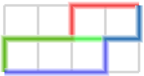
The magic number conjecture for the $m = 2$ amplituhedron and Parke-Taylor identities

Lauren K. Williams, Harvard

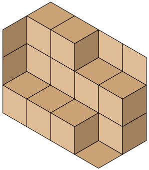
noncrossing lattice paths



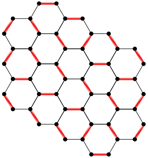
plane partition

3	3	2	2
1	1	1	

rhombic tiling



perfect matching



Based on: arXiv:2404.03026,

joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler

Outline

- Partial cyclic orders and tricolored subdivisions
- Applications to Parke-Taylor identities and Parke-Taylor polytopes
- What is the amplituhedron?
- Magic number conjecture for the amplituhedron
- Proof of Magic number conjecture when $m = 2$

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Partial and total cyclic orders

A *(partial) cyclic order* on a finite set X is a ternary relation $C \subset X^3$ such that for all $a, b, c, d \in X$:

$$\begin{aligned} (a,b,c) \in C &\implies (c,a,b) \in C && \textit{cyclicity} \\ (a,b,c) \in C &\implies (c,b,a) \notin C && \textit{asymmetry} \\ (a,b,c) \in C \text{ and } (a,c,d) \in C &\implies (a,b,d) \in C && \textit{transitivity} \end{aligned}$$

Ex: The triples $\{(2,5,7),(5,7,6),(1,8,7),(8,7,2)\}$ determine a partial cyclic order on $[8]$.

A cyclic order C is *total* if for all $a, b, c \in X$, either $(a,b,c) \in C$ or $(a,c,b) \in C$.

Informally, a total cyclic order C on $[n]$ is a way of placing $1, \dots, n$ on a circle, just as a total order is a way of placing $1, \dots, n$ on a line.

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Tricolored subdivisions and cyclic orders

- A *tricolored subdivision* τ of an n-gon is a subdivision of the polygon into smaller polygons (black, grey, or white) in which every edge connects two vertices of the n-gon.
- From each τ, can read off a *cyclic order* C_τ. To get C_τ from τ, read vertices of white (resp black) polygons clockwise (resp counterclockwise), and ignore the grey.
- The C_τ from our example requires that $(2, 5, 7)$, $(5, 7, 6)$, and $(1, 8, 7, 2)$ are circularly ordered.
- A *circular extension* of C_τ is a total circular order compatible with C_τ. E.g. one circular extension of our example is: $(2\,5\,1\,8\,7\,6\,3\,4)$.

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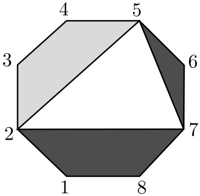

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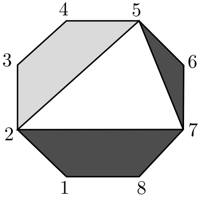

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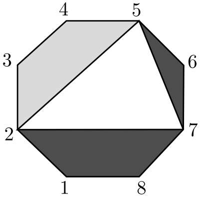

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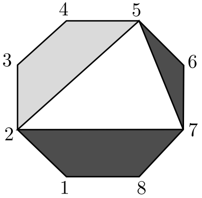

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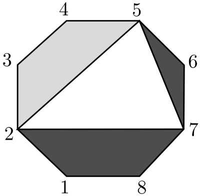

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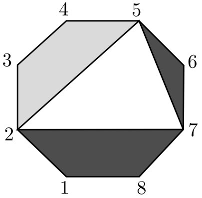

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The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix C.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(C)$ is the minor of the $k \times k$ submatrix of C in column set I.

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Grassmannian identities from tricolored subdivisions

- Given a permutation $w = w_1 \dots w_n$, define the *Parke-Taylor function*

$$\mathrm{PT}(w) := \frac{1}{P_{w_1 w_2} P_{w_2 w_3} \dots P_{w_n w_1}},$$

where the P_{ij} are Plücker coordinates on the Grassmannian $\mathrm{Gr}^{\circ}_{2,n}$.

