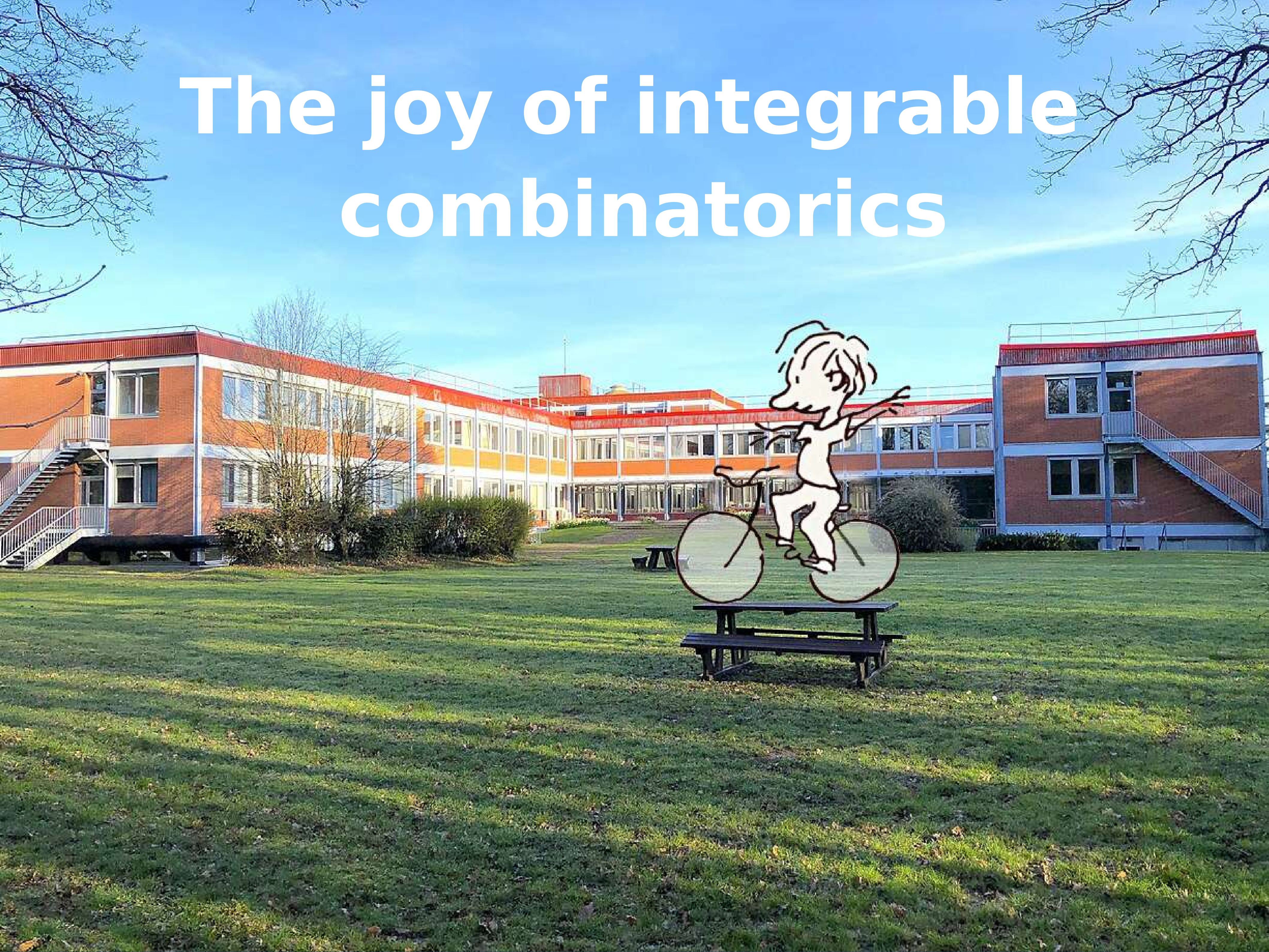
The joy of integrable combinatorics


The joy of integrable
combinatorics

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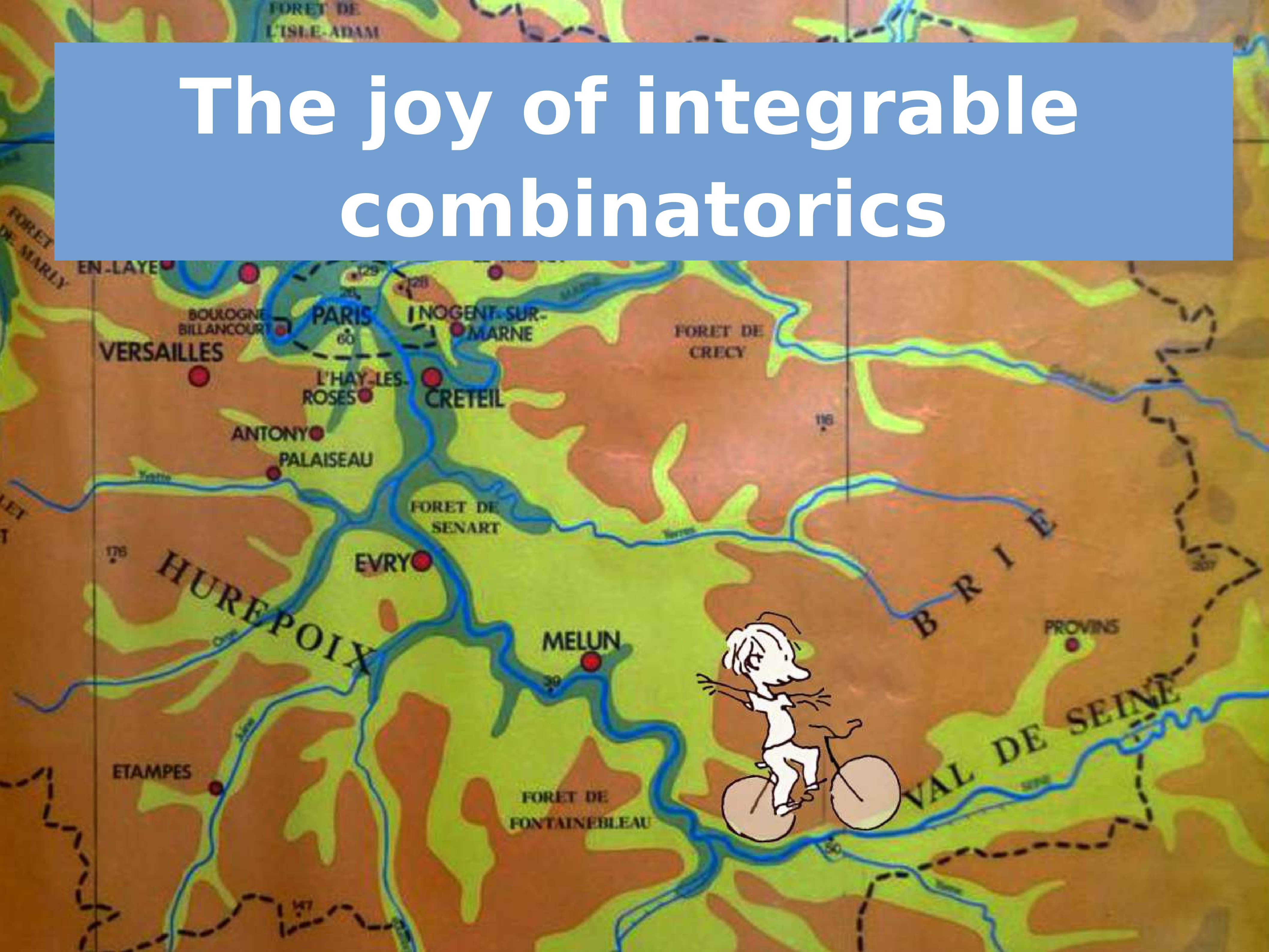
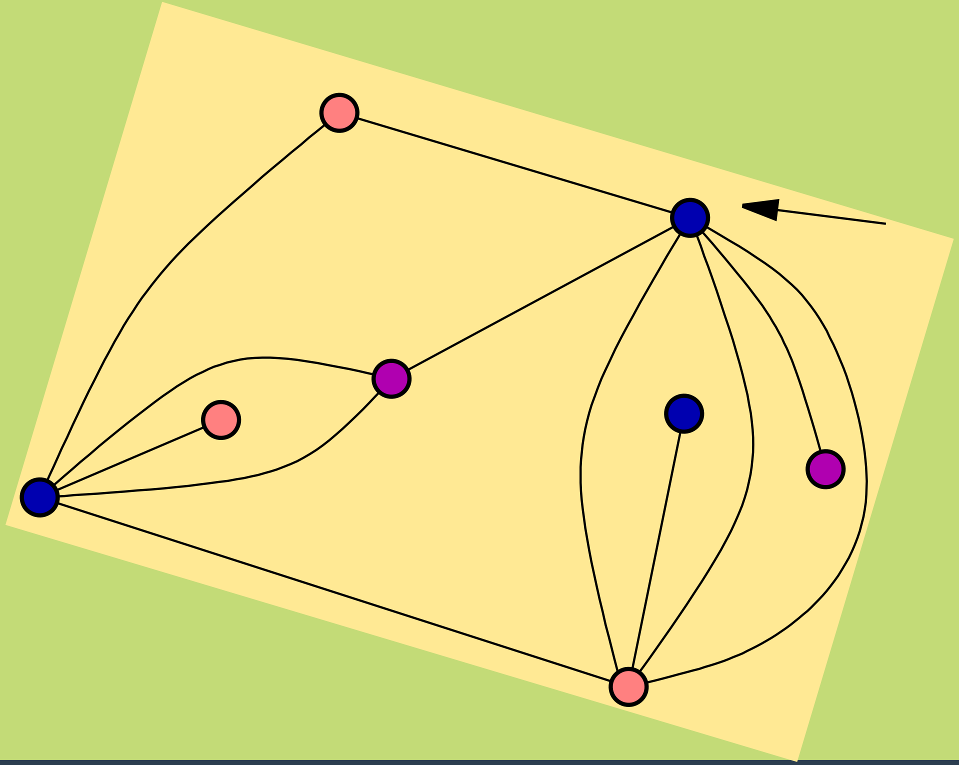


Combinatorics of 3-coloured quadrangulations



Mireille Bousquet-Mélou
CNRS, Université de Bordeaux, France

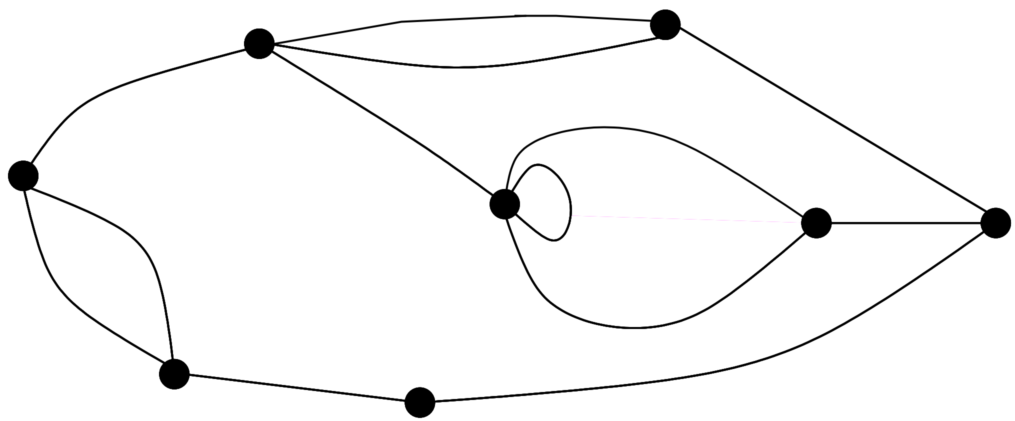
Andrew Elvey Price
CNRS, Université de Tours, France

Planar maps

Def. A connected planar (multi)graph, given with an embedding in the plane, taken up to continuous deformation.

