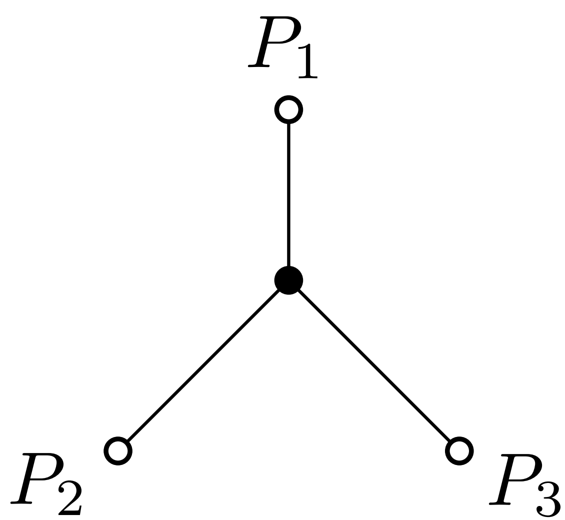


From TCD maps to dimers

- One can associate edge variables to a TCD map by lifting the points $P_i \in \mathbb{C}P^m$ to vectors $v_i \in \mathbb{C}^{m+1}$.



$P_1, P_2, P_3 \in \mathbb{C}P^m$ collinear

↓

$v_1, v_2, v_3 \in \mathbb{C}^{m+1}$ coplanar

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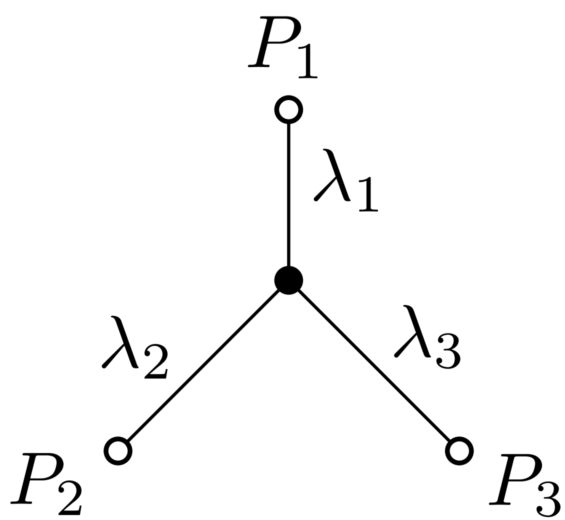


$P_1, P_2, P_3 \in \mathbb{C}P^m$ collinear

$\downarrow$

$v_1, v_2, v_3 \in \mathbb{C}^{m+1}$ coplanar

- There exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0.$$

- There is some ambiguity (gauge freedom) for the choice of edge variables.

- One defines face weights on the bipartite graph by taking the alternating products of such edge variables (up to some technical sign).

- Each face weight can be expressed as a multi-ratio of points attached to white vertices.

- The evolution of these face weights under the two moves defined for TCD maps is the same as the evolution of the face weights under the counterpart moves for the dimer model.

- Part 1: bipartite graphs with face weights, evolving according to some moves.

- Part 2: bipartite graphs with a point in $\mathbb{C}P^m$ attached to each white vertex. These decorated graphs evolve according to some moves.

- From the points in $\mathbb{C}P^m$, one can compute face weights, which evolve like in part 1.

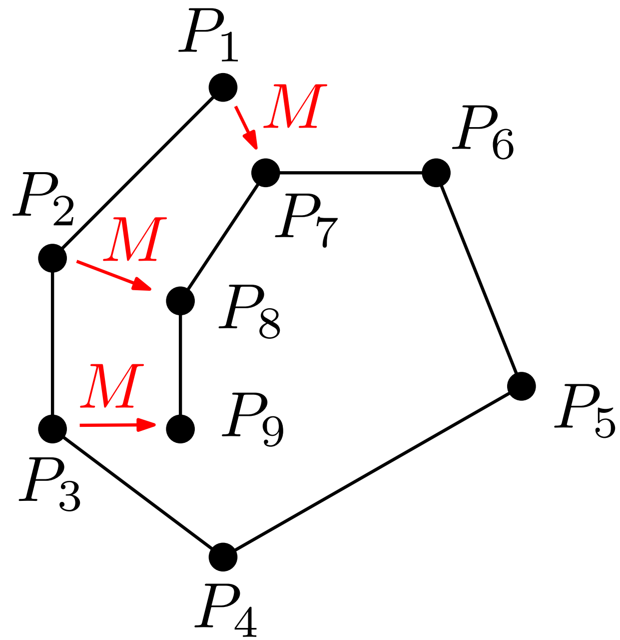
- Dimer integrable systems (part 1): bipartite graph on the torus.

- Geometric integrable systems (part 2): bipartite graph on an infinite cylinder, the graph is periodic, the points are quasi-periodic.

3 Application: dynamics on polygons

Twisted polygons

- If $M \in PGL_{m+1}(\mathbb{C})$, a *twisted n-gon with monodromy* M is defined as a bi-infinite sequence of points $(\ldots, P_{-1}, P_0, P_1, P_2, \ldots)$ in $\mathbb{C}P^m$ such that for every $i \in \mathbb{Z}$, $P_{i+n} = M.P_i$.