We get the following identity.

Theorem (Parisi–ShermanBennett–Tessler–W)

Let τ be a tricolored subdivision with at least one grey polygon, and let C_τ be the cyclic partial order. Then

$$\sum_{w} \mathrm{PT}(w) = 0,$$

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Theorem (P–SB–T–W)

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Example:
The circular extensions of C_τ are $(1234), (1243), (1423)$,
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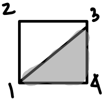

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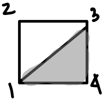
Example: 

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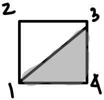

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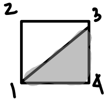

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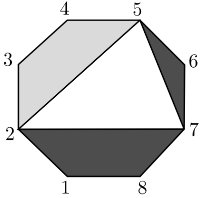
Tricolored subdivisions and Parke-Taylor polytopes



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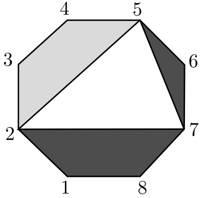

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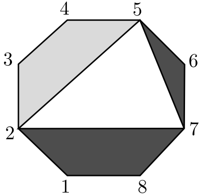

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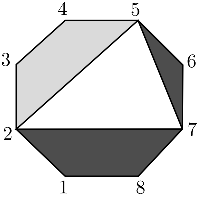

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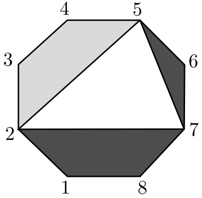

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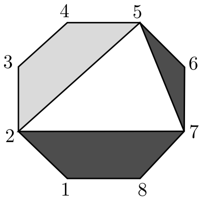

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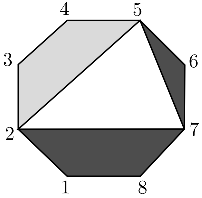

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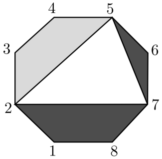

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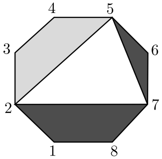

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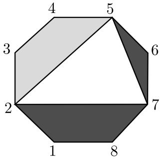

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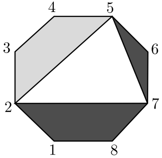

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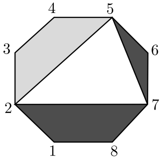

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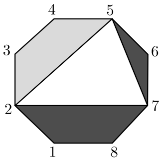

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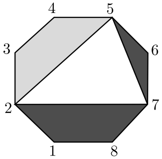

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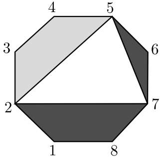

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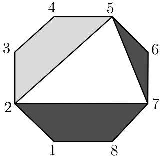

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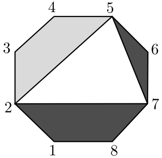

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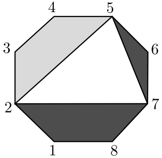

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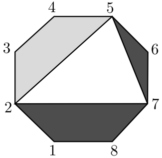

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- the recurrence of Britto–Cachazo–Feng–Witten (2005) expresses scattering amplitudes as sums of rat'l functions of momenta. Indiv terms have "spurious poles" – singularities not present in amplitude.
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- the recurrence of Britto–Cachazo–Feng–Witten (2005) expresses scattering amplitudes as sums of rat'l functions of momenta. Indiv terms have "spurious poles" – singularities not present in amplitude.
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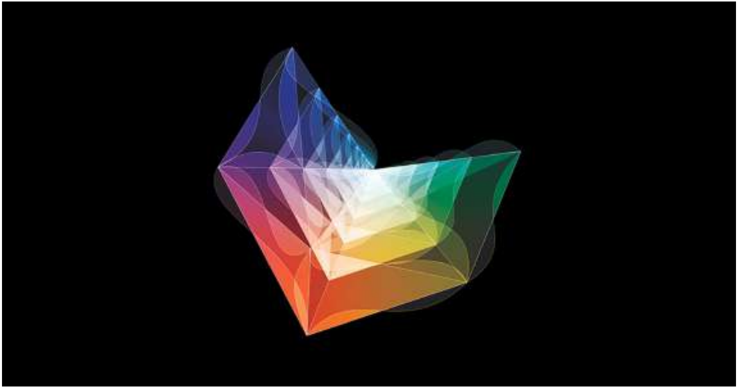
What is the amplituhedron?

- A "jewel at the heart of quantum physics" – Wired Magazine.



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- #10 among the 100 top stories of 2013, Discover Magazine.

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Special cases:

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We'd like to "triangulate" or "tile" the amplituhedron

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map onto km-dim'l amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

A *tiling* of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of km-dimensional cells, such that:

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We will work with all-Z tilings, coming from collections of cells that give tilings for all Z.

Motivation:

the "volume" of the amplituhedron computes scattering amplitudes;
AH-T conjectured that certain "BCFW cells" give a tiling of $\mathcal{A}_{n,k,4}(Z)$;
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AH-T conjectured that certain "BCFW cells" give a tiling of $\mathcal{A}_{n,k,4}(Z)$;
(proved for the "standard" BCFW tiling by EvenZohar–Lakrec–Tessler and generalized to all BCFW tilings by EZ–L–P–SB–T–W.)

We'd like to "triangulate" or "tile" the amplituhedron

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map onto km-dim'l amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

A *tiling* of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of km-dimensional cells, such that:

- $\tilde{Z}$ is injective on each S_π for $\pi \in \mathcal{C}$ ($\overline{\tilde{Z}(S_\pi)}$ a *tile*)
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Tilings of the amplituhedron

Tilings have been studied in special cases. Their cardinalities are interesting!

special case	cardinality of tiling of $\mathcal{A}_{n,k,m}$	explanation
$m = 0$ or $k = 0$	1	$\mathcal{A}$ is a point
$k + m = n$	1	$\mathcal{A} \cong \mathrm{Gr}_{k,n}^{\geq 0}$
$m = 1$	$\binom{n-1}{k}$	Karp-W.
$m = 2$	$\binom{n-2}{k}$	AH-T-T, Bao-He, P-SB-W
$m = 4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	AH-T, EZ–L–T, EZ–L–P–SB–T–W
$k = 1$, m even	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n, m)$

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$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

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noncrossing lattice paths

plane partition

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1	1	1	

rhombic tiling

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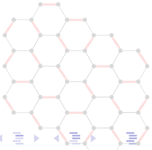

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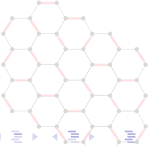

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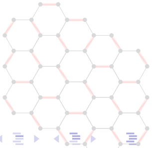

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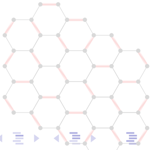

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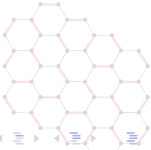

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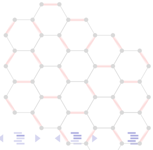

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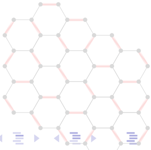

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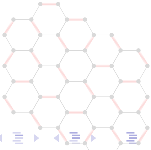

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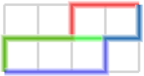

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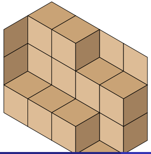

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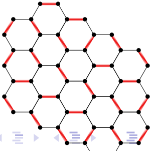

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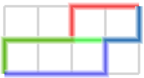

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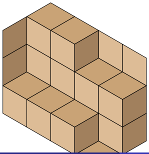

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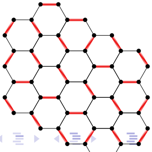

The magic number theorem for the $m = 2$ amplituhedron

Magic Number Theorem (P–SB–T–W)

All tilings of ampl. $\mathcal{A}_{n,k,2}(Z)$ have size $M(k, n-k-2, 1) = \binom{n-2}{k}$.

$k = 1$: Thm says that all triangulations of an n-gon have size $n-2$.
Ideas of the proof:

- There is a classification of tiles for the $m = 2$ amplituhedron using *bicolored subdivisions* (P–SB–W).
- Just as each Parke-Taylor polytope has a decomposition into w-simplices where w ranges over certain circular extensions, each tile has a decomposition into "w-chambers" where w ranges over certain circular extensions.
- Use above decompositions to define the P-T function of $\mathcal{A}_{n,k,2}(Z)$ and each tile, and show that this function is the same for ALL tiles.
- Therefore each tiling of $\mathcal{A}_{n,k,2}(Z)$ has the same size.
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The magic number theorem for the $m = 2$ amplituhedron

Magic Number Theorem (P–SB–T–W)

All tilings of ampl. $\mathcal{A}_{n,k,2}(Z)$ have size $M(k, n-k-2, 1) = \binom{n-2}{k}$.

$k = 1$: Thm says that all triangulations of an n-gon have size $n - 2$.

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Recall: $\overline{\tilde{Z}(S_\pi)}$ is a *tile* for $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on km-dim'l cell S_π. Lukowski–Parisi–Spradlin–Volovich conjectured:

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The tiles for $\mathcal{A}_{n,k,2}(Z) \leftrightarrow$ collections of **bicolored** subdivisions of an n-gon with total "area" k. To construct the cell S_π:

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\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
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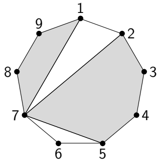

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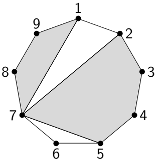

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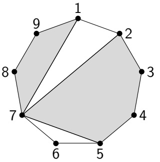

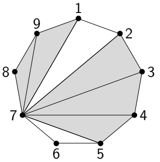

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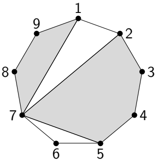

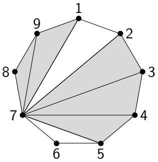

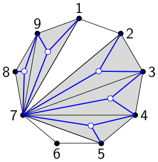

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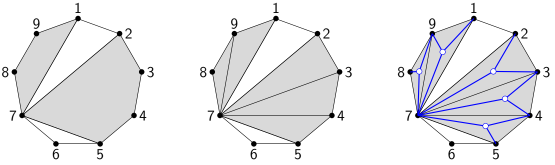

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Let $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+2}$ sending $C \mapsto CZ$.
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- Inspired by matroid stratification, we define the *amplituhedron sign stratification* – decompose $\mathcal{A}_{n,k,2}(Z)$ into pieces based on the signs of twistor coordinates. (Parisi–Sherman-Bennett–W.; Karp-W.)
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- Given any region R of $\mathcal{A}_{n,k,2}(Z)$ that admits a tiling, we define its *weight function*

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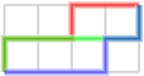
Thank you, and HAPPY BIRTHDAY Philippe!

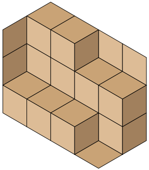
noncrossing lattice paths



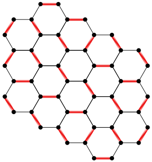
plane partition

3	3	2	2
1	1	1	

rhombic tiling



perfect matching



- The magic number conjecture for the $m = 2$ amplituhedron and Parke-Taylor identities `arXiv:2404.03026`, joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler.

- "The $m = 2$ amplituhedron and the hypersimplex: signs, clusters, triangulations, Eulerian numbers, Communications of the AMS, 2023, joint with Matteo Parisi and Melissa Sherman-Bennett.