Components:

- vertices
- edges
- faces



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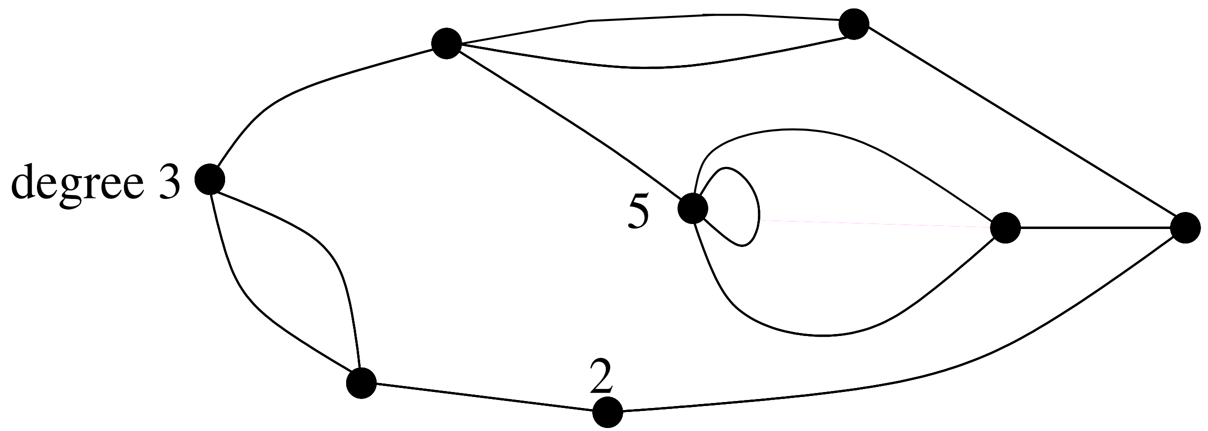

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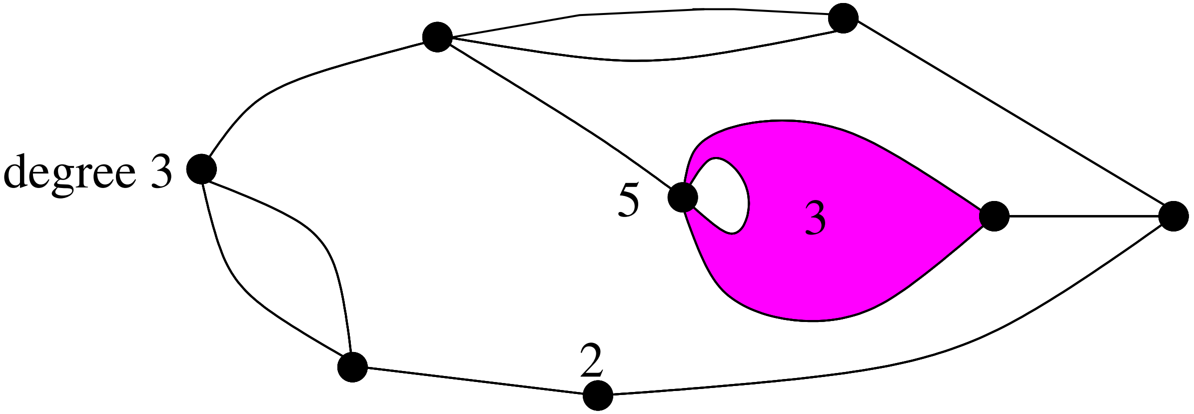

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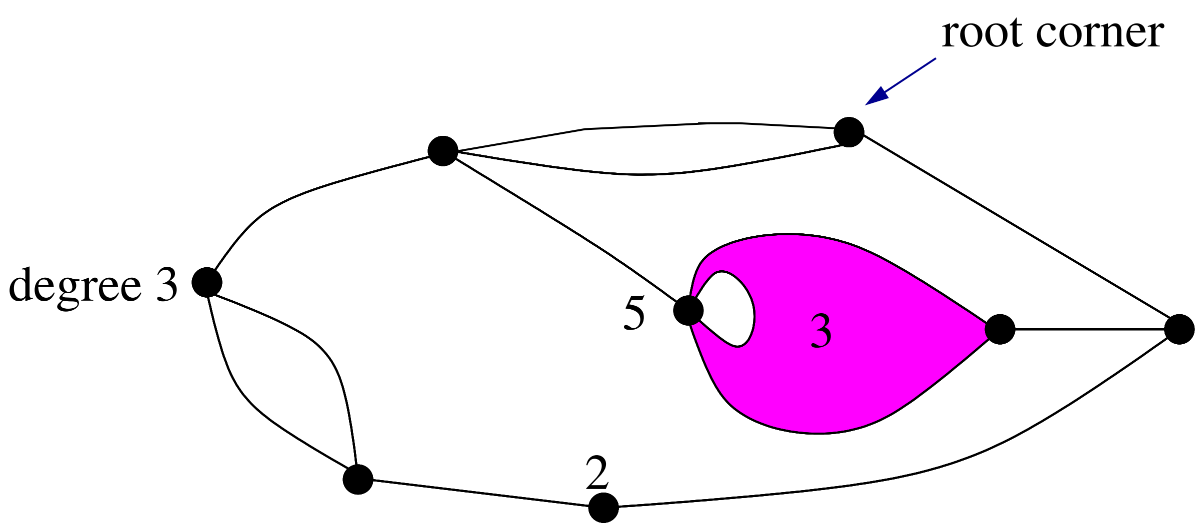

Rooted map: a distinguished corner in the outer face

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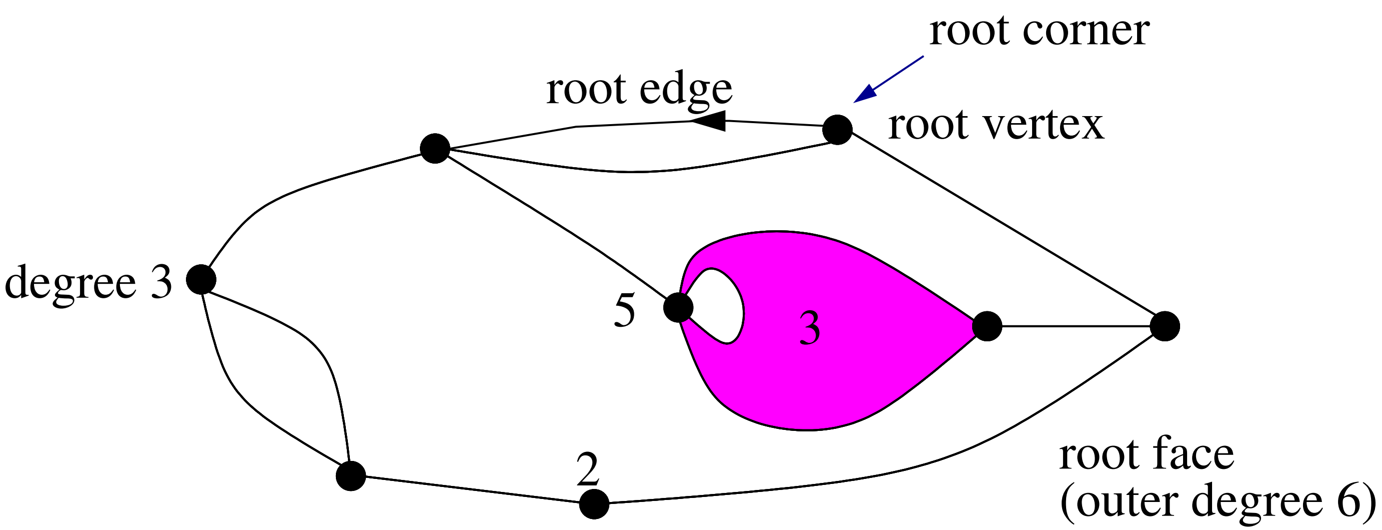

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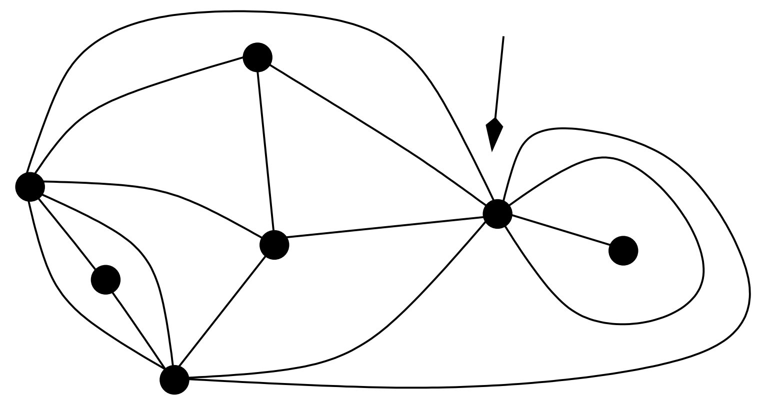

Rooted map: a distinguished corner in the outer face

Triangulation: all faces have degree 3

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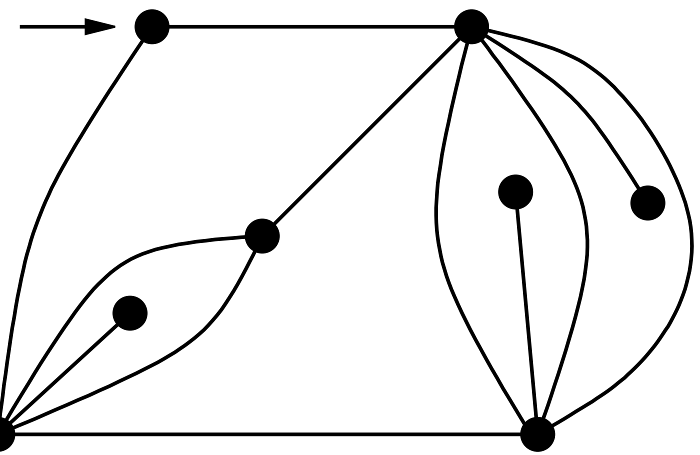


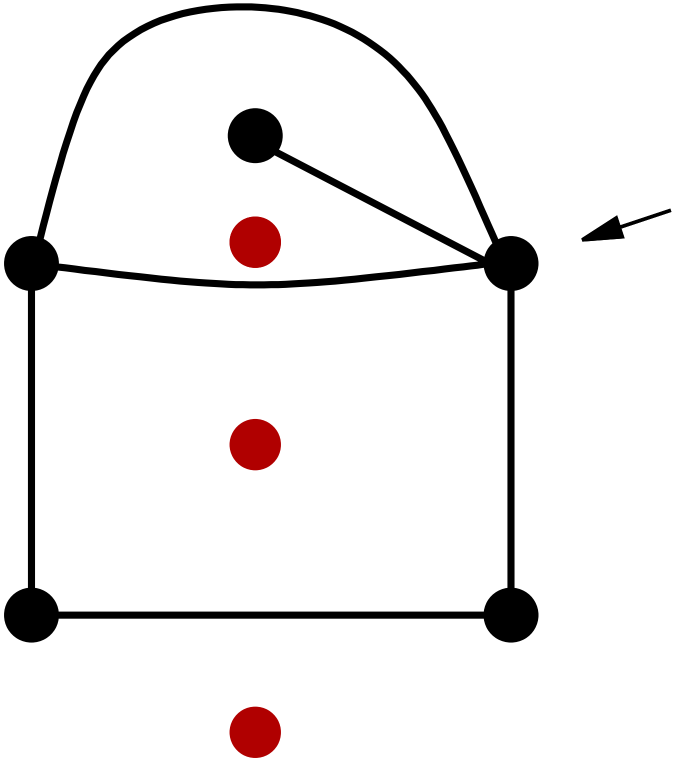
Rooted map: a distinguished corner in the outer face

Triangulation: all faces have degree 3

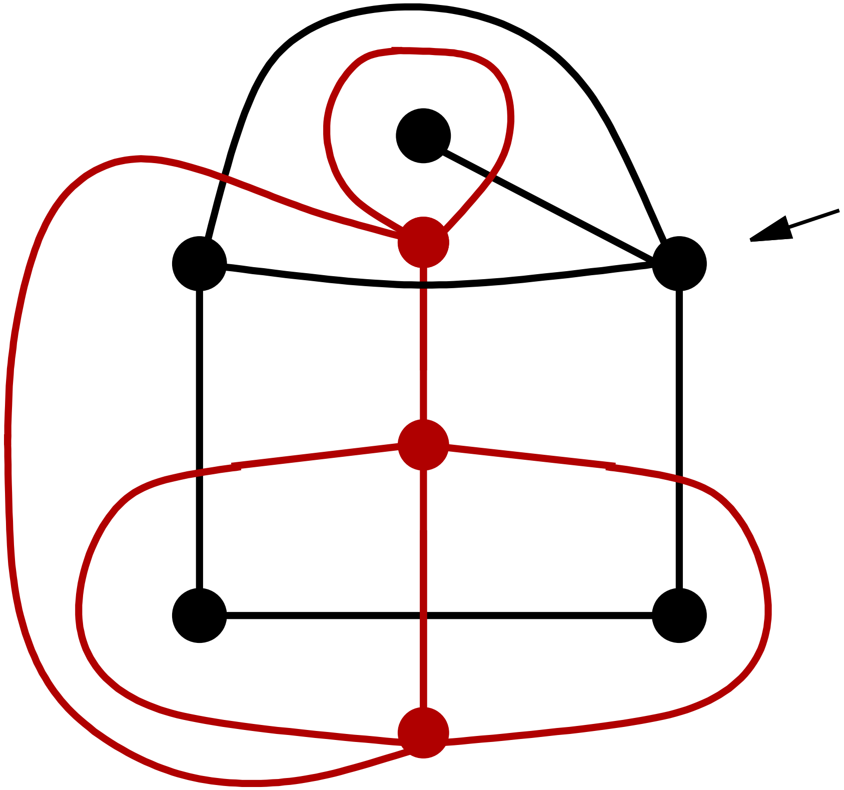
Quadrangulation: all faces have degree 4

