Categorical symmetries in loop soups

loops

H. Saleur, IPhT CEA Saclay

(NOT spaghettis - sorry Philippe)

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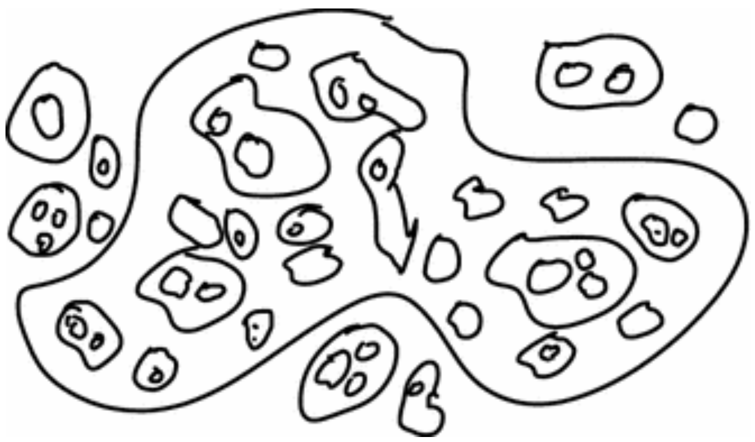
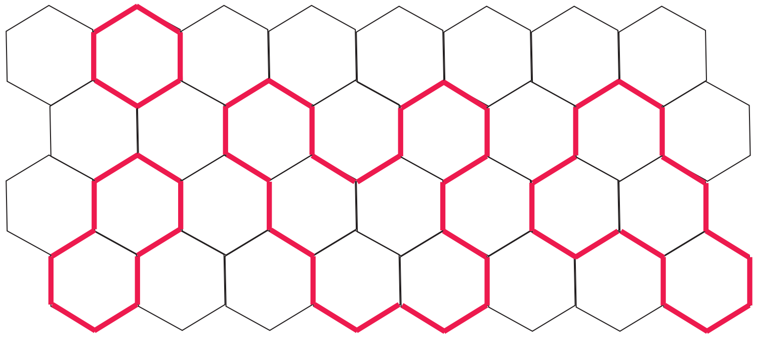
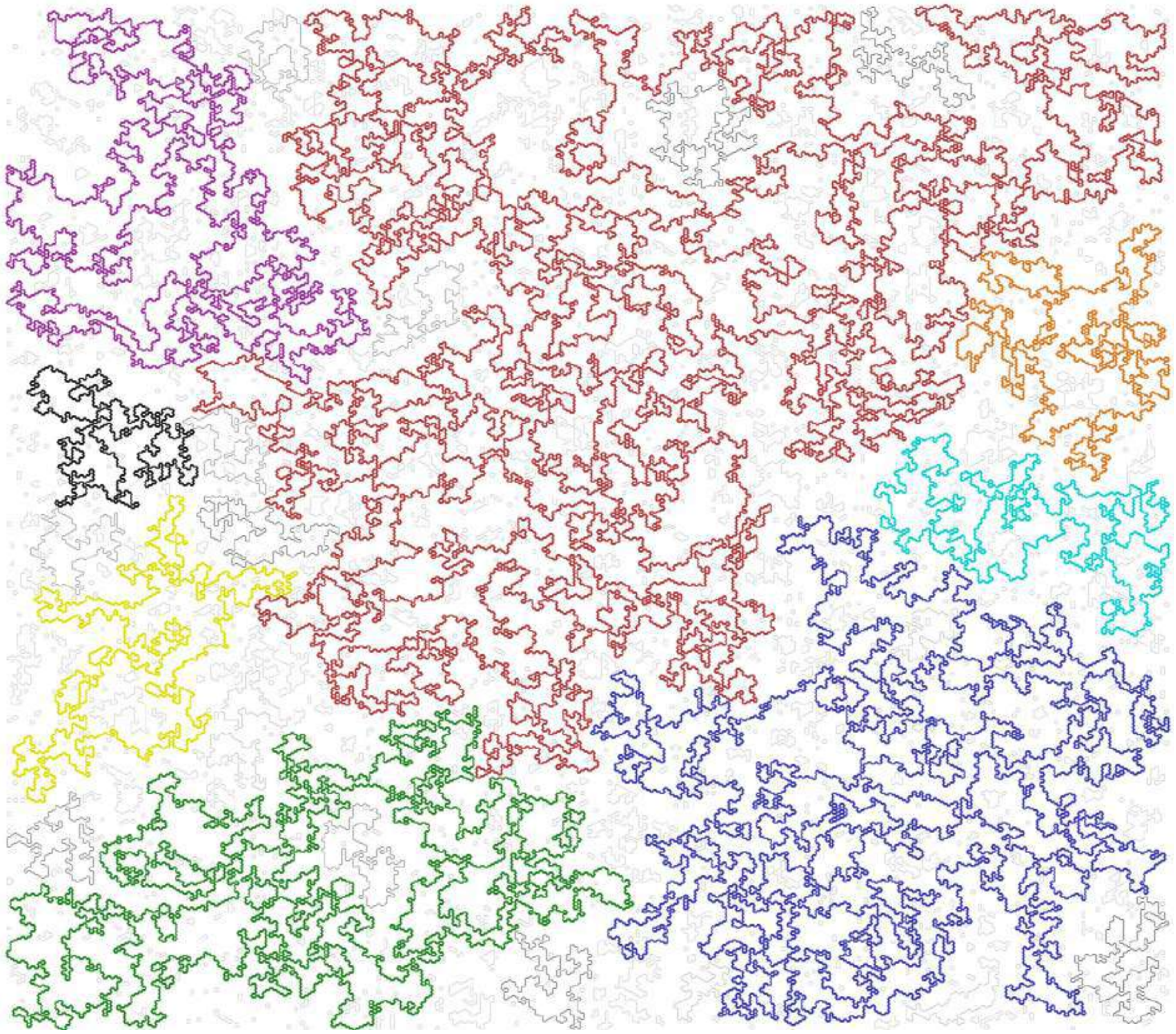
Conformal loop ensembles or loop soups