Twisted polygons

- Many integrable dynamics on polygons are defined on spaces of twisted polygons.

- These twisted polygons can be realized as twisted TCD maps.

- A generating function for the conserved quantities of these integrable systems on twisted polygons is usually given as a simple function of the monodromy M.

Results of Affolter-George-R. '22

- For twisted TCD maps, we show that the monodromy matrix M can be obtained from the Kasteleyn matrix of a fundamental block of the infinite cylinder.

- This relates the generating function of conserved quantities for dimer integrable systems to the generating function of conserved quantities for dynamics on twisted polygons.

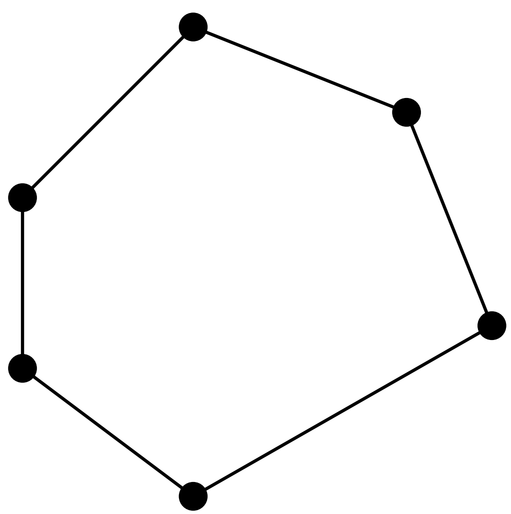
Results of Affolter-George-R. '22

- We define twisted TCD maps for the pentagram map and cross-ratio dynamics.

- We thus recover the Ovsienko-Schwartz-Tabachnikov conserved quantities for the pentagram map and the Arnold-Fuchs-Izmestiev-Tabachnikov conserved quantities for cross-ratio dynamics as conserved quantities for the dimer model.

- The iterations of both dynamics are realized as some sequences of moves on the respective TCD maps.

The pentagram map (Schwartz '92)

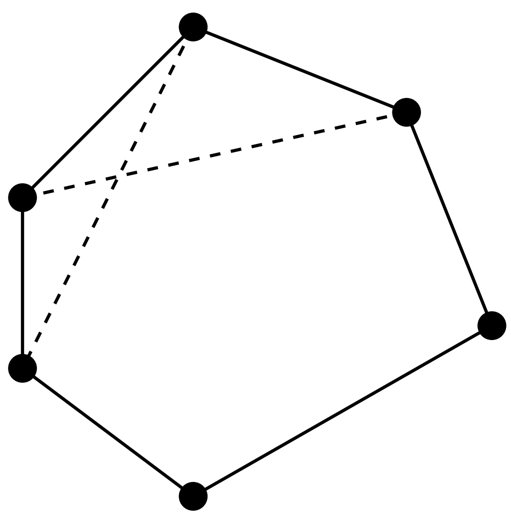


$n = 6$

- points at time 0

- Discrete-time dynamics on twisted n-gons in $\mathbb{C}P^2$ considered up to $PGL_3(\mathbb{C})$.

The pentagram map (Schwartz '92)



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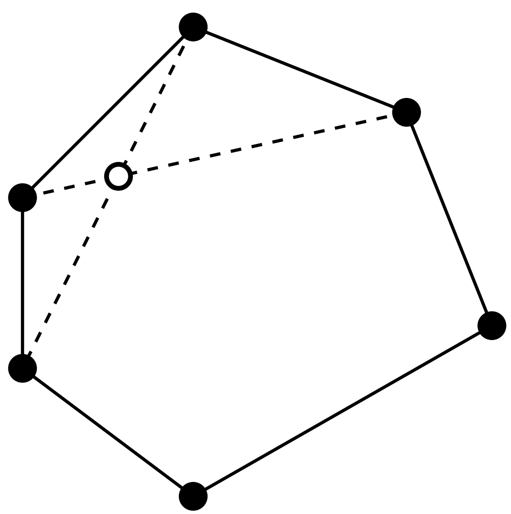

$n = 6$

- points at time 0
- points at time 1

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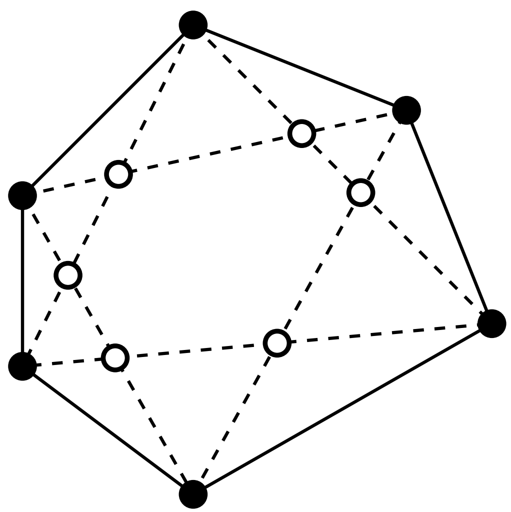

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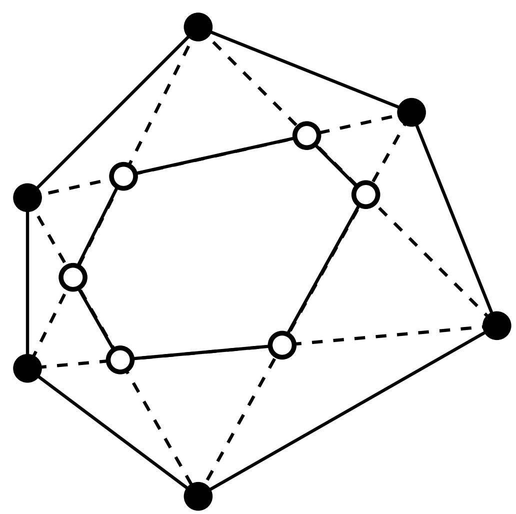

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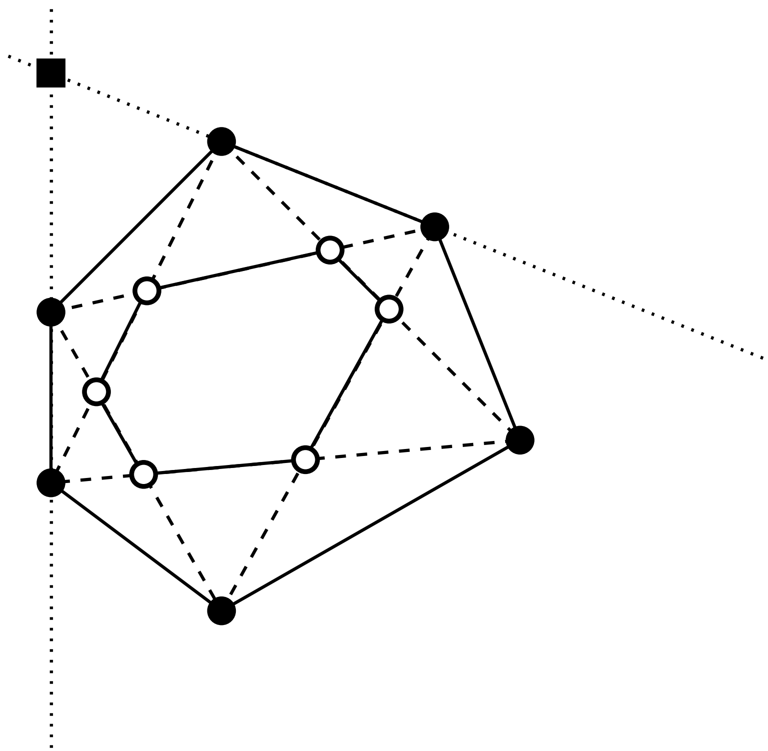
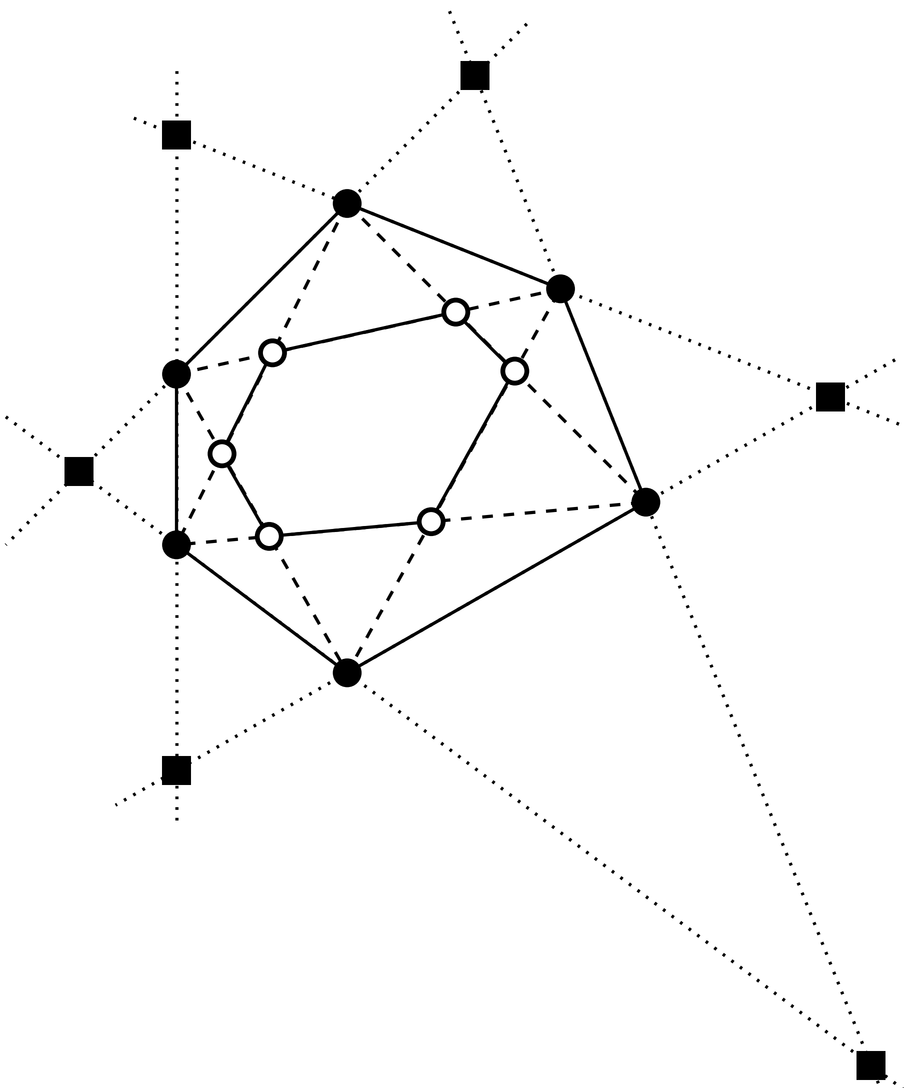


$n = 6$

- points at time 0
- points at time 1

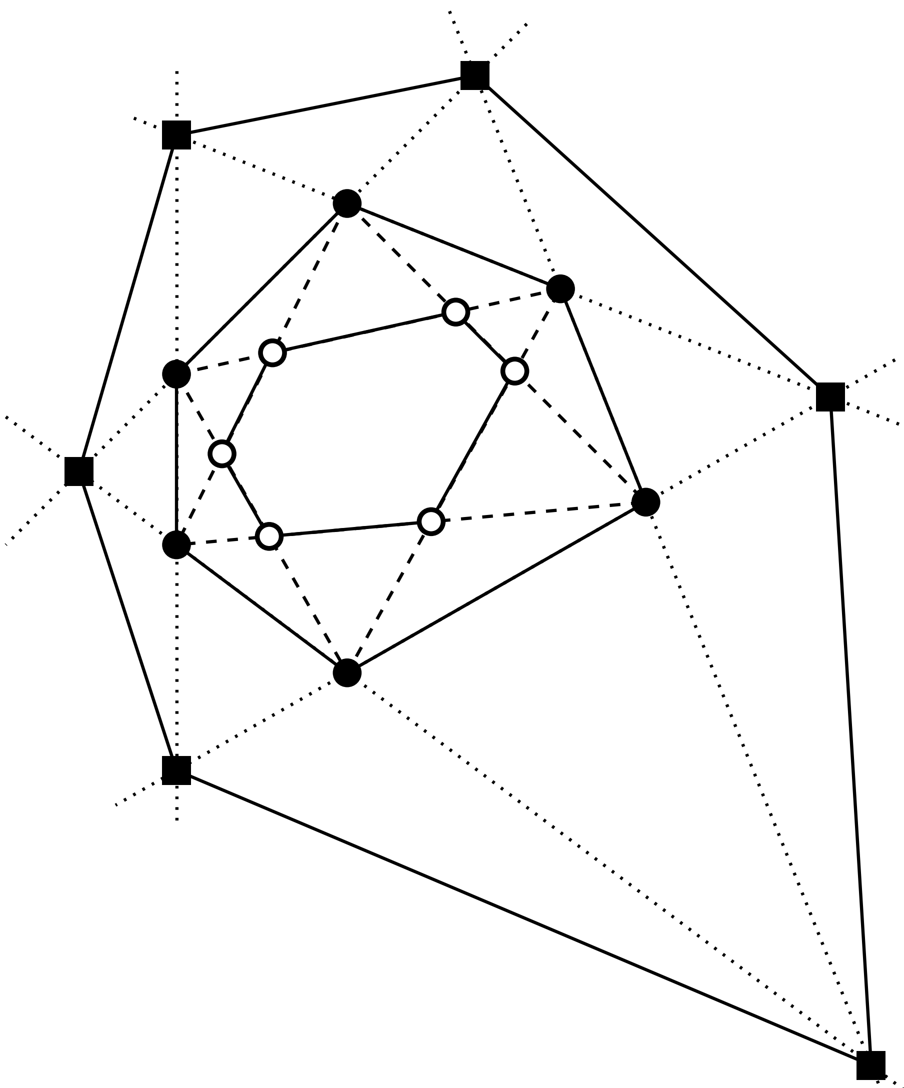
The pentagram map (Schwartz '92)



$n = 6$

- ■ points at time -1
- ● points at time 0
- ○ points at time 1

The pentagram map (Schwartz '92)