Duality

Exchange faces and vertices



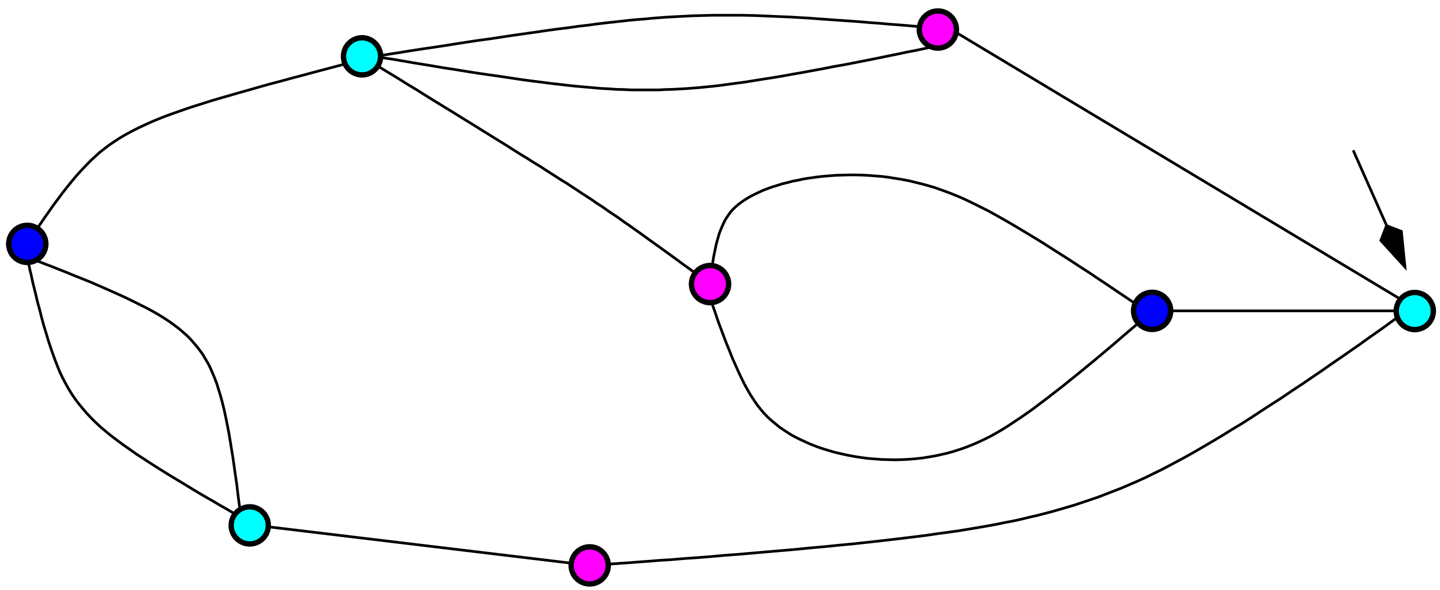
Quadrangulation



Quartic (or: 4-valent) map

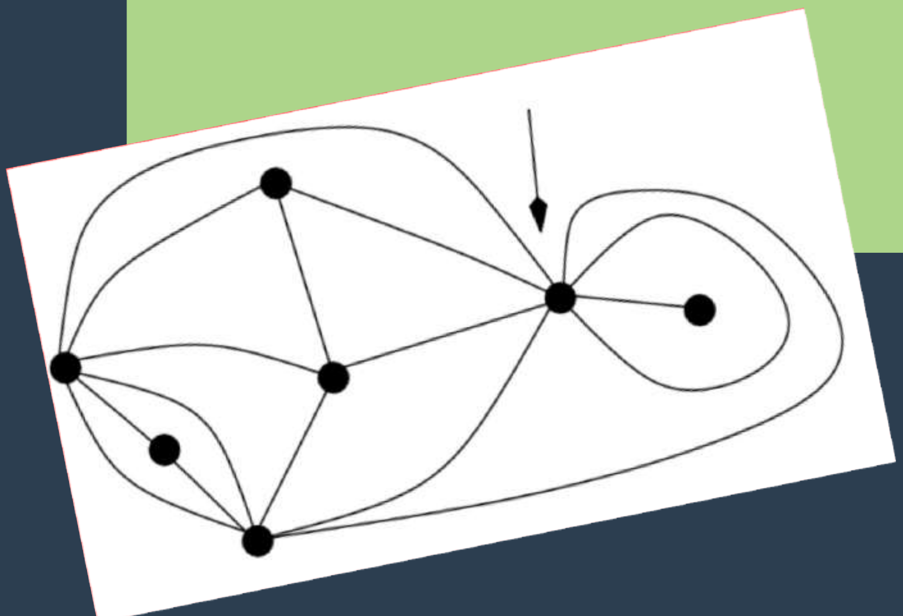
Proper colourings of maps

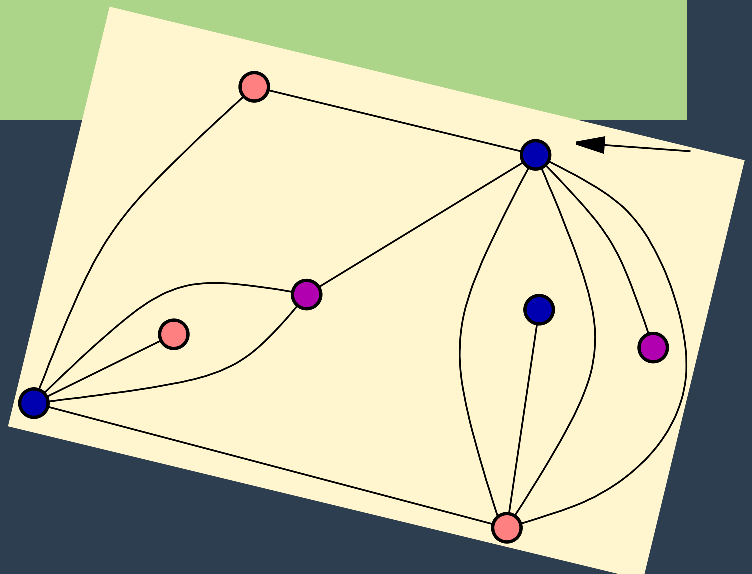
Def. Vertices are coloured in q colours, and two neighbour vertices get different colours.



q=3

I. Map enumeration





Enumeration of maps: a typical result

Let m(n) be the number of (planar) maps with n edges. Then:

$$m(n) = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \sim \kappa\, 12^n n^{-5/2}$$

[Tutte 63]
[BDG 02]

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The associated **generating function**,

$$M = \sum_{n \geq 0} m(n) t^n = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}$$

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$$16t - 1 + (1 - 18t)\, M + 27t^2 M^2 = 0$$

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$$16t - 1 + (1-18t)\, M + 27t^2 M^2 = 0$$

with a **rational parametrisation**: if

$$t = \frac{A}{3(1+A)^2}, \qquad \text{i.e.} \qquad A = 3t(1+A)^2$$

then

$$M = A - tA^3.$$

[Tutte 63]
[BDG 02]

Enumeration of maps: a typical result

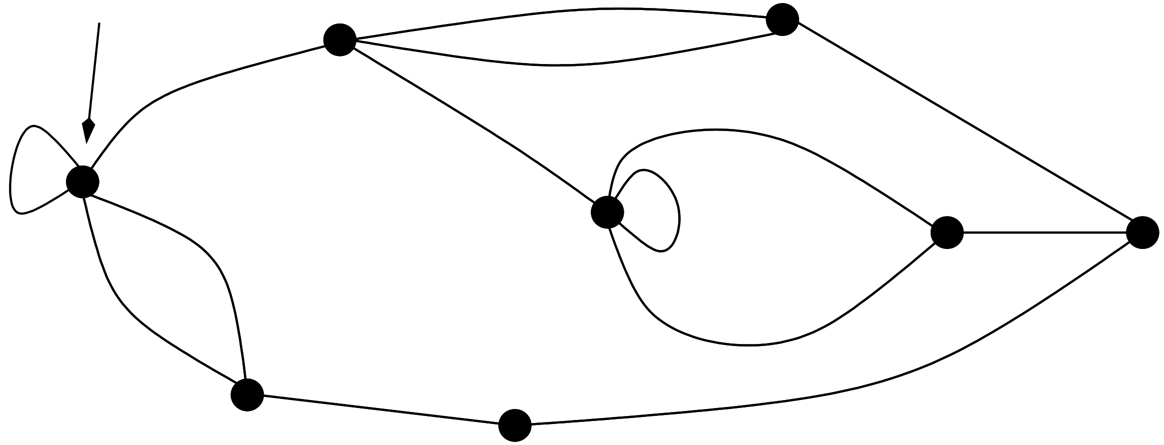
- The generating function M of maps (counted by edges) is **algebraic of degree 2**.

It has a **rational parametrisation**:

$$t = \frac{A}{3(1+A)^2}, \qquad M = A - tA^3.$$

- **Asymptotics:**

$$m(n) \sim \kappa\, 12^n\, n^{-5/2}.$$



[Tutte 63]
[BDG 02]

Triangulations

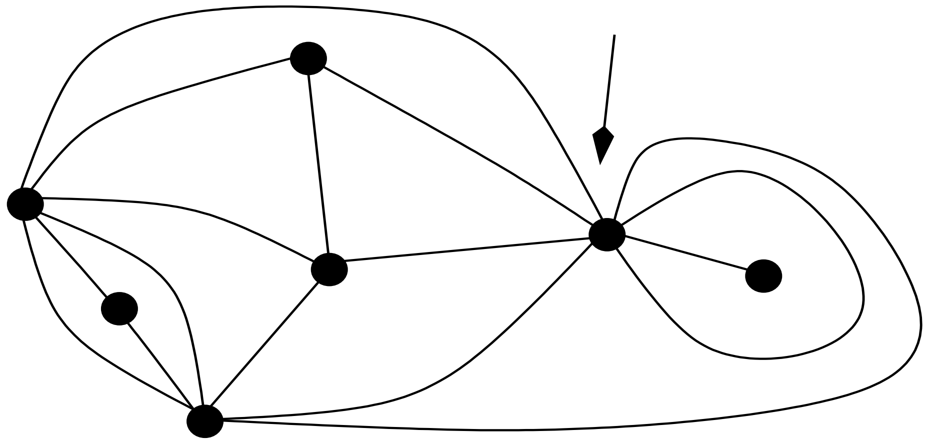
- The generating function T of triangulations (counted by vertices) is **algebraic of degree 3**.

It has a **rational parametrisation**:

$$t = \frac{A(1+A)}{2(1+2A)^3}, \qquad T = \frac{A(1-A)}{2(1+2A)}.$$

- **Asymptotics:**

$$t(n) \sim \kappa \left(12\sqrt{3}\right)^n n^{-5/2}.$$



[Mullin, Nemeth & Schellenberg 70]

Two-coloured maps

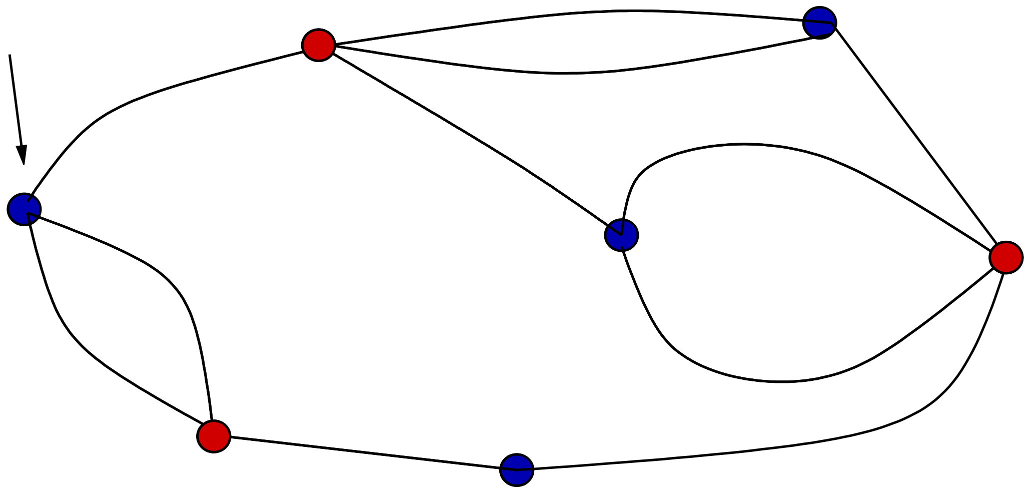
- The generating function M_2 of bicoloured maps (counted by edges) is **algebraic of degree 2**.