(ensembles of self-avoiding mutually avoiding loops with fugacities per loop and bond)

- Are “generically” critical CFT

$$Z = \sum_{\text{dilute loop gas}} K_c^B n^L$$

- and arise in the study of $O(n)$ lattice spin models

n-component vectors $\vec{S}_i$ with $O(n)$ symmetric $\vec{S}_i.\vec{S}_j$ couplings

$$Z \propto \int \prod d\vec{S}_i \prod_{<ij>} (1 + K\vec{S}_i.\vec{S}_j)$$

- Properties like central charge and (some) critical exponents have been known for a long time

(Dotsenko Fateev 1984)

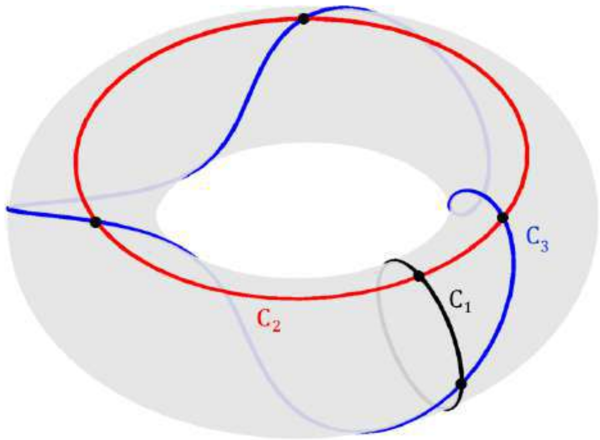
$$n \in [-2,2]; \; n = 2\cos\frac{\pi}{x}, \; x \in [1,\infty]$$

$$c = 1 - \frac{6}{x(x+1)}$$

$$\kappa = \frac{4x}{x+1}, \quad \kappa = \frac{4(x+1)}{x}$$

(Den Nijs, Nienhuis, Duplantier Saleur, late 80's)

- Our old work: torus partition function



Relations between the Coulomb Gas Picture and Conformal Invariance of Two-Dimensional Critical Models

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is a sort of bizarre elliptic function

$$\mathscr{Z}_n \to \hat{Z}[g, e_0] = \sum_{M', M \in \mathbb{Z}} Z_{M',M}(g/4) \cos(\pi e_0 M' \wedge M)$$

$\wedge \equiv \text{gcd}$

$$Z_{m',m}(g) = Z_1(g) \exp\left[-\pi g \frac{m'^2 + m^2(\tau_R^2 + \tau_I^2) - 2\tau_R m m'}{\tau_I}\right] \qquad Z_1(g) = \frac{\sqrt{g}}{\tau_I^{1/2} \eta(q)\, \eta(\bar{q})}$$

$$\eta(q) = q^{1/24} \prod_{N=1}^{\infty} (1 - q^N) \qquad g = \frac{x+1}{x}, \; e_0 = \frac{1}{x}$$

$q = e^{2i\pi\tau}$, τ the torus modular parameter

nicely modular invariant but not very explicit

37 years ago!

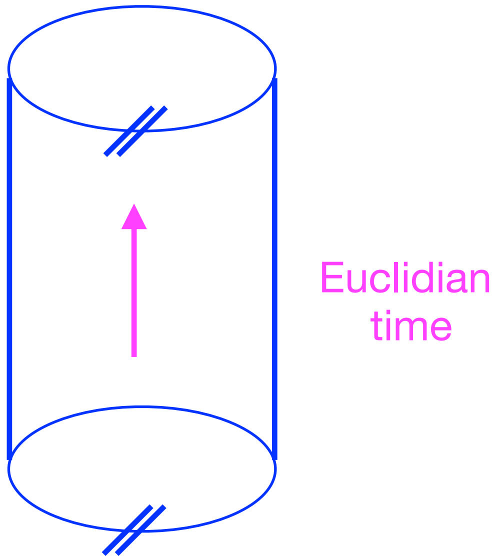
Why should we care?

- From CFT the result should have the form *(Cardy 1988)*

$$\left(Z = \text{Tr } q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\right) \qquad Z = \sum_{h, \bar{h}} \text{degeneracy} \times q^{h - c/24} \bar{q}^{\bar{h} - c/24}$$

A good way to think of this is to observe that

$$Z = \text{Tr (transfer matrix)}^{\text{power}} = \sum \underline{\text{degeneracies}} \times \text{eigenvalues}^{\text{power}}$$



The degeneracies should be integer for n integer and in general correspond to (the dimensions of) the irreducible representations of the symmetry

E.g. the order parameter comes with [1] the vector representation

So the first technical challenge is to perform a Poisson resummation in order to write the partition function as a sum over powers of $q, \bar{q}$

- This was done in *(Read Saleur 2007)* Set $h_{rs} = \frac{[(x+1)r - xs]^2 - 1}{4x(x+1)}$ $n = 2\cos\frac{\pi}{x}$

$$Z_{O(n)} = \sum_{s\in 2\mathbb{N}+1} \chi^D_{\langle 1,s\rangle} + \sum_{r\in\frac{1}{2}\mathbb{N}^*} \sum_{s\in\frac{1}{r}\mathbb{Z}} \left(E_{r,s} + \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1}\right) \chi^N_{(r,s)}$$

this gives the spectrum of the theory, crucial to determine correlation functions using the bootstrap

(Grans-Samuelsson, He, Jacobsen, Nivesvivat, Ribault, Saleur)

$\chi^D_{\langle r,s\rangle}$, $r, s \in \mathbb{N}^*$ characters of irreducible diagonal representations $K_{h_{rs}} \otimes \overline{K}_{h_{r,s}}$

$$\chi^D_{\langle r,s\rangle} = q^{h_{rs}-\frac{c}{24}} \frac{1-q^{rs}}{P(q)} \times \text{h.c.}$$

$\chi^N_{(r,s)}$ characters of non-diagonal potentially reducible representations $V_{h_{rs}} \otimes \overline{V}_{h_{r,-s}}$

$$\chi^N_{(r,s)} = \frac{q^{h_{rs}-\frac{c}{24}}}{P(q)} \times \frac{\bar{q}^{h_{r,-s}-\frac{c}{24}}}{P(\bar{q})}$$

The degeneracies are given by the mysterious formula

$$E_{r,s} = \frac{1}{2r}\sum_{r'=0}^{2r-1} e^{i\pi r' s} x_{(2r)\wedge r'}(n), \quad E_{r,s} = E_{r,s \bmod 2\mathbb{Z}}$$

polynomials in n

$$n = \mathfrak{q} + \mathfrak{q}^{-1}$$

$$x_d(n) = \mathfrak{q}^d + \mathfrak{q}^{-d}$$

(r,s)	$E_{r,s}$
$(\frac{1}{2},0)$	n
$(1,0)$	$\frac{1}{2}(n+2)(n-1)$
$(1,1)$	$\frac{1}{2}n(n-1)$
$(\frac{3}{2},0)$	$\frac{1}{3}n(n^2-1)$
$(\frac{3}{2},\frac{2}{3})$	$\frac{1}{3}n(n^2-4)$
$(2,0)$	$\frac{1}{4}n\,(n^3-3n+2)$
$(2,\frac{1}{2})$	$\frac{1}{4}\,(n^4-5n^2+4)$
$(2,1)$	$\frac{1}{4}(n-2)n(n+1)^2$
$(2,\frac{3}{2})$	$\frac{1}{4}\,(n^4-5n^2+4)$
$(3,0)$	$\frac{1}{6}\,(n^6-6n^4+n^3+11n^2-n-6)$

$$Z_{O(n)} = \sum_{s\in 2\mathbb{N}+1} \chi^D_{\langle 1,s\rangle} + \sum_{r\in\frac{1}{2}\mathbb{N}^*} \sum_{s\in\frac{1}{r}\mathbb{Z}} \underbrace{\left(E_{r,s}+\delta_{r,1}\delta_{s\in 2\mathbb{Z}+1}\right)}_{\Lambda_{r,s}} \chi^N_{(r,s)}$$

multiplicity spaces

- Note: while the formula for Z was derived in the context of CFT and for $n\in[-2,2]$, it is in fact true for all n (integer or not), in finite size, and not necessarily at criticality

Therefore the question of interpreting the multiplicities in the language of $O(n)$ representation theory makes perfect sense

(for $n\notin\mathbb{N}$ it still makes sense in a (Deligne) categorical sense see below) *(Binder, Rychkov 2019)*

The problem was solved in *(Jacobsen, Ribault, Saleur 2022)*

$$\Lambda_{(\frac{1}{2},0)} = [1]$$
$$\Lambda_{(1,0)} = [2]$$
$$\Lambda_{(1,1)} = [11]$$
$$\Lambda_{(\frac{3}{2},0)} = [3] + [111]$$
$$\Lambda_{(\frac{3}{2},\frac{2}{3})} = \Lambda_{(\frac{3}{2},\frac{4}{3})} = [21]$$
$$\Lambda_{(2,0)} = [4] + [22] + [211] + [2] + []$$
$$\Lambda_{(2,\frac{1}{2})} = \Lambda_{(2,\frac{3}{2})} = [31] + [211] + [11]$$
$$\Lambda_{(2,1)} = [31] + [22] + [1111] + [2]$$
$$\Lambda_{(\frac{5}{2},0)} = [5] + [32] + 2[311] + [221] + [11111] + [3] + 2[21] + [111] + [1]$$
$$\Lambda_{(\frac{5}{2},\frac{2}{5})} = [41] + [32] + [311] + [221] + [2111] + [3] + 2[21] + [111] + [1]$$

$$\vdots$$

(The exact decomposition for the $\Lambda_{(r,s)}$ is known explicitely)

So the degeneracies correspond to groupings of $O(n)$ irreducible representations into blocks

The formula:

$$\Lambda_{(r,s)} = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1}[] + \frac{1}{2r}\sum_{r'=0}^{2r-1} e^{\pi i r' s} U_{\gcd(2r,r')}\left(\lambda_{\frac{2r}{\gcd(2r,r')}}\right)$$

$$U_0(z) = 2 \quad , \quad U_1(z) = z \quad , \quad zU_d(z) = U_{d-1}(z) + U_{d+1}(z)$$

$$\lambda_r = \delta_{r\in 2\mathbb{N}}[] + \sum_{k=0}^{r-1}(-1)^k[r-k, 1^k]$$

$$\begin{aligned}
\lambda_1 &= [1] \\
\lambda_2 &= [2] - [11] + [] \\
\lambda_3 &= [3] - [21] + [111] \\
\lambda_4 &= [4] - [31] + [211] - [1111] + []
\end{aligned}$$

$$\mathrm{Tr}_{\lambda_r}(g) = \mathrm{Tr}_{[1]}(g^r)$$

where $\lambda^2 \equiv \lambda \otimes \lambda$ in $O(n)$...

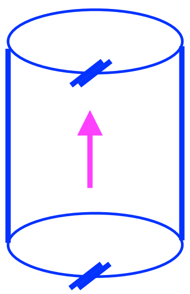
What explains these (very large) degeneracies?

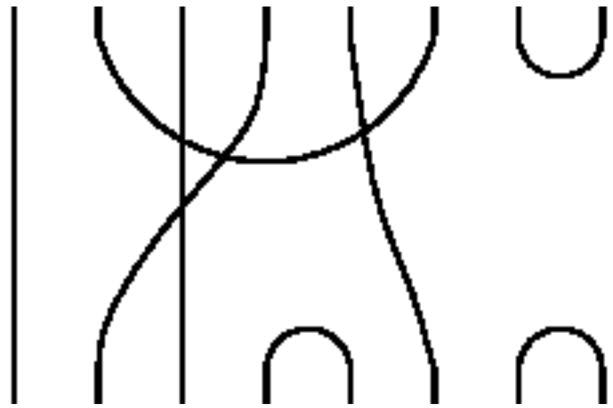
- The idea of an extended symmetry (Hopf algebra) doesn't seem to work., The set of $\Lambda_{(r,s)}$ is not stable under the $O(n)$ tensor product. Periodic boundary conditions seem to play a crucial role (degeneracies would be different for open b.c.).
- On the other hand, with p.b.c. it is difficult indeed to give a meaning to tensor products of the $\Lambda_{(r,s)}$ (cut and ew?)
- It is useful to start to understand why there are such degeneracies

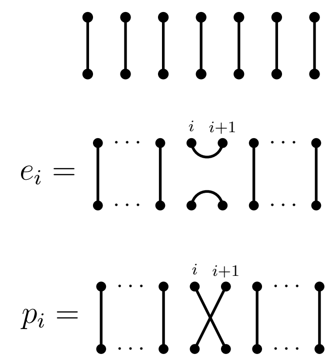
The culprit is the non-crossing constraint

which is better understood using a bit of algebra

- The natural relationship between loops and $O(n)$ is of Schur-Weyl duality between $O(n)$ acting on the tensor product of fundamental (vector) representations $[1]^{\otimes L}$ and the Brauer algebra.







(there are 3 diagrams/operators because $[1]^{\otimes 2} = [] \oplus [1] \oplus [2]$)

Strictly speaking, Brauer is relevant only for the case where every edge is occupied by a "monomer", which corresponds to the limit $K \to \infty$ and the low-temperature (dense) phase. Algebraically however, the following makes sense for the dilute versions of the problem: $([] \oplus [1])^{\otimes L}$ and "dilute" Brauer. I will often not specify this.

- Schur-Weyl for a general $O(n)$ interaction would lead to

$$[1]^{\otimes L} \underset{\mathscr{B}_L(n)\times O(n)}{=} \bigoplus_{\substack{|\lambda|\leq L \\ |\lambda|\equiv L \bmod 2}} B_\lambda^{(L)} \otimes \lambda$$

For example, $[1]^{\otimes 3} = [3] + 2[21] + [111] + 3[1]$ tells us that $\dim B_{[21]}^{(3)} = 2$.

Brauer modules

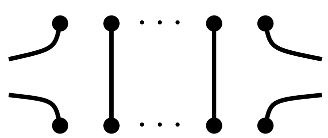
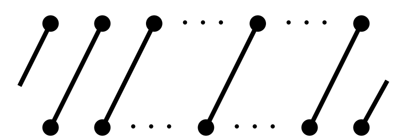
sum over partitions

So what happens to Schur-Weyl when we forbid crossings?

- Well the algebra becomes smaller - technically it is now the unoriented Jones-Temperley-Lieb algebra $u\mathscr{JTL}_L(n)$

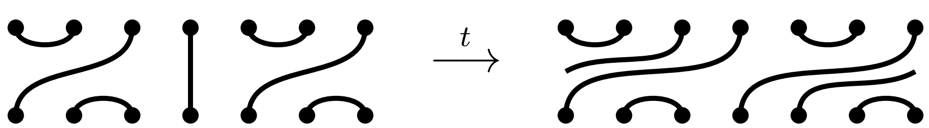
This is ordinary Temperley-Lieb plus an element contracting around (we want periodic boundary conditions to understand the bulk critical theory) and a translation

(V. Jones)

$e_L =$  $u =$ 

All non-contractible loops have weight n. Through lines winding fully around do not acquire any weight

Translation of through lines (pseudo-translation) t



$$t^{2r} \underset{u\mathscr{JTL}_L(n)}{=} 1$$

If $2r$ is the number of such lines

Irreducible modules $W^{(L)}_{(r,s)}$ with $2r \in \mathbb{N}$ and

$$\left(t - e^{\pi i s}\right) W^{(L)}_{(r,s)} = 0$$

(Martin Saleur…Graham Lehrer)

A smaller algebra means a bigger centralizer

- So what we would like are the branching rules $\mathscr{B}_L(n) \downarrow u\mathscr{JTL}_L(n)$

$$B_\lambda^{(L)} \underset{u\mathscr{JTL}_L(n)}{=} \bigoplus_{r=\frac{|\lambda|}{2}}^{\frac{L}{2}} \bigoplus_{\substack{s \in \frac{1}{r}\mathbb{Z} \\ -1 < s \leq 1}} c^\lambda_{(r,s)} W^{(L)}_{(r,s)}$$

This problem is well defined even if $n \in \mathbb{C}$

We have a combinatorial formula for these
Note they don't depend on n nor L

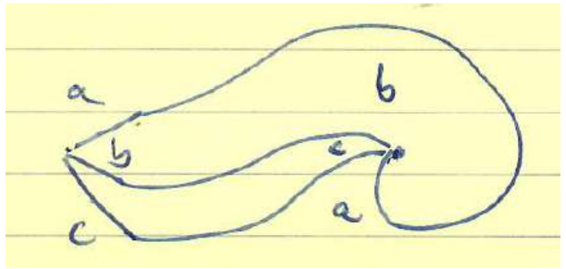
(Jacobsen Ribault Saleur)

The problem was solved earlier in the case or ordinary Temperley-Lieb $\mathscr{TL}_L(n)$ *(Benkart Moon)*

For $\lambda = [L]$ this amounts to decomposing Specht modules of the permutation group $S_{|\lambda|}$ into representations of a cyclic subgroup (Stembridge)

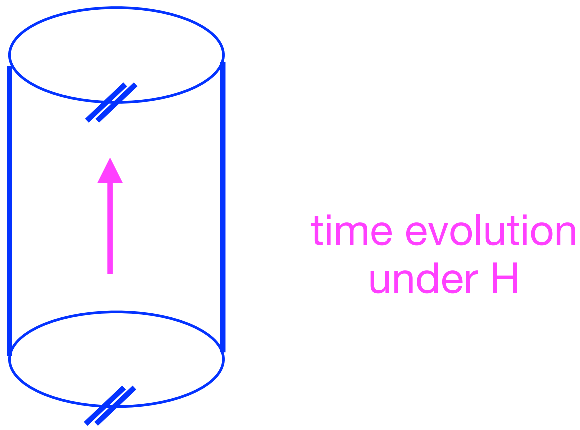
- And from these branching rules the $\Lambda_{(r,s)}$ follow

all this means is that



In 2D not all tensors can be realized without crossings (of course)
e.g. [111] and [3] can't be distinguished

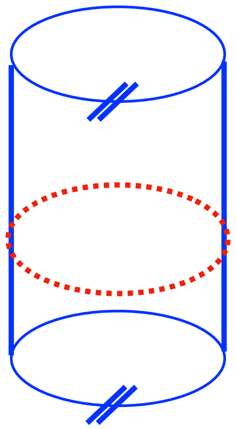
So what's the "symmetry"?



- Apart from the generators of $O(n)$ there are other operators commuting with the Hamiltonian.

(Jacobsen Saleur)

- They are in fact topological defect lines (TDLs) operators, and commute with the full left and right Virasoro algebras

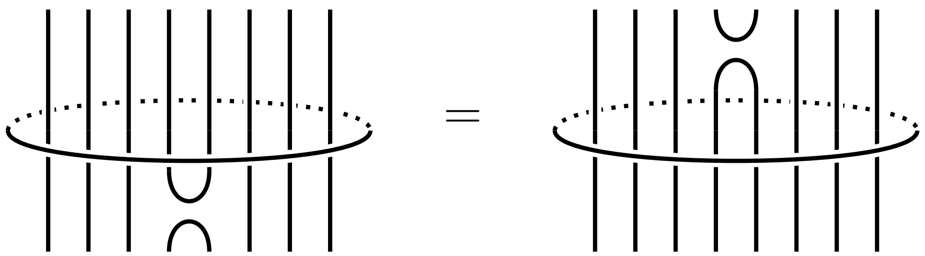


$$[\mathrm{Vir}, \mathsf{D}] = [\overline{\mathrm{Vir}}, \mathsf{D}] = 0$$

- This happens because they commute with the full lattice algebra

$$[u\mathcal{JTL}_L(n), \mathsf{D}] = [u\mathcal{JTL}_L(n), \overline{\mathsf{D}}] = 0$$

$$
\begin{aligned}
\mathsf{D} &= (-q)^{-L/2} u (1 - q e_{L-1}) \cdots (1 - q e_2)(1 - q e_1) + (-q)^{N/2} (1 - q^{-1} e_1) \cdots (1 - q^{-1} e_{L-1}) u^{-1} \\
\overline{\mathsf{D}} &= (-q)^{L/2} \tau (1 - q^{-1} e_{L-1}) \cdots (1 - q^{-1} e_2)(1 - q^{-1} e_1) + (-q)^{-L/2} (1 - q e_1) \cdots (1 - q e_{L-1}) u^{-1}
\end{aligned}
$$



- In theories with a global symmetry group G there exists invertible topological defects associated with an element g. They act as $\mathsf{D}_g|\Phi\rangle = g|\phi\rangle$, if $|\phi\rangle$ is a state in the Hilbert space, and of course $\mathsf{D}_g \mathsf{D}_{g^{-1}} = 1$.

In the microscopic models, degrees of freedom are acted on by g when the defect line is crossed (e.g. $G = Z_2$ and spin flips)

Here the defect is TDL is not invertible
(technically it is a Verlinde line associated with operator $(h_{21}, 0)$)

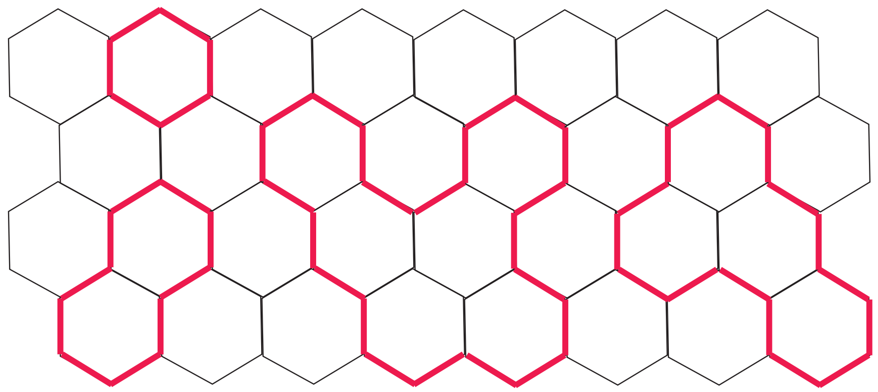
- The $\Lambda_{(r,s)}$ are eigenspaces of the TDL operator

$$Z_{O(n)} = \sum_{s\in 2\mathbb{N}+1} \chi^D_{\langle 1,s\rangle} + \sum_{r\in\frac{1}{2}\mathbb{N}^*} \sum_{s\in\frac{1}{r}\mathbb{Z}} \underbrace{\left(E_{r,s} + \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1}\right)}_{\Lambda_{r,s}} \chi^N_{(r,s)}$$

What is it good for?

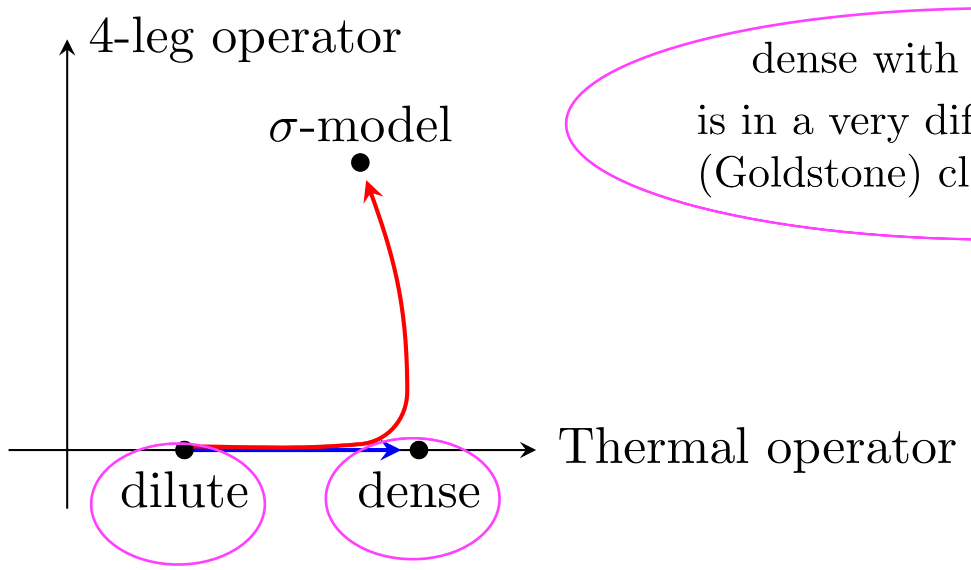
- There's in fact the dilute and the dense critical points

$$Z = \sum_{\text{dilute loop gas}} K_c^B n^L$$



- The dilute universality class is very robust: crossings don't matter

- But things are different for the dense (critical) phase $(K > K_c)$



dense with crossings
is in a very different universality
(Goldstone) class

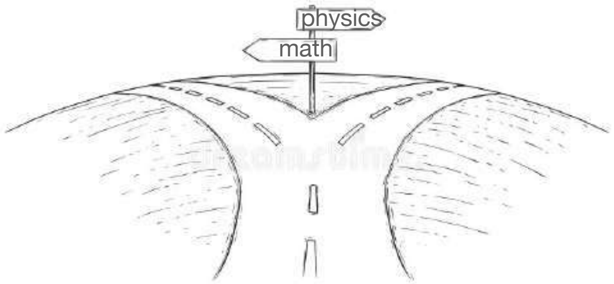
- Any amount of coupling to the 4-leg operator drives the system away from the dense fixed point (operator is dangerously irrelevant) and such coupling is not forbidden by $O(n)$ symmetry... so nothing prevents it a priori from appearing as a counter term (from being generated in the RG)

But it is prevented by topological symmetry (all terms/ counter-terms generated by the RG are in the topological sector of the identity, not of the 4-leg operator)

- In other words, $[\mathsf{D}, \times] \neq 0$: crossings break the topological symmetry and thus can't appear if the latter is conserved

At the crossroads of physics and mathematics

At the crossroads of physics and mathematics



"when you come to a fork in the road, take it!" (Yogi Berra)

Happy Birthday Philippe!