$n = 6$

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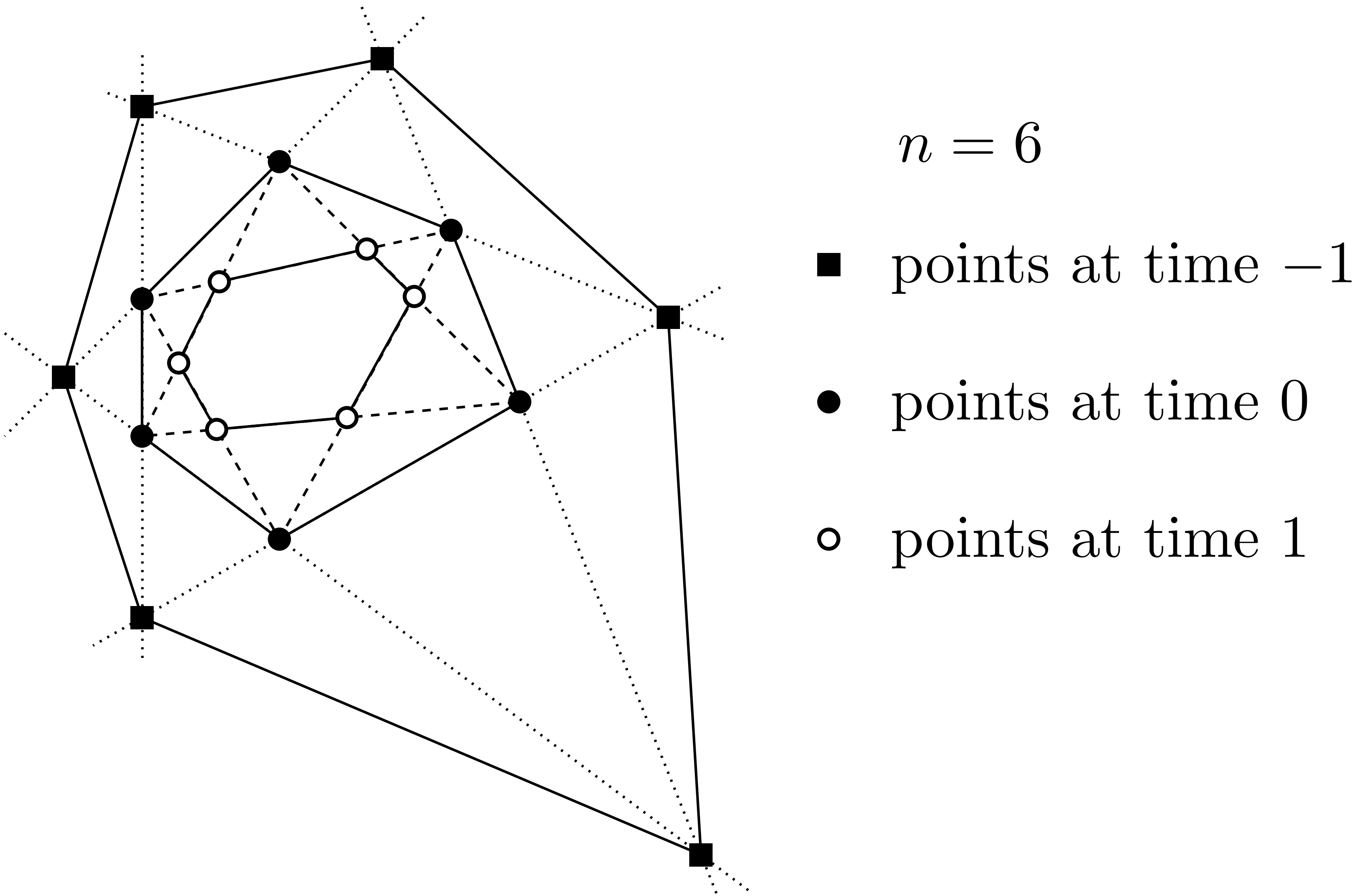
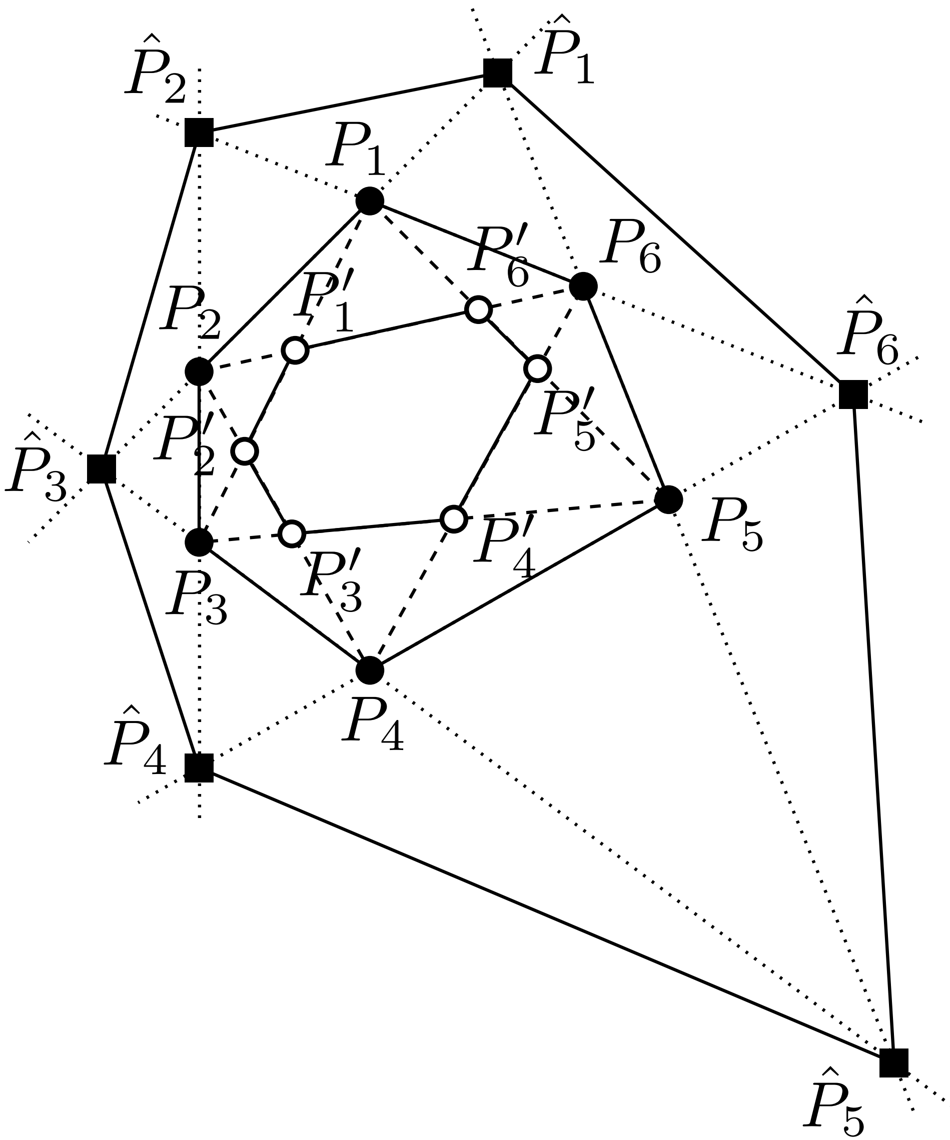
- ● points at time 0
- ○ points at time 1



TCD map for the pentagram map



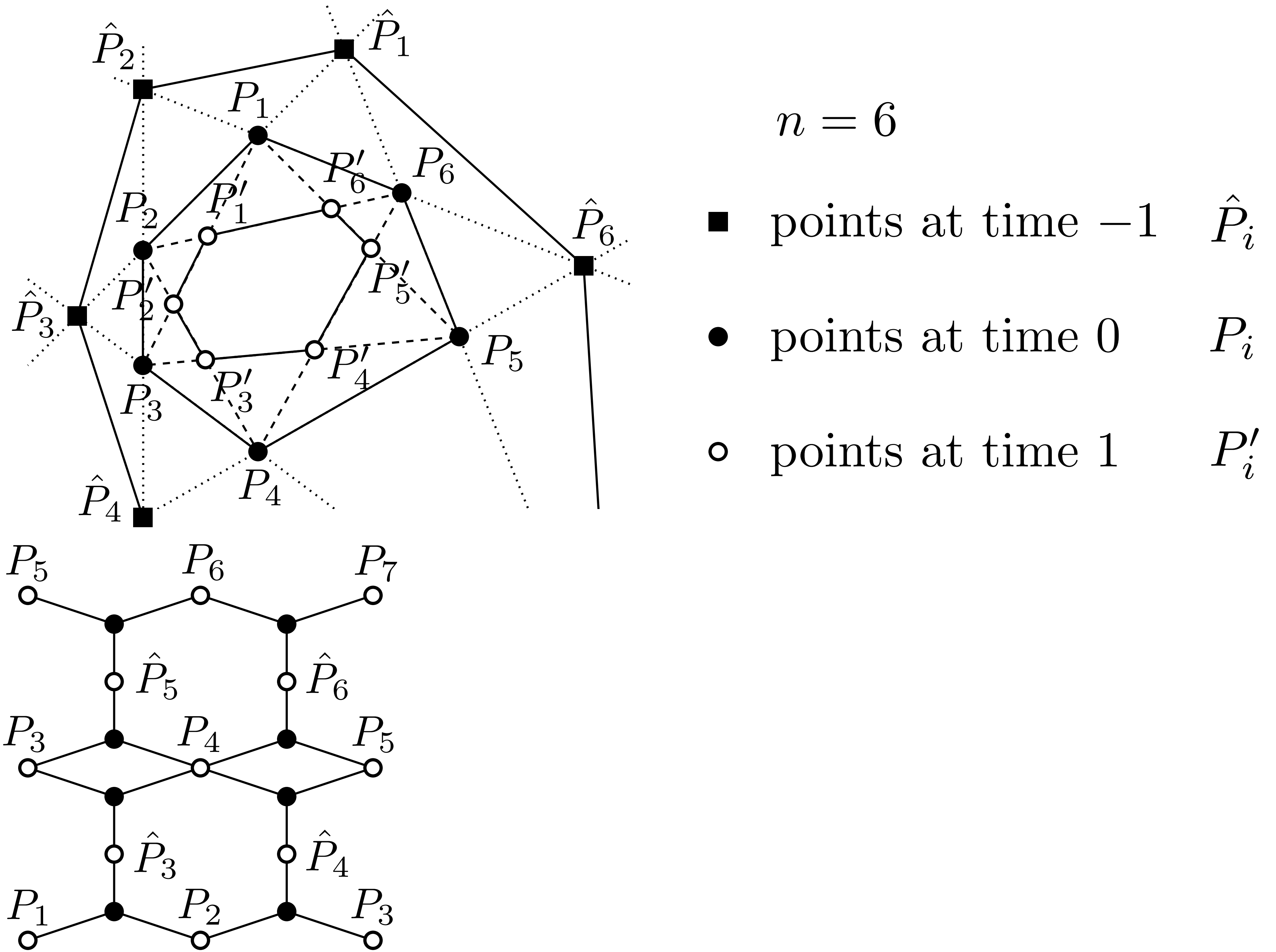
$n = 6$

■ points at time -1 $\hat{P}_i$

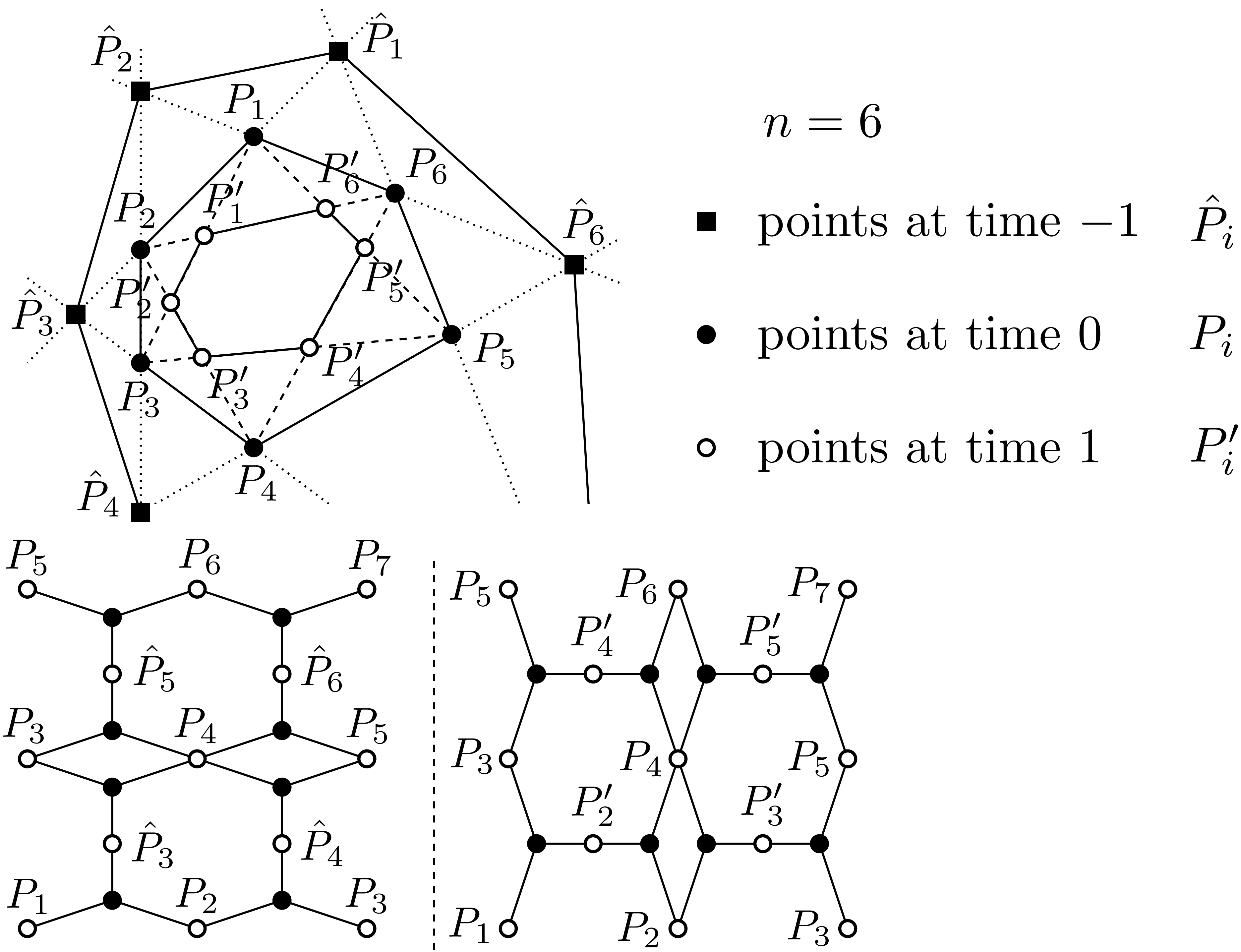
● points at time 0 P_i

○ points at time 1 P'_i

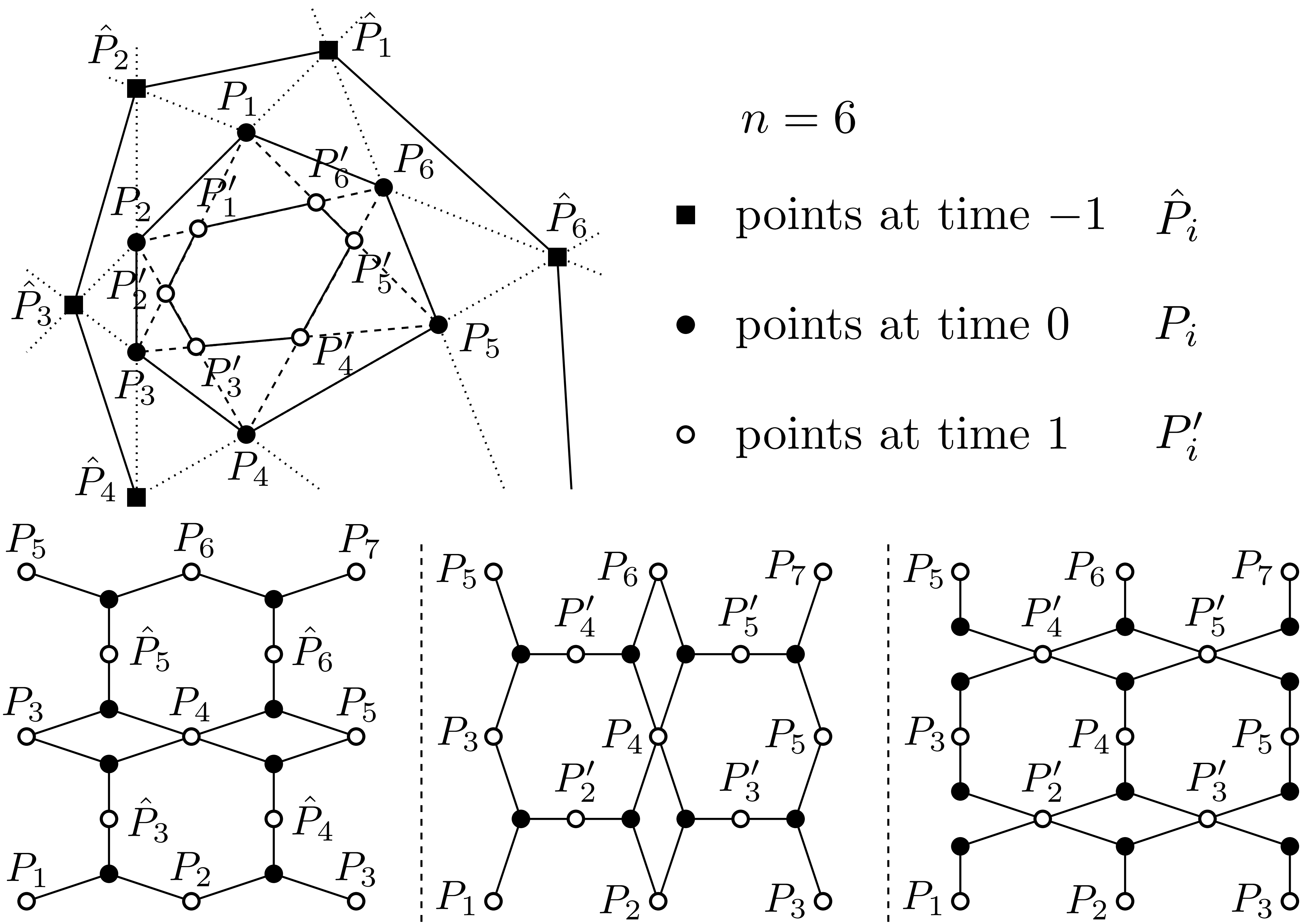
TCD map for the pentagram map



$\hat{P}_1$
$\hat{P}_2$
$\hat{P}_3$
$\hat{P}_4$
$\hat{P}_6$
P_1
P_2
P_3
P_4
P_5
P_6
P'_1
P'_2
P'_3
P'_4
P'_5
P'_6
$n = 6$
points at time -1 $\hat{P}_i$
points at time 0 P_i
points at time 1 P'_i
P_5 P_6 P_7
$\hat{P}_5$ $\hat{P}_6$
P_3 P_4 P_5
$\hat{P}_3$ $\hat{P}_4$
P_1 P_2 P_3



$n = 6$
points at time -1 $\hat{P}_i$
points at time 0 P_i
points at time 1 P_i'
$\hat{P}_1$ $\hat{P}_2$ $\hat{P}_3$ $\hat{P}_4$ $\hat{P}_6$
P_1 P_2 P_3 P_4 P_5 P_6
P_1' P_2' P_3' P_4' P_5' P_6'
P_5 P_6 P_7 $\hat{P}_5$ $\hat{P}_6$ P_3 P_4 P_5 $\hat{P}_3$ $\hat{P}_4$ P_1 P_2 P_3
P_5 P_6 P_7 P_4' P_5' P_3 P_4 P_5 P_2' P_3' P_1 P_2 P_3



$n = 6$
points at time -1 $\hat{P}_i$
points at time 0 P_i
points at time 1 P'_i
$\hat{P}_1$ $\hat{P}_2$ $\hat{P}_3$ $\hat{P}_4$ $\hat{P}_6$
P_1 P_2 P_3 P_4 P_5 P_6
P'_1 P'_2 P'_3 P'_4 P'_5 P'_6
P_5 P_6 P_7 $\hat{P}_5$ $\hat{P}_6$ P_3 P_4 P_5 $\hat{P}_3$ $\hat{P}_4$ P_1 P_2 P_3
P_5 P_6 P_7 P'_4 P'_5 P_3 P_4 P_5 P'_2 P'_3 P_1 P_2 P_3
P_5 P_6 P_7 P'_4 P'_5 P_3 P_4 P_5 P'_2 P'_3 P_1 P_2 P_3

Conclusion

- Dimer integrable systems possess a unified theory, valid for any bipartite graph on the torus.
- Taking TCD maps for different choices of graphs, we recover a wealth of seemingly disparate examples, coming from either geometric dynamics or from discrete differential geometry.
- Provides a powerful machinery to prove integrability and find a cluster algebra structure for even more geometric systems.

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Happy birthday Philippe!