It has a **rational parametrisation**:

$$t = A(1-2A), \qquad t^2 M_2 = A^2(1-3A+A^2).$$

- **Asymptotics:**

$$m_2(n) \sim \kappa\, 8^n\, n^{-5/2}.$$



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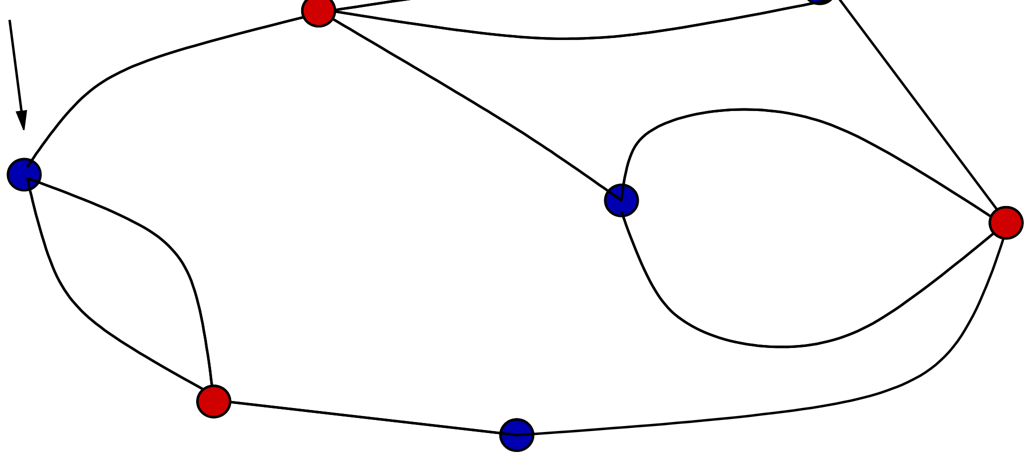

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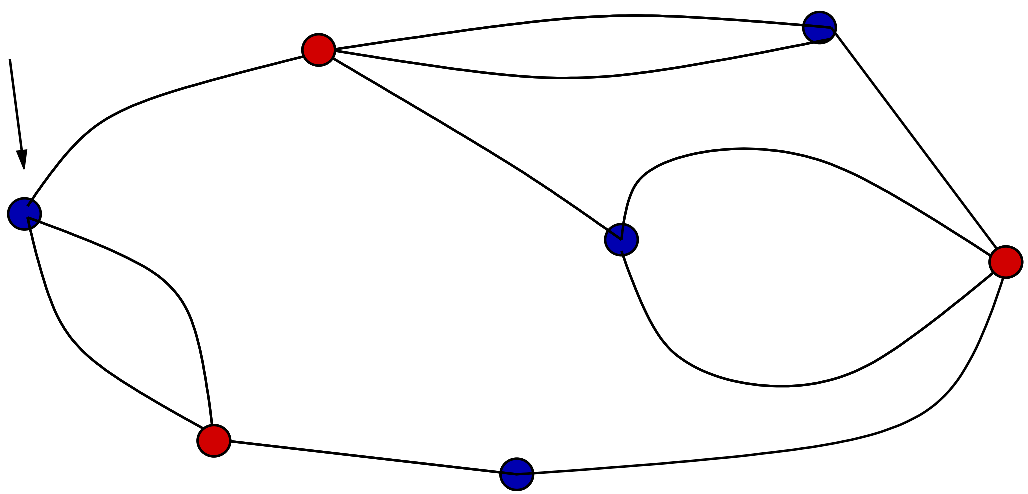

[Tutte 63]

Three-coloured triangulations

- The generating function T_3 of 3-coloured triangulations (counted by vertices) is **algebraic of degree 2**.

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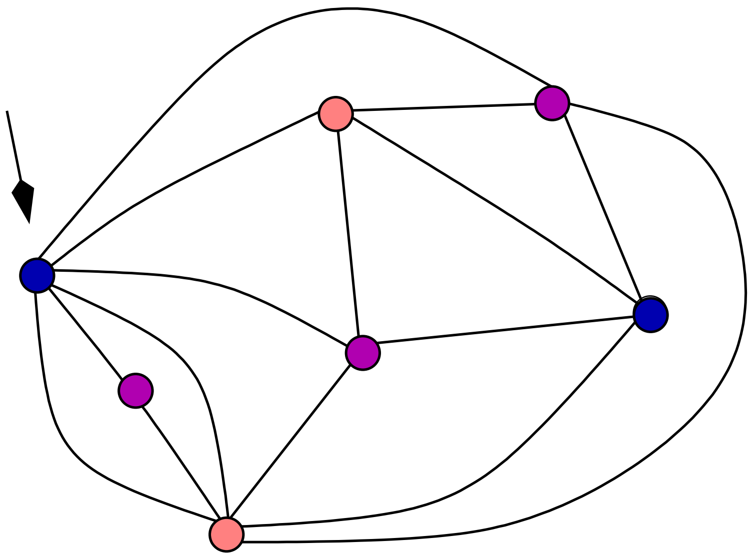

[Tutte 63]

[DF, Eynard, Guitter 98, BDG 02]

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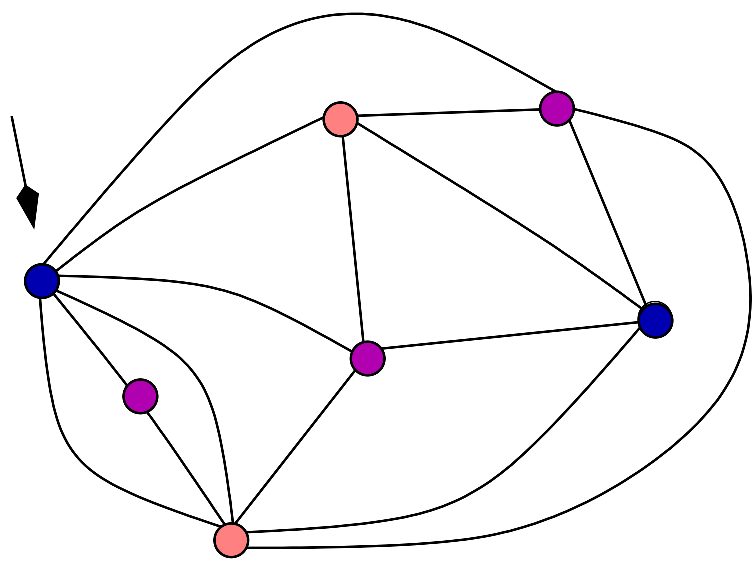

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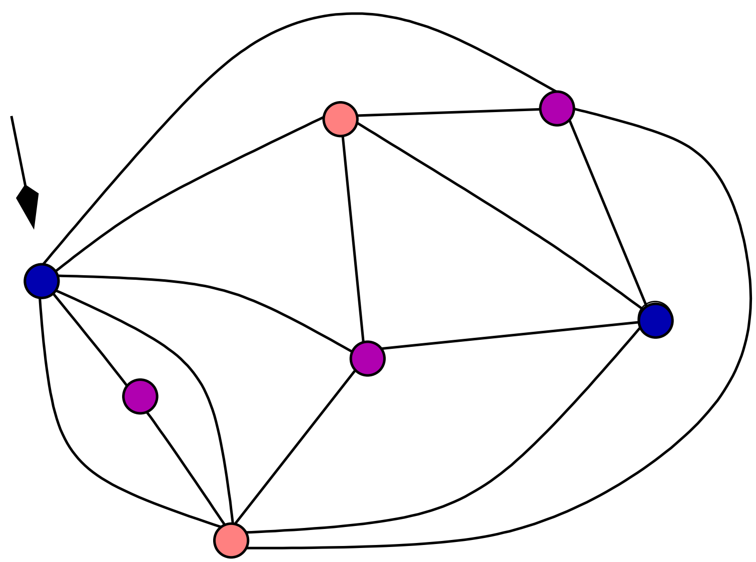

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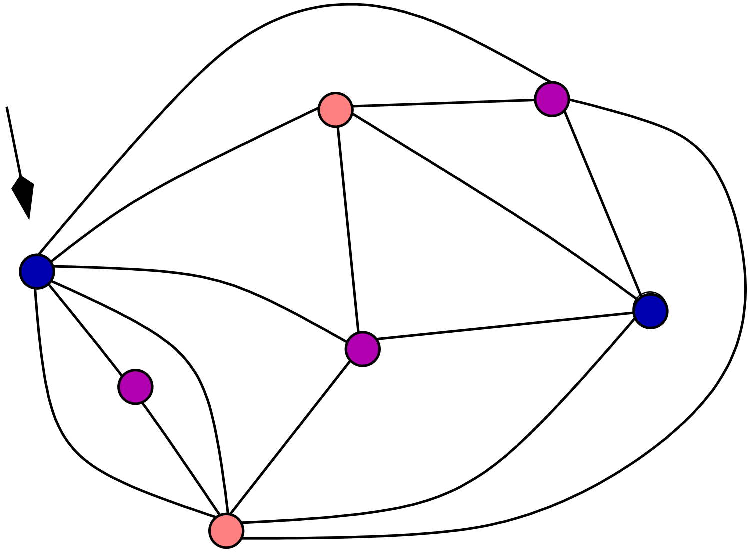


All vertices have even degree

~ bicoloured maps

[Tutte 63]

[DF, Eynard, Guitter 98, BDG 02]

Three-coloured maps

- The generating function M_3 of 3-coloured maps (counted by edges) is algebraic of degree 4.

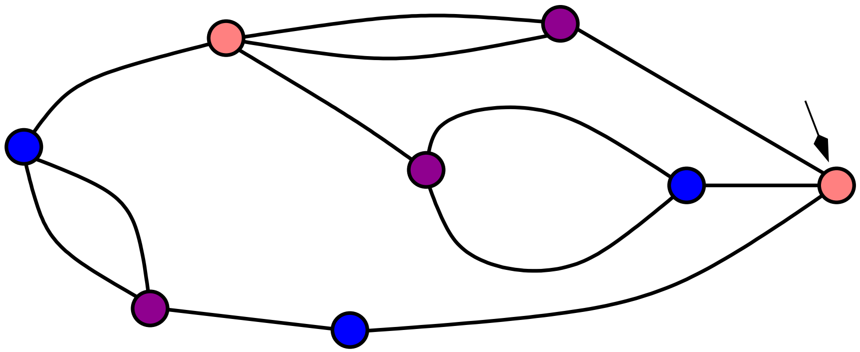
It has a rational parametrisation:

$$t = A\frac{(1-2A^3)}{(1+2A)^3}, \qquad M_3 = \frac{(1+2A)\left(1-2A^2-4A^3-4A^4\right)}{(1-2A^3)^2}.$$

- Asymptotics:

$$m_3(n) \sim \kappa \left(\frac{22+8\sqrt{6}}{3}\right)^n n^{-5/2}.$$

A true colouring problem



[Bernardi-mbm 11]

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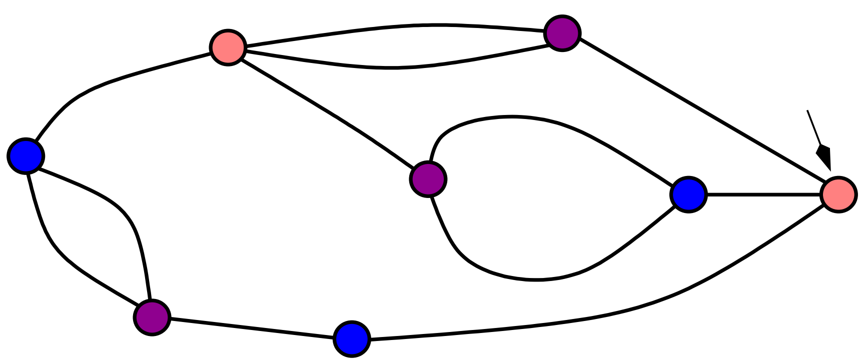
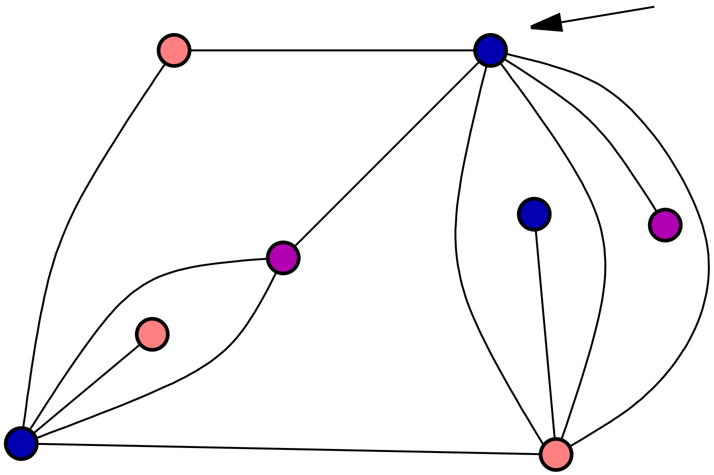


A mysterious result
Bijection with some trees?

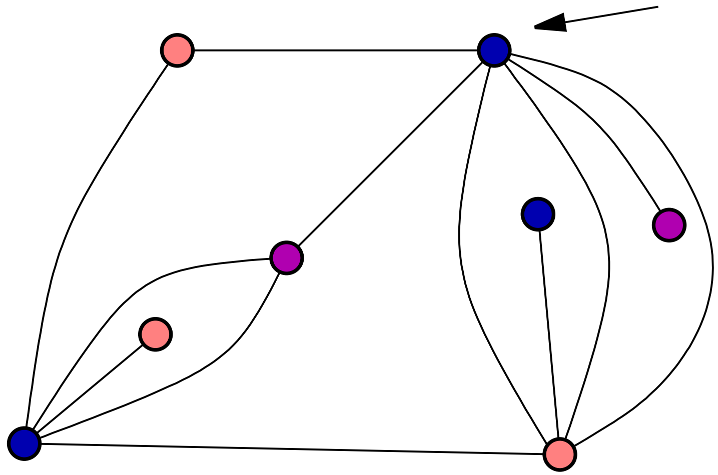
[Bernardi-mbm 11]

Three-coloured quadrangulations



Three-coloured quadrangulations

- The generating function Q_3 of 3-coloured quadrangulations (counted by faces) is NOT ALGEBRAIC.



Explicit 2^{nd} order DE (degree 3)

Three-coloured quadrangulations

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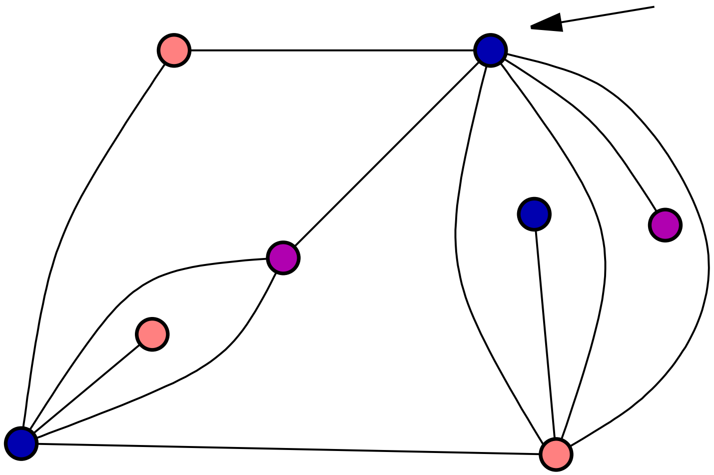
It has a D-FINITE parametrisation:

$$t = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{3n}{n} A^{n+1}, \qquad Q_3 = \frac{t-A}{3t^2} - 1.$$



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[mbm & Elvey Price 20]

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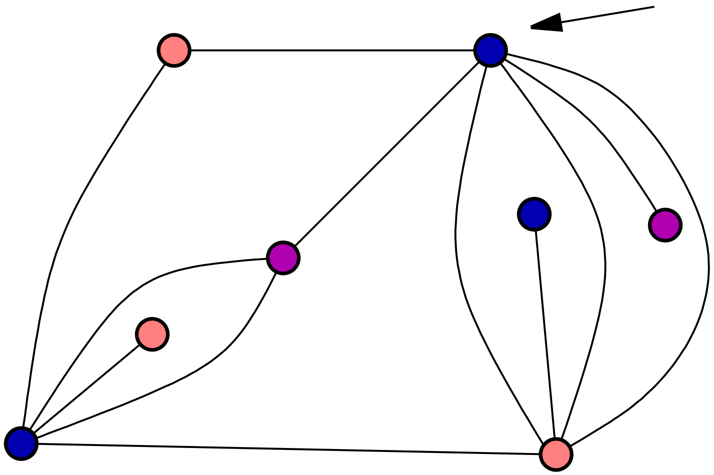
- **Asymptotics:**

$$q_3(n) \sim \kappa \left(4\sqrt{3}\pi\right)^n (n \log n)^{-2}.$$



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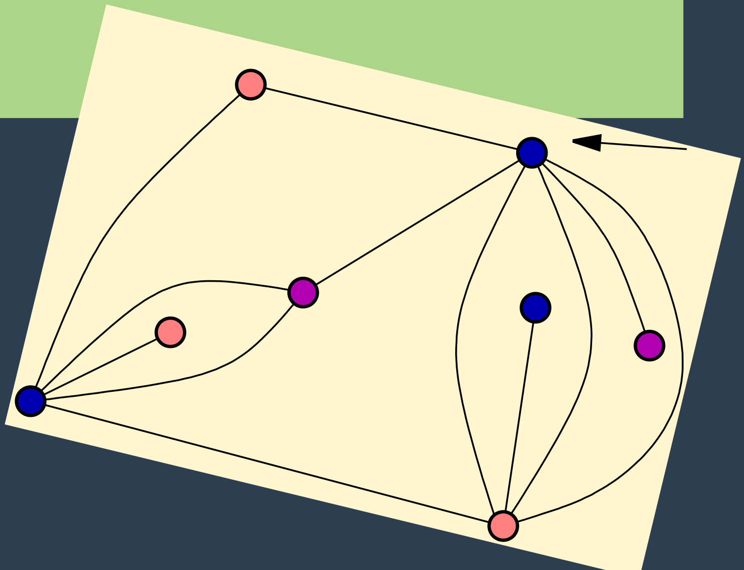
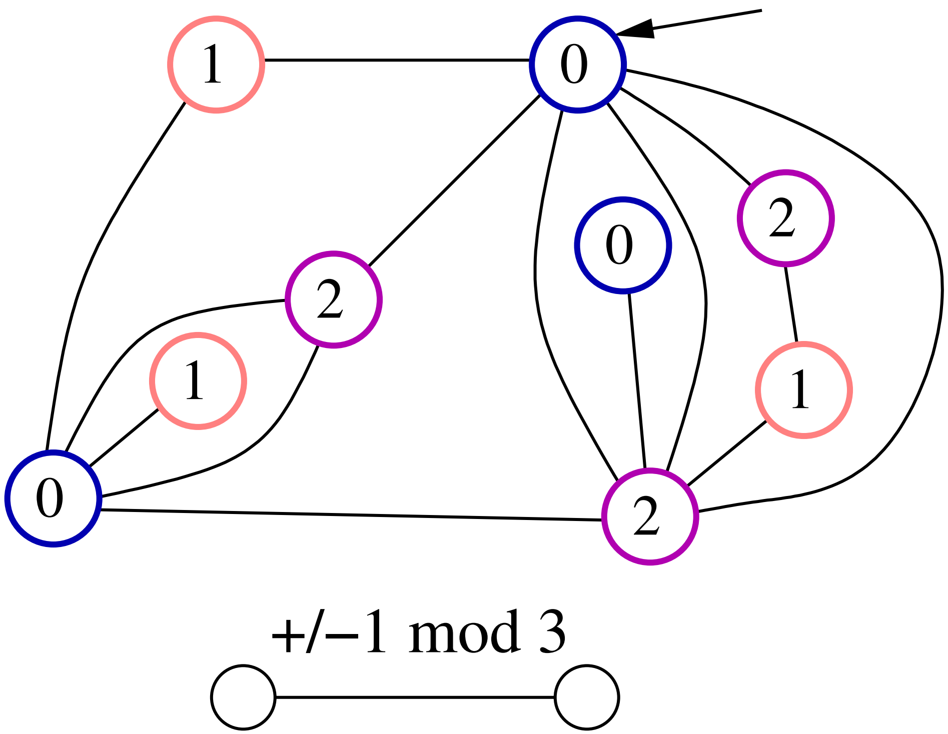
II. Three-coloured quadrangulations: a rich model



Three bijections

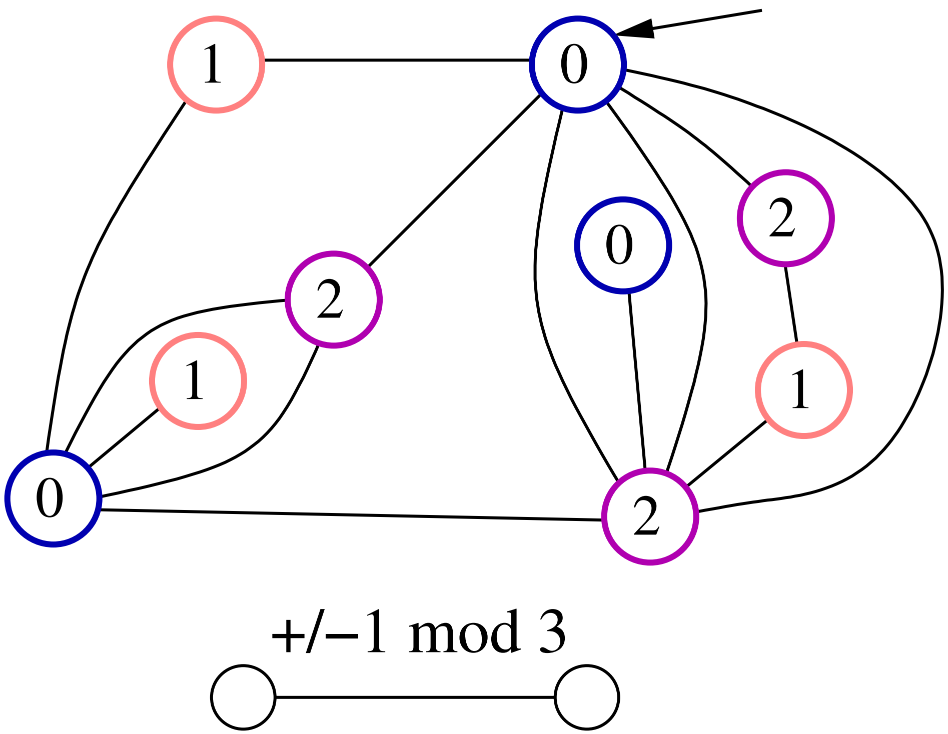
Three-coloured quadrangulations as a height model



[EP & Guttmann 18 + Welsh]

Three-coloured quadrangulations as a height model

Enforce variations of ± 1 along edges: a **height model**



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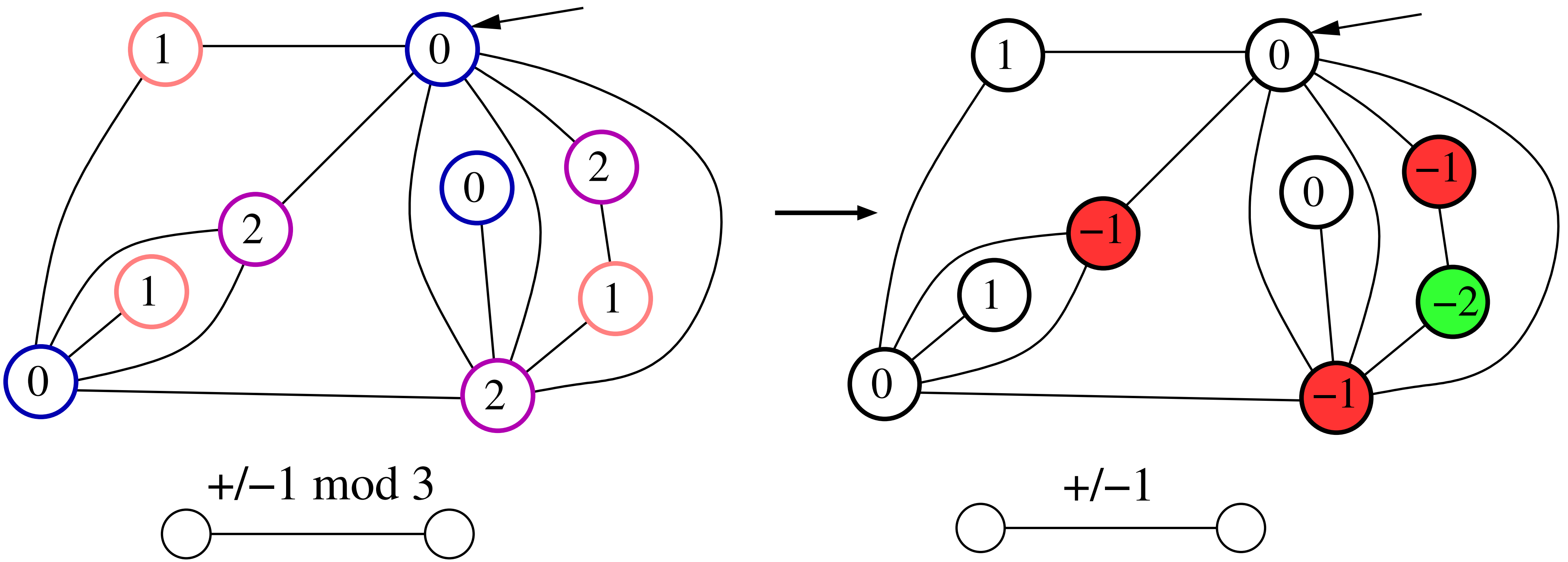

[EP & Guttmann 18 + Welsh]

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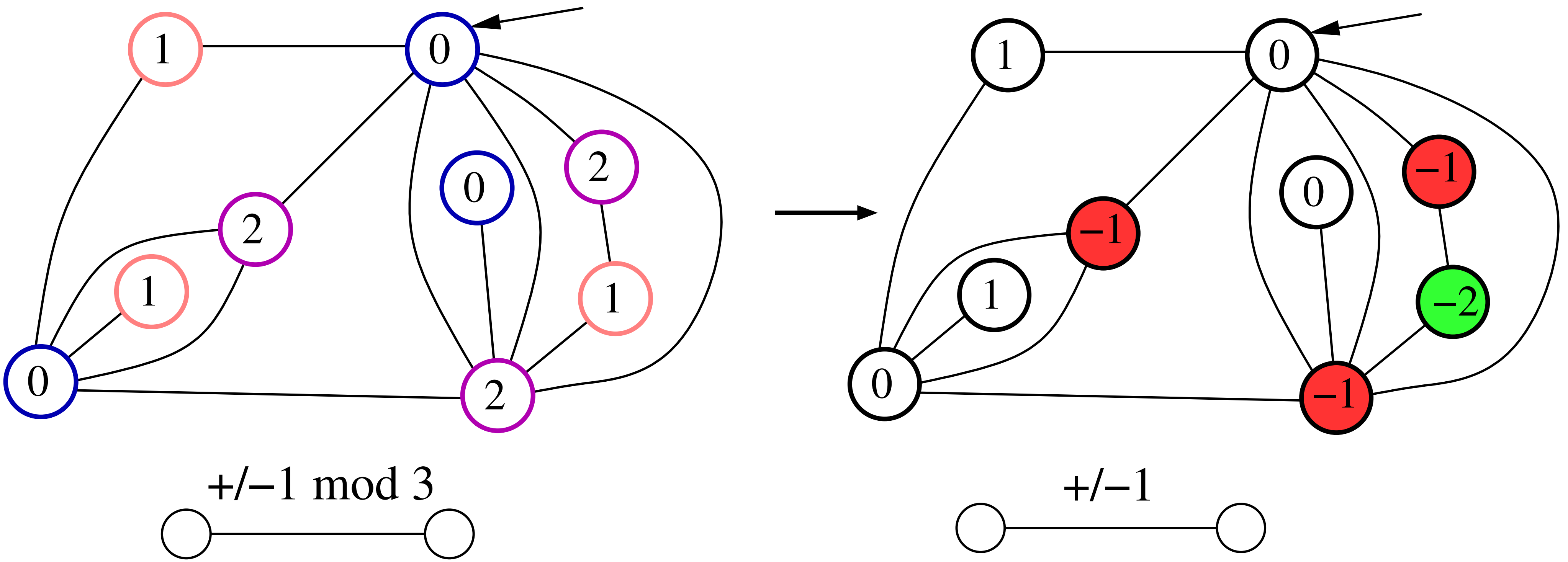
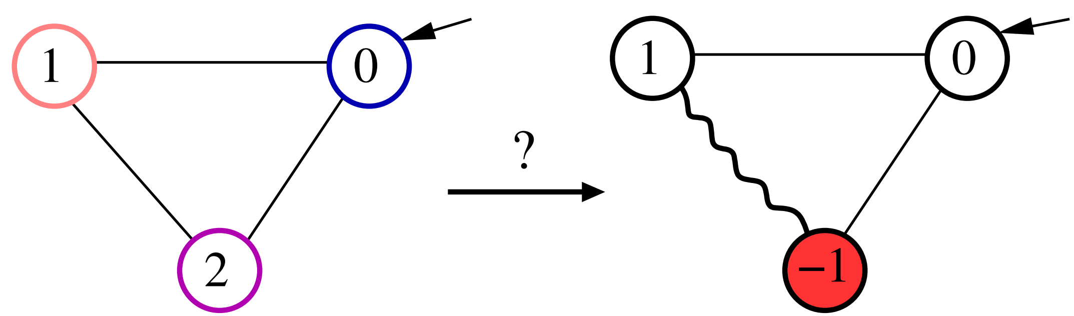
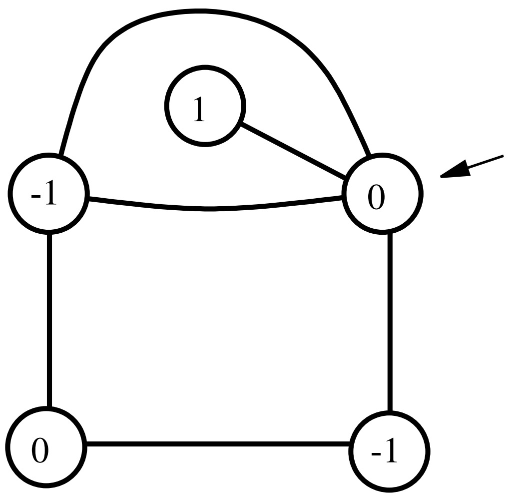


Only works for quadrangulations!



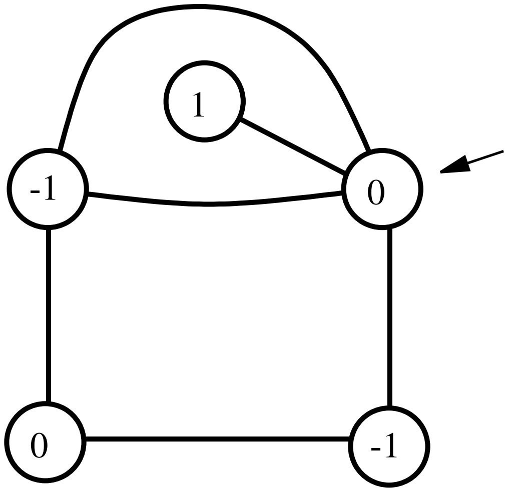
[EP & Guttmann 18 + Welsh]

The many faces of height labelled quadrangulations



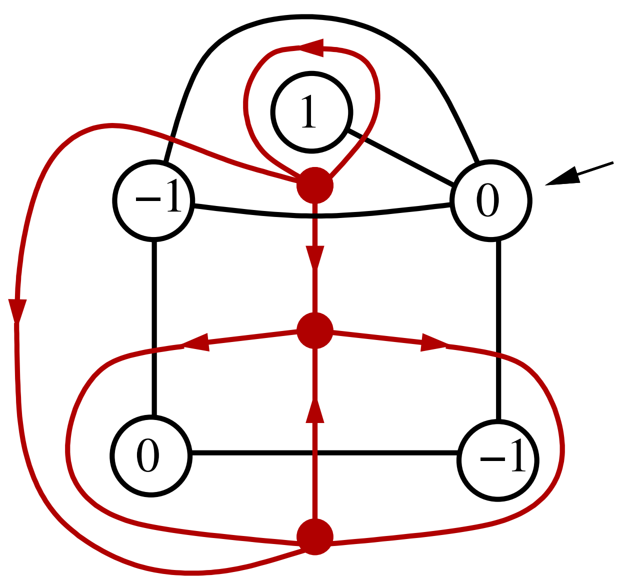
Labelled quadrangulation

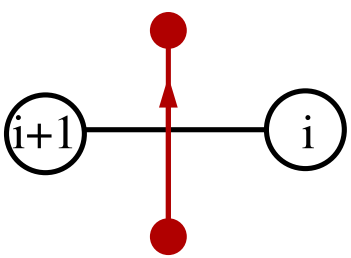
The many faces of height labelled quadrangulations



Labelled quadrangulation

duality





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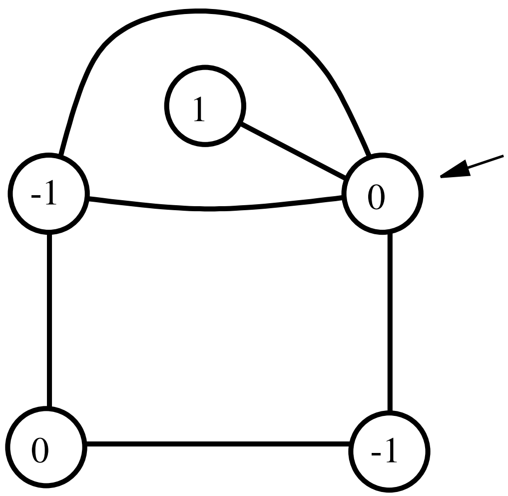
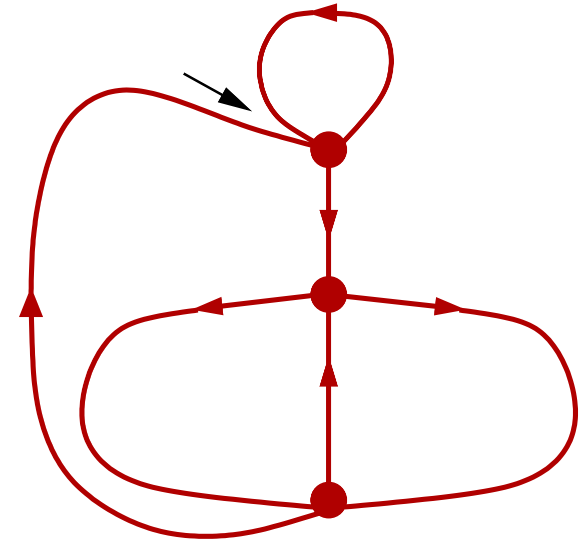
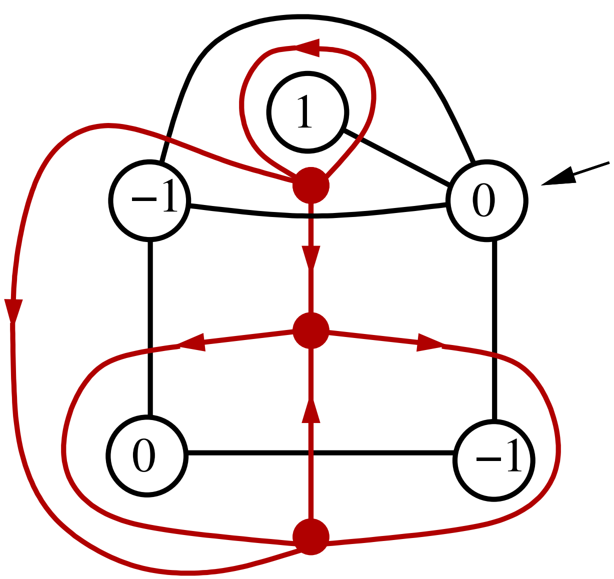


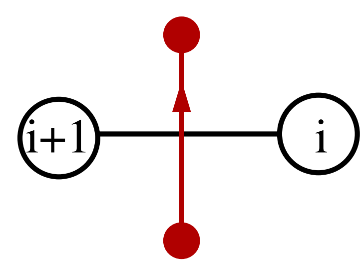


Labelled quadrangulation

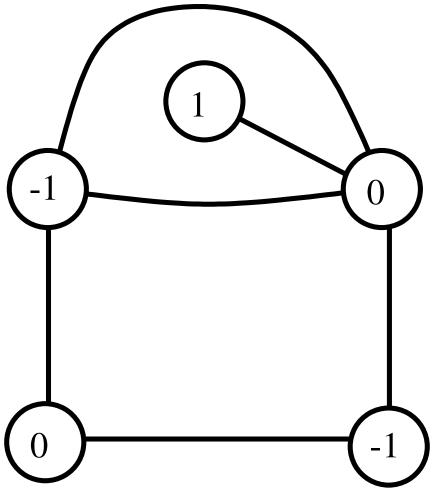
duality

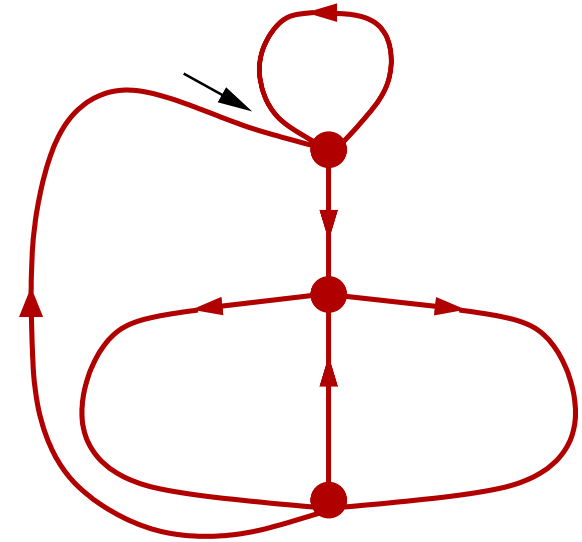
Quartic Eulerian orientation
(6 vertex model)





The many faces of height labelled quadrangulations



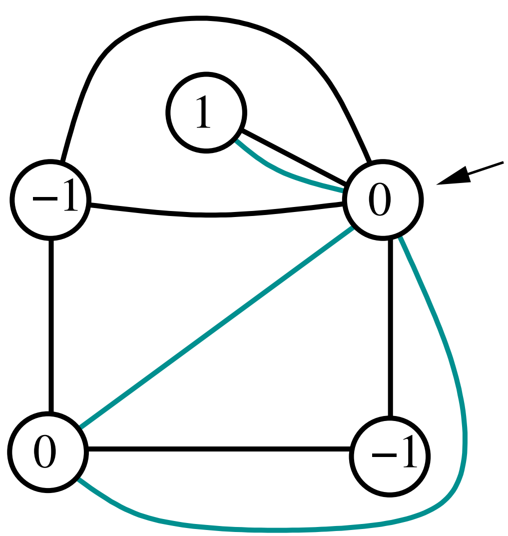


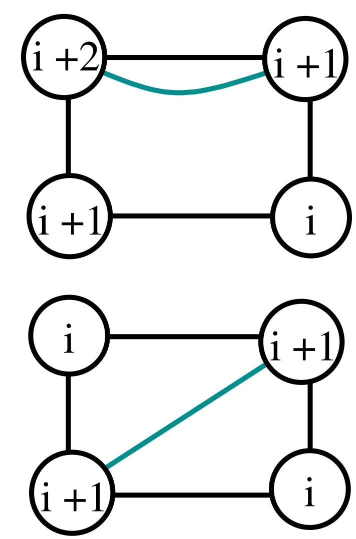
Labelled quadrangulation

duality

Quartic Eulerian orientation
(6 vertex model)

Ambjorn-Budd 13





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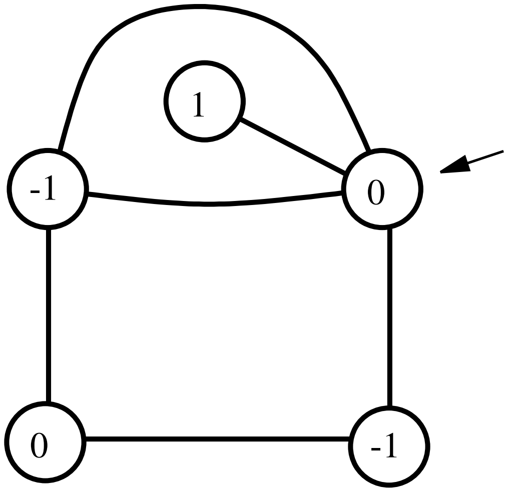

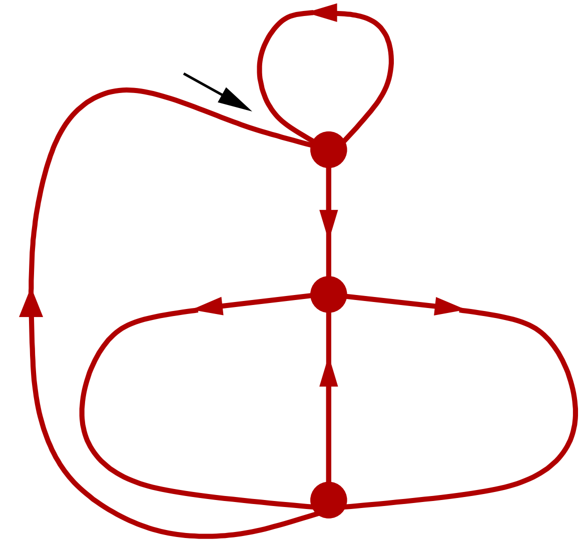

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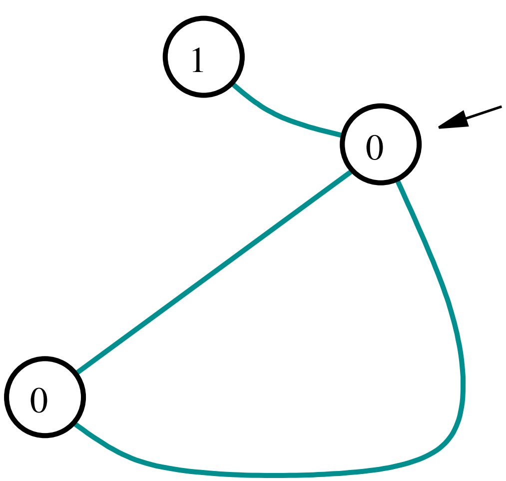
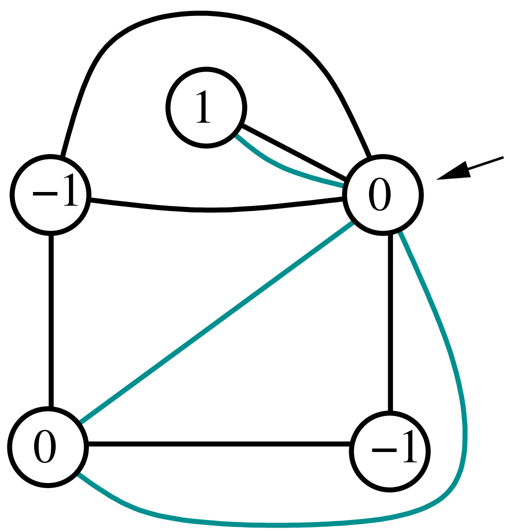
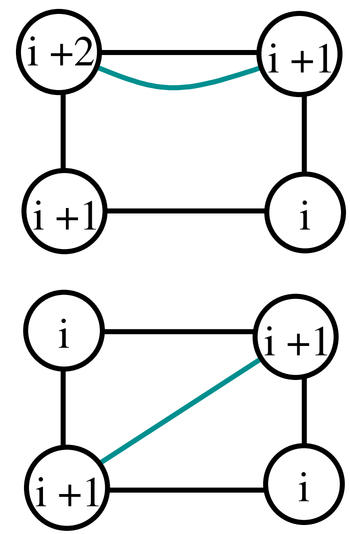
Weakly labelled map







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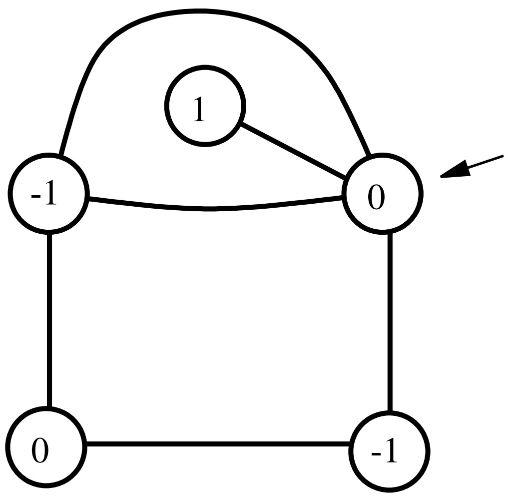

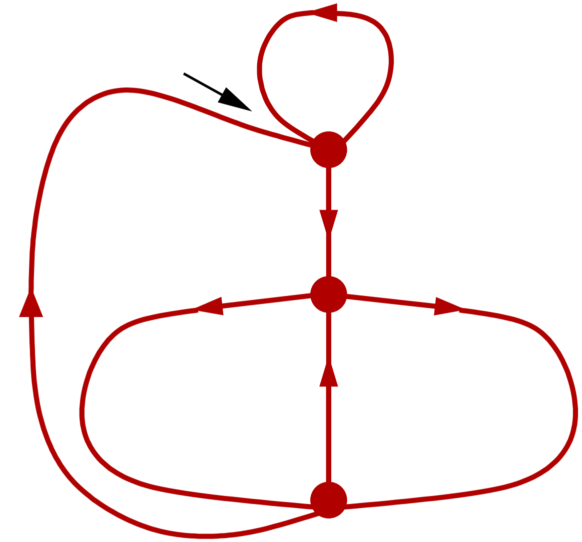

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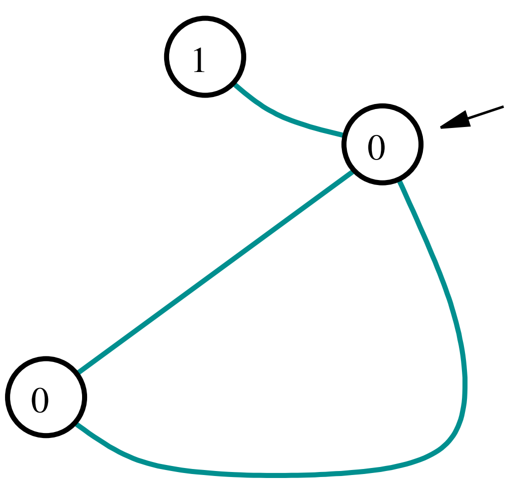
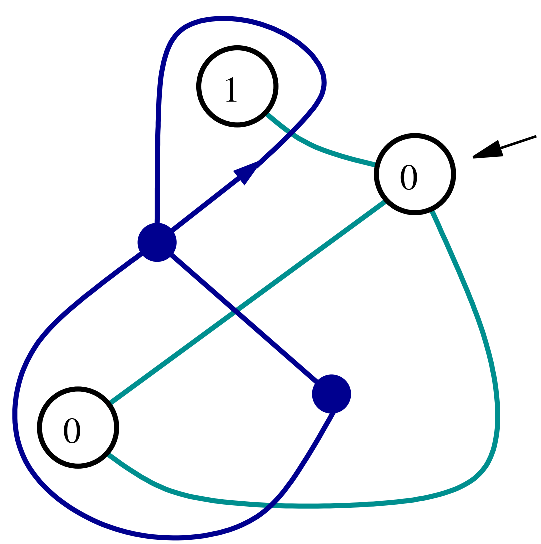
Ambjorn-Budd 13

Weakly labelled map ⟷ *duality*





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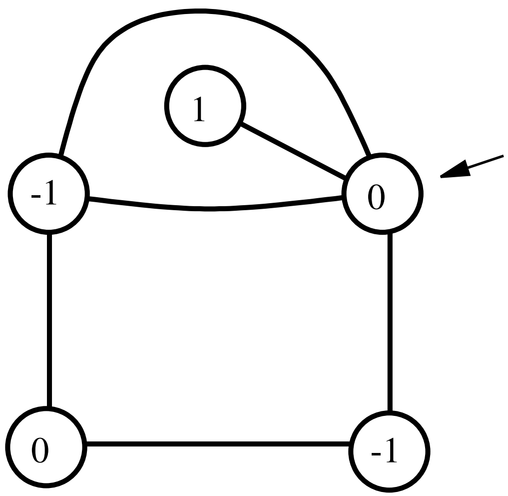
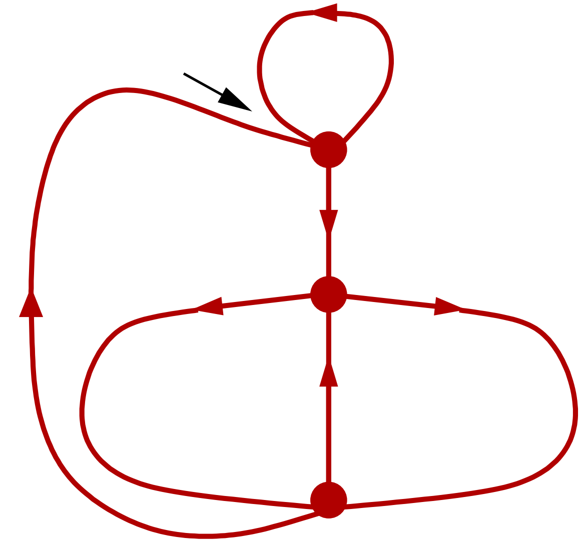
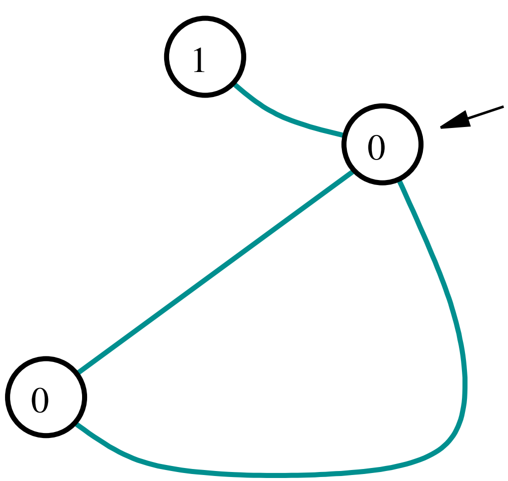


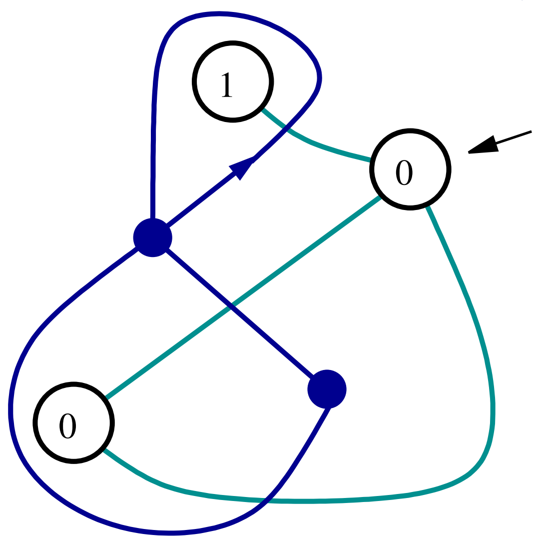


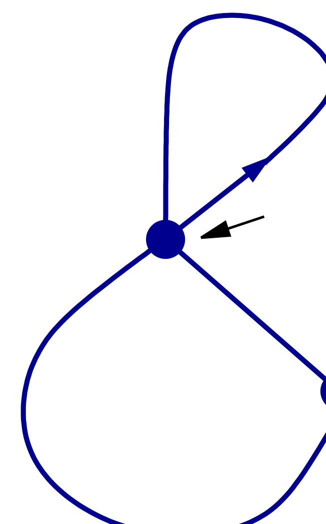
Labelled quadrangulation ⟷ *duality* ⟷ Quartic Eulerian orientation (6 vertex model)

↕ *Ambjorn-Budd 13*

Weakly labelled map ⟷ *duality* ⟷ *Partial* Eulerian orientation







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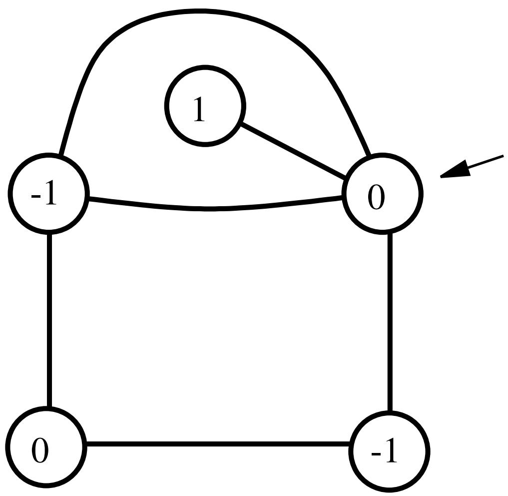

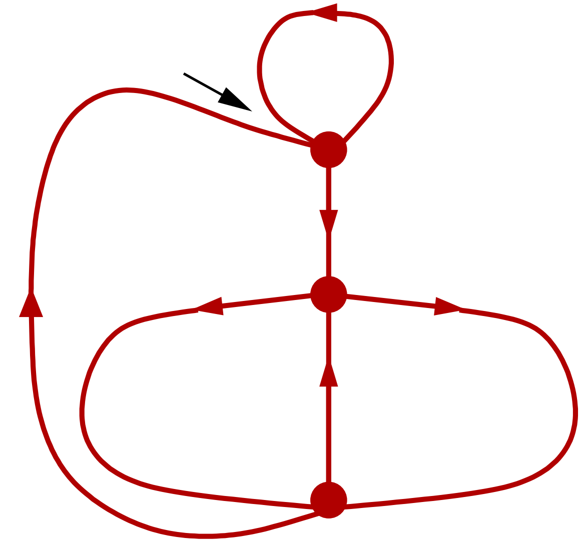

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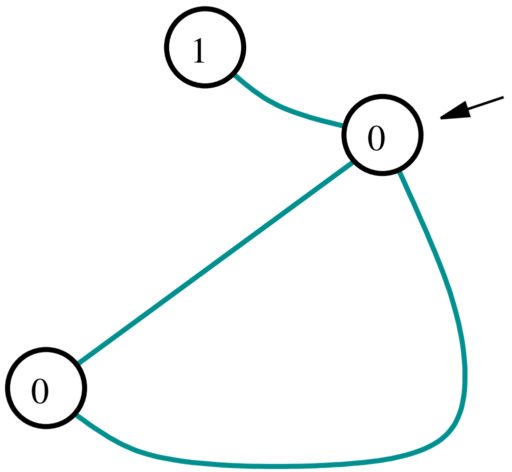

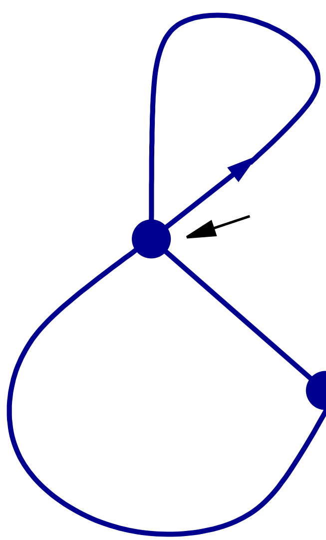

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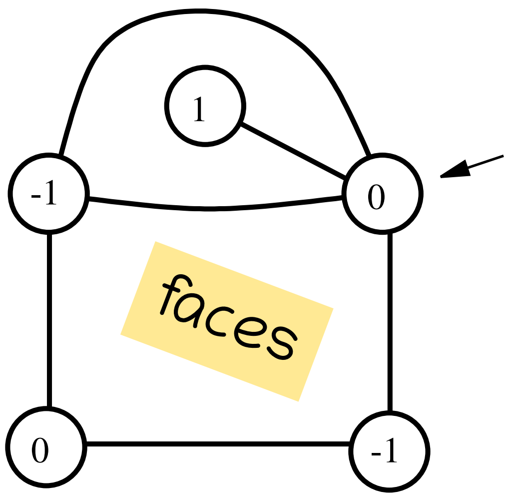

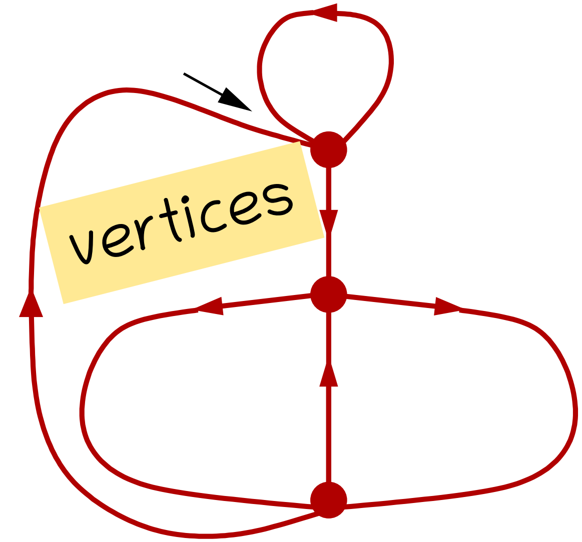

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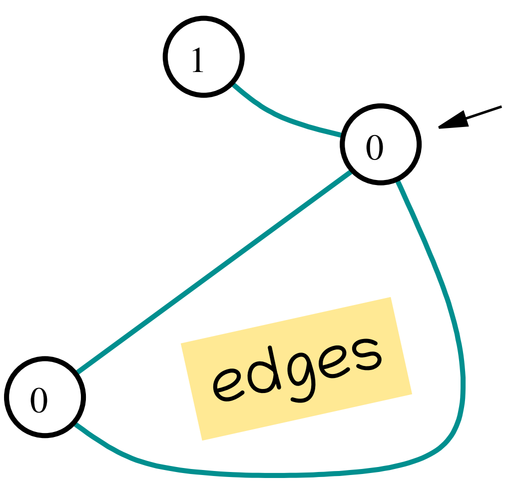
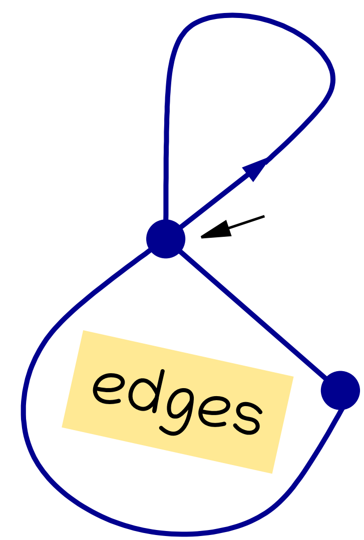
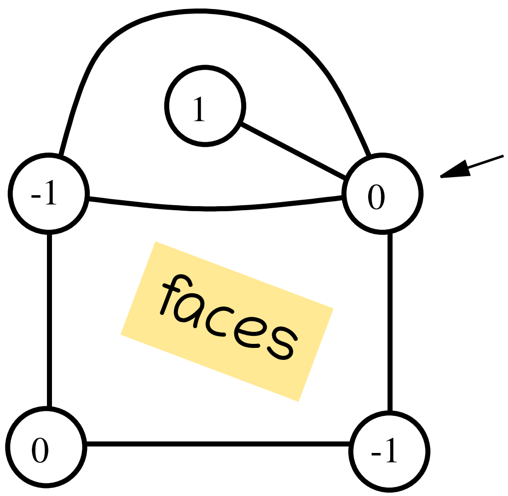
Ambjorn-Budd 13

Weakly labelled map ⟷ *duality* ⟷ *Partial* Eulerian orientation

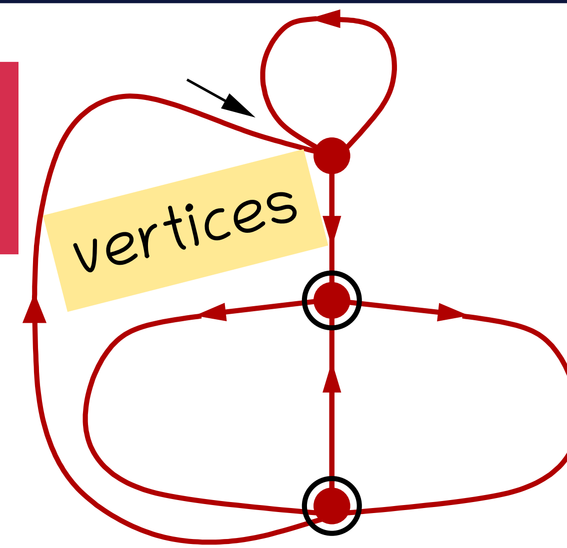




Two more statistics



Alternating vertices



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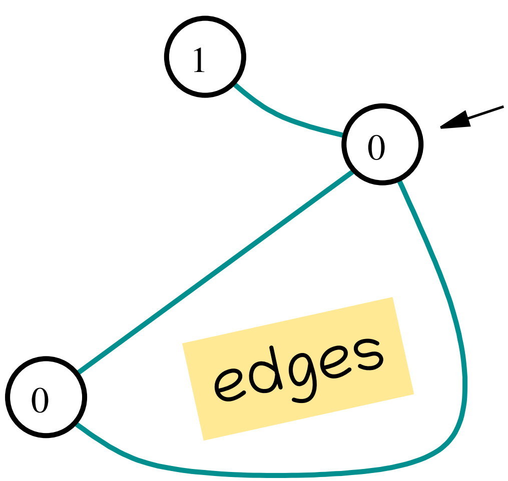
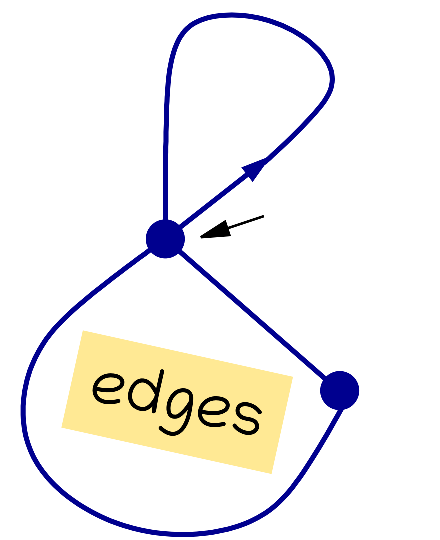
Weakly labelled map ⟷ *duality* ⟷ *Partial* Eulerian orientation





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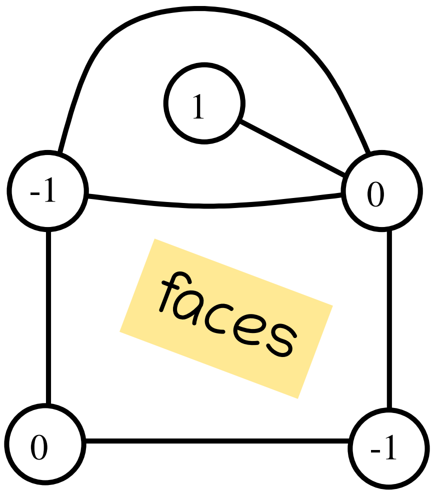
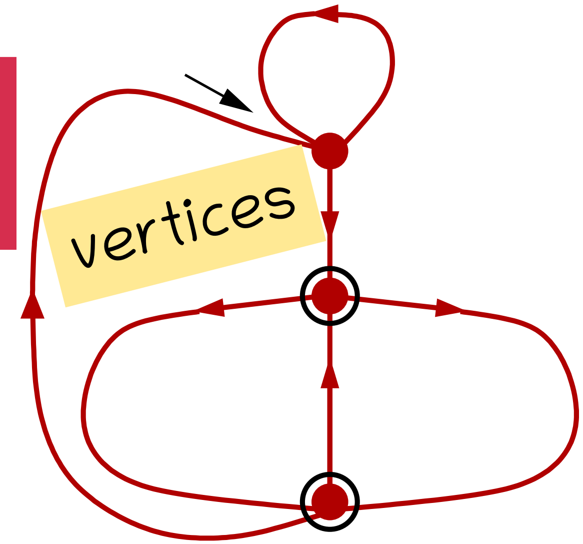


Bicoloured faces

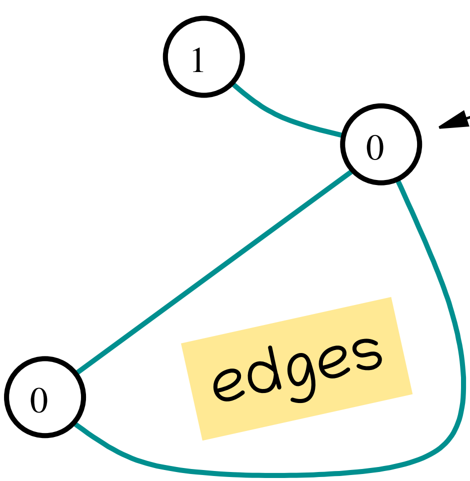
Alternating vertices



Labelled quadrangulation ← *duality* → Quartic Eulerian orientation (6 vertex model)

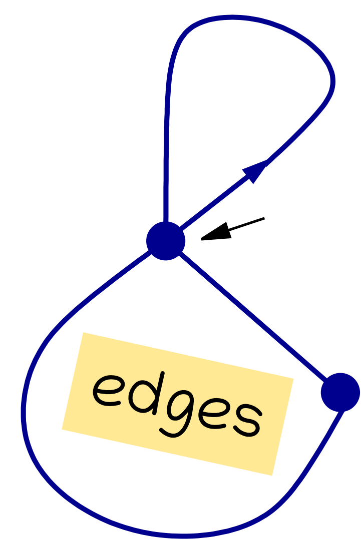
↕ *Ambjorn-Budd 13*

Weakly labelled map ← *duality* → *Partial* Eulerian orientation



Mono-chromatic edges

Unoriented edges



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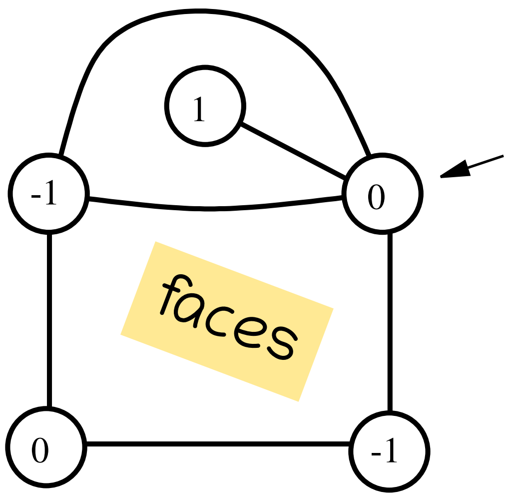

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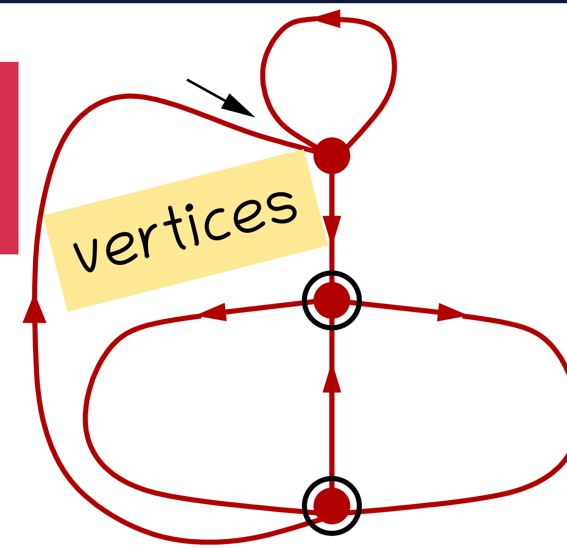

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