Philippe's Fest !

where is Charlie ?

J.-B.

Playing with random matrices and free probability in noisy many-body systems

(A small — possibly entertaining — exercice)

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(work done in collaboration with Ludwig Hruza)

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(Standard) random matrices :

with impact on modeling many physical phenomenons (say random surfaces, chaotic systems, etc)

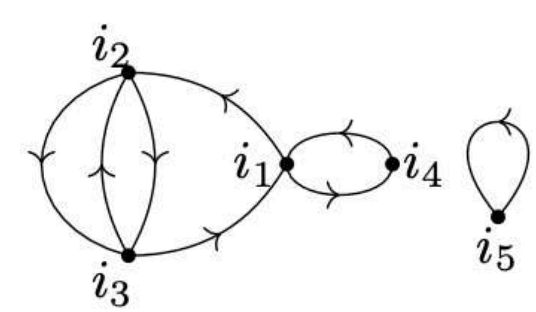
— Basic examples :

« Wigner's matrices » $M = (M_{ij})$ hermitian, NxN
with Gaussian entries $\mathbb{E}[M_{ij}M_{ji}] = N^{-1}$

« Haar orbits » $M = UDU^\dagger$ with U unitary Haar distributed, D diagonal
with HCIZ integral as cumulant generating function $\mathbb{E}[e^{tr(AM)}] = \int dU\, e^{tr(AUDU^\dagger)}$

— Properties : - U(N), hence U(1)^N invariance
- Unstructured matrices (invariance under permutation, in law)
—> Expectation values are « topological »

—> Look at structured matrices ?…

$M_{i_1i_2}|M_{i_2i_3}|^2 M_{i_2i_3}M_{i_3i_1}M_{i_1i_4}M_{i_4i_1}M_{i_5i_5} =$  .

Some computational experiments…

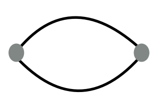
—> Originates from noisy many-body systems, in particular **QSSEP** (more later…)

— U(1) invariance (by choice) : $M_{jk} \equiv_{\text{in law}} e^{i\theta_j} M_{jk} e^{-i\theta_k}$ (—> charge conservation)

$\mathbb{E}[M_{ij}] = 0 \ , \quad (i \neq j)$ ⟵ non-trivial graphs are Eulerian

$\mathbb{E}[M_{ii}] = g_1(x) \ , \quad x = i/N$ ⟵ echo of the « structure »

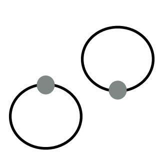
— Scaling of the loop/simple cumulants (—> limiting spectrum and large deviation function)



$$\mathbb{E}[M_{ij}M_{ji}] = N^{-1} g_2(x,y) \ , \quad x = i/N, \ y = j/N, \ (i \neq j)$$

↖ with smaller 1/N dependence for bigger loops (moments)

— Factorisation of disconnected graphs (at leading order)



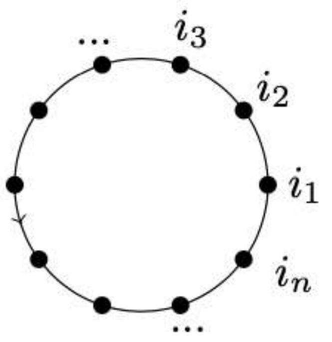
$$\mathbb{E}[M_{ii}M_{jj}] = g_1(x)g_1(y) \ , \quad (i \neq j)$$

(up to 1/N correction)

Ensemble of 'structured' random matrices :

— Ensemble of large NxN matrices M with four defining properties :

- U(1) invariance : $M_{jk} \equiv_{\text{in law}} e^{i\theta_j} M_{jk} e^{-i\theta_k}$



- Scaling of the loop expectation values : $(x_k = i_k/N, \text{distincts})$

$$\mathbb{E}[M_{i_1 i_n} \cdots M_{i_3 i_2} M_{i_2 i_1}] \sim N^{1-n}$$

- Factorisation of product of disconnected graphs (at leading order) :

$$\mathbb{E}[M_{j_1 j_n} \cdots M_{j_2 j_1} \cdot M_{i_1 i_p} \cdots M_{i_2 i_1}] = \mathbb{E}[M_{j_1 j_n} \cdots M_{j_2 j_1}]\, \mathbb{E}[M_{i_1 i_p} \cdots M_{i_2 i_1}]$$

- Continuity of the loop cumulants at coincident points (at leading order) :

$$N^{n-1} \mathbb{C}[M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1}] \text{ continuous at } x_{i_p} = x_{i_q}$$

——> Allow to compute the expectation of any graph (at leading order)

The role of non-crossing partitions (i) :

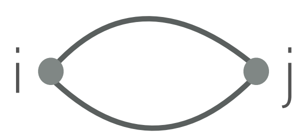
Compute the moments / loop expectation values
(with possible coincident points) via the cumulant expansion

Recall the moment-cumulant formula : $\mathbb{E}[X_1 \cdots X_n] = \sum_{\pi \in P(n)} \mathbb{E}_\pi [X_1 \cdots X_n]^c$

— N=1 : $\mathbb{E}[M_{ii}] = g_1(x)$

$\delta_{ij}\mathbb{E}[M_{ii}]$ $\quad$ $\delta_{ij}\mathbb{E}[M_{jj}]$

— N=2 : $\mathbb{E}[M_{ij}M_{ji}] = \mathbb{E}[M_{ij}M_{ji}]^c + \mathbb{E}[M_{ij}]^c\mathbb{E}[M_{ji}]^c$



$$= N^{-1}\,[g_2(x,y) + \delta(x-y)g_1(x)g_1(y)]$$

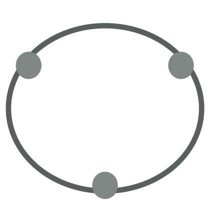
$(with\ \delta_{ij} \rightsquigarrow N^{-1}\delta(x-y)\ )$

(or $\mathbb{E}[\mathrm{Tr}(h_1 M h_2 M \cdots h_n M)], h_k$ diag)

— N=3 : (idem...)

δ_{ij} $\quad$ $\delta_{ij}N^{-1}$ $\quad$ δ_{ij} $\quad$ δ_{jk} $\quad$ δ_{ki}

$$\mathbb{E}[M_{ij}M_{jk}M_{ki}] = \mathbb{E}[M_{ij}M_{jm}M_{ki}]^c + \mathbb{E}[M_{ij}]^c\mathbb{E}[M_{jk}M_{ki}]^c + \circlearrowleft + \mathbb{E}[M_{ij}]^c\mathbb{E}[M_{jk}]^c\mathbb{E}[M_{ki}]^c$$

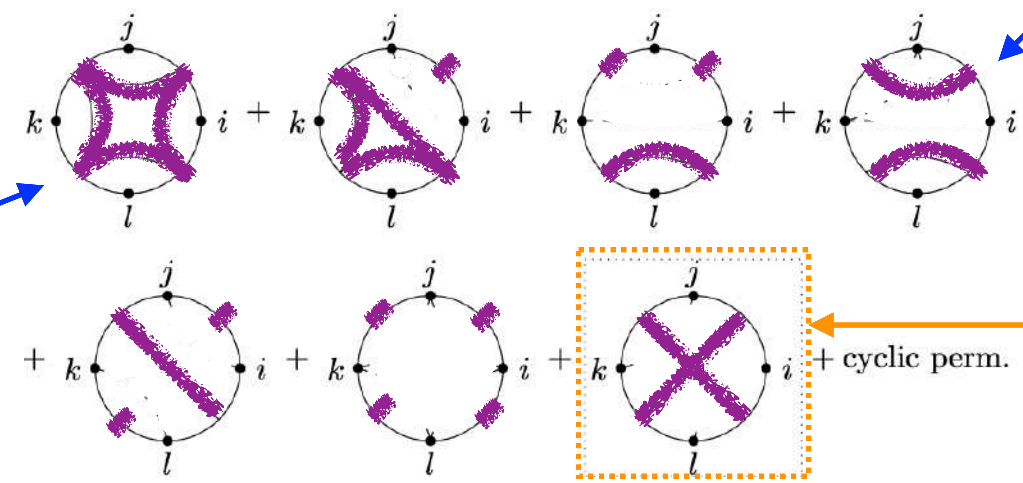


$$= N^{-2}\,[g_3(x_1,x_2) + \delta(x_1-x_2)g_1(x_1)g_2(x_2,x_3) + \circlearrowleft$$
$$+ \delta(x_1-x_2)\delta(x_2-x_3)g_1(x_1)g_1(x_2)g_1(x_3)]$$

The role of non-crossing partitions (ii) :

— N=4 : the role of non-crossing partitions…

$\mathbb{E}[M_{ij}M_{jk}M_{kl}M_{li}] =$



cumulants
(partition of edges)

$\mathbb{E}[M_{ij}M_{jk}]^c\mathbb{E}[M_{kl}M_{li}]^c$

$\delta_{ik}N^{-2}$

$\mathbb{E}[M_{ij}M_{kl}]^c\mathbb{E}[M_{jk}M_{li}]^c$

$\delta_{ij}\delta_{jk}\delta_{kl}N^{-2}$

negligeable !

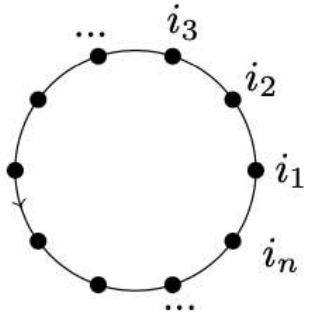
$$= N^{-3}\,[g_4(x_1,x_2,x_3,x_4) + \delta(x_1,x_2)g_1(x_1)g_3(x_2,x_3,x_4) + \circlearrowleft$$
$$+\,\delta(x_1,x_3)g_2(x_1,x_2)g_2(x_3,x_4) + \circlearrowleft$$
$$+\cdots \text{"only non-crossing partitions"}]$$

Only the **non-crossing partitions** contribute. $\Longrightarrow$ **Free probability !**

Structured RMT & Loop expectations values:

——> Dominant role of the **loop expectation values.**

$$g_n(x_1, x_2, \cdots, x_n) := \lim_{N\to\infty} N^{n-1}\, \mathbb{E}[M_{i_1 i_2} M_{i_2,i_3} \cdots M_{i_n i_1}]^c$$



$(x_k = i_k/N, \text{distincts})$

- They code for all structures and correlations (at leading order)

- Scaling with N ensures the existence of a large deviation function (formal series)

- We call them the '**local free cumulants**'

- They have a natural interpretation as free cumulants in operator valued free probability (see below)

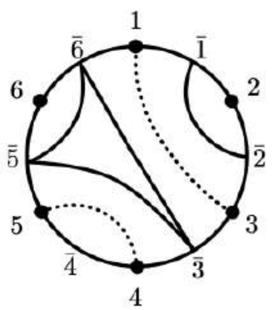
Loop expectations values & and non-crossing partition :

— For loop of any size, we have (by similar arguments) :

$$\lim_{N\to\infty} N^{n-1}\mathbb{E}[M_{i_1 i_2} M_{i_2 i_3}\cdots M_{i_n i_1}] = \sum_{\pi\in NC(n)} g_\pi(\vec{x})\,\delta_{\pi^*}(\vec{x})\ .$$

non-crossing partition (on edges) → $g_\pi(\vec{x})$

its Kreweras dual (on points) → $\delta_{\pi^*}(\vec{x})$



$$\pi = \{\{\bar{1},\bar{2}\},\ \{\bar{3},\bar{5},\bar{6}\},\ \{\bar{4}\}\}$$

$$\pi^* = \{\{1,3\},\ \{2\},\ \{4,5\},\{6\}\}$$

or $$N^{-1}\mathbb{E}[\mathrm{Tr}(M^n)] = \sum_{\pi\in NC(n)} \int g_{\pi^*}(\vec{x})\,\delta_\pi(\vec{x})d\vec{x}$$

— Operator valued probability alias « conditioned expectations over the diagonal »

$$\text{e.g.}\quad \mathbb{E}[\langle x|\, M h_1 M\cdots h_n M\, |x\rangle],\quad h_k\ \text{diag}$$

Non-linear transformations

[Thanks to R. Speicher]

The 'axioms' (i)-(iv) are closed under some non-linear transformations

—> A way of constructing new ensembles from simple ones.
Applications (say in ML) ??...

— Polynomial transformations

> If M satisfies the axioms, so does $P(M) = \sum_{n>0} a_n \, M^n$.

(with an explicit expression for the new local free cumulants)

— Point-wise transformations

> Given M, define Y with entries $Y_{ij} = M_{ij} f_{ij}(N|M_{ij}|^2)$, with non-linear functions f_{ij} which may be different for each entries. Then
> If M satisfies the axioms, so does Y.

(with an explicit expression for the new local free cumulants)

Spectrum of sub-blocks (i)

$$M = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix} \quad \rightsquigarrow \quad \hat{M}_I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & * & 0 \\ 0 & * & * & * & 0 & * & 0 \\ 0 & * & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

—> What is the spectral distribution of any (collection of) sub-blocks ?

— For the usual unstructured ensemble is simple or known:

- sub-blocks of Wigner matrices —> again Winger semi-circle law.
- sub-blocks of Haar rotated matrices UDU*

  —> free dilatation of the spectral measure of D

Spectrum of sub-blocks (ii)

→ (Using the moment methods (from which the spectral measure can be deduced…))

Let $F[h](z) := \mathbb{E}\, N^{-1} \mathrm{Tr} \log(z - M_h)$ with $M_h = h^{\frac{1}{2}} M h^{\frac{1}{2}},\ e.g.\ h(x) = \mathbb{I}_{x \in I}$

[D.B.-L. Hruza]

Theorem

$$F[h](z) = \min_{a_z, b_z} \left[\int [\log(z - h(x) b_z(x)) + a_z(x) b_z(x)]\, dx - F_0[a_z] \right]$$

« generating function of single loop expectation values »

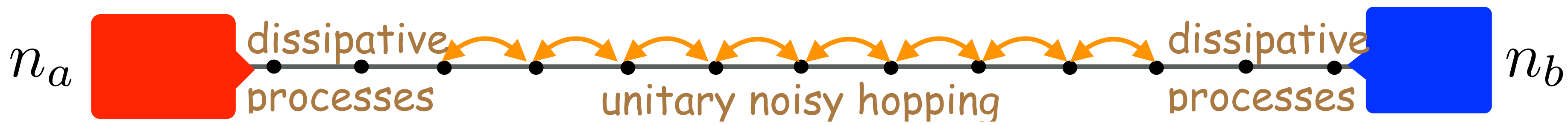
$$F_0[a] = \sum_{n \geq 1} \frac{1}{n} \int d\vec{x}\, g_n(\vec{x})\, a(x_1) \cdots a(x_n)$$

Proof Via nested structured of non-crossing partition or via operator valued free probability

$$b_z(x) = \frac{\delta F_0[a_z]}{\delta a_z(x)}\ \&\ a_z(x) = \frac{h(x)}{z - h(x) b_z(x)}, \quad \rightsquigarrow \quad G[h](z) = \int \frac{dx}{z - h(x) b_z(x)}$$

Origin in / Relation with Q-SSEP :

— Q-SSEP = Quantum Symmetric Simple Exclusion Process : [D.B.-T.Jin, 2019]



$$dH_t = \sqrt{D}\, \sum_j \left(c^\dagger_{j+1} c_j\, dW^j_t + c^\dagger_j c_{j+1}\, d\overline{W^j_t}\right)$$

{ + boundary terms…
  + injection/extraction…

— Stochastic many-body quantum system (quadratic but noisy) :

—> « Coherences » alias « matrix of two-point functions »

$$G_{ij} = \langle c^\dagger_j c_i \rangle_t = \mathrm{Tr}(\rho_t\, c^\dagger_j c_i) \quad \Longrightarrow$$

random quantum states

Large « structured » random matrices

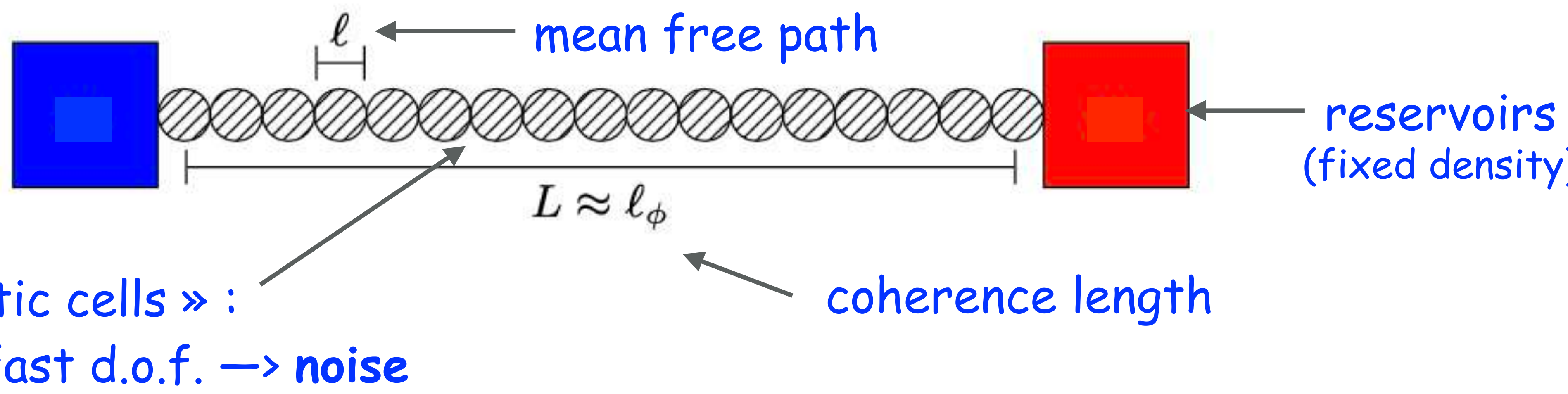
— <u>Claim</u> :

> At any time (with the measure E induced by the above SDE), and hence in particular in the infinite time steady measure :
>
> The matrix of coherences G belongs to the above 'structured' RMT class.

⟶ Many (physical) consequences : { e.g. long range correlations, volume law for entanglement mutual information, …

Application to /Relation with mesoscopic physics :

— « Mesoscopic » (diffusive + coherent) systems out-of-equilibrium



— <u>Conjecture</u> : [D.B.-L. Hruza]

> If the fast dynamics on the small scale degrees of freedom is ergodic,
> If the mean dynamics follows the classical 'macroscopic fluctuation theory',
> Then
> The matrix of two point functions belongs to the above 'structured' RMT class.

Bon Anniversaire Philippe !!!

Q-SSEP Spectral density from free probability

— Let $a_z(x) := \mathbb{E}\langle x|\frac{h}{z-G_h}|x\rangle$ then $zh(x)^{-1} = R_0[a_z](x) + a_z(x)^{-1}$

Let $b_z(x) := R_0[a_z](x) = \delta F_0[a_z]/\delta a_z(x)$ (local R-transform)

— For Q-SSEP:

$$\begin{cases} [\log(z - b_z(x))]'' = 0, & x \in I, \\ b_z(x)'' = 0, & x \notin I, \end{cases}$$

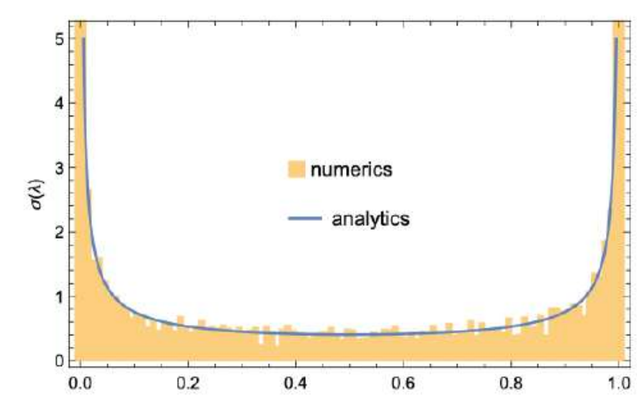
with $b_z(0) = 0,\ b_z(1) = 1.$

because Ro[a](x) is constructed from free cumulants w.r.t. the Lebesgue measure.

— For the complete interval (i.e. for G)

$$d\sigma_{[0,1]}(\lambda) = \frac{d\lambda}{\lambda(1-\lambda)} \frac{1}{\pi^2 + \log^2(\frac{1-\lambda}{\lambda})}.$$

which is a Lorentzian distribution for $\nu = \log(\frac{1-\lambda}{\lambda})$



Hidden free probability in classical SSEP :

[M. Bauer, D.B. ,Ph. Biane]

— **Large deviation Q-SSEP**

$$\mathbb{P}\left[\mathfrak{n}(\cdot) \approx n(\cdot)\right] \asymp_{N\to\infty} e^{-N\, I_{\rm ssep}[n]},$$

—> In terms of free probability data

$$I_{\rm ssep}[n] = \max_{g(\cdot),q(\cdot)} \Big(\int_0^1 dx \Big[n(x) \log\big(\frac{n(x)}{g(x)}\big) + (1-n(x)) \log\big(\frac{1-n(x)}{1-g(x)}\big) + q(x) g(x) \Big] - F_0^{\rm ssep}[q] \Big)$$

with $$F_0^{\rm ssep}[a] = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} R_n(\mathbb{I}_{[a]}),$$

n-th free cumulants of $\mathbb{I}_{[a]}(x) = \int_x^1 dy\, a(y)$ w.r.t. to the Lebesgue measure on [0,1]

— **Three steps** :

(i) : Q-SSEP /classical SSEP relation

(ii) : Free probability description of the Q-SSEP steady measure

(iii) : A relation between cumulants and « non-coincidants » cumulants for Bernoulli variables

Emergence of free probability in noisy systems

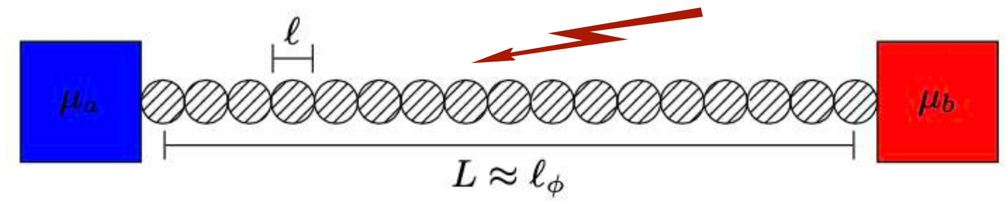
— **Coarse-grained description** (at mesoscopic scale)

(i) separation of time scales :

fast, closed dynamics within ballistic cells for $t < t_\ell$

—> unitary dynamics within each cells for $t < t_\ell$

« ballistic cells »



(ii) ergodicity of the fast dynamics (—> noise) :

$$\mathbb{E}_t[G_{jk}] := \frac{1}{t_\ell}\int_t^{t+t_\ell} dt'\, G_{jk}(t') = \mathrm{Tr}(\rho_t\, [c_i^\dagger c_j]_U)$$

(local) Haar measure

— **Validity of U(1) sym. + MFT in mean —> the « universality three conditions »**

- U(1) invariance :

  $G_{jk} \equiv_{\mathrm{in\ law}} e^{i\theta_j} G_{jk} e^{-i\theta_k}$

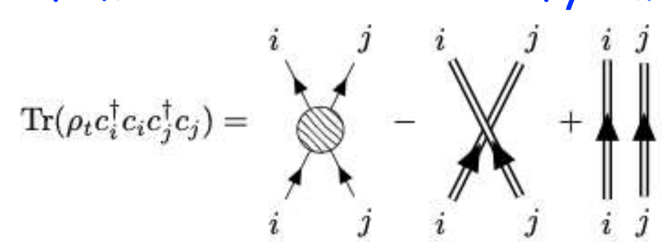
  <— local conservation and closed unitary dynamics at short time

- Scaling of the loop expectation values :

  $\mathbb{E}[G_{j_1 j_n} \cdots G_{j_3 j_2} G_{j_2 j_1}] \sim N^{1-n}$

  If some perturbation theory is valid $(H = H_0 + V)$

  <— If mean densities satisfy MFT



- Factorisation of loop expectations :

  <— closed fast dynamics / cells independence

— Validation / Violation of these three conditions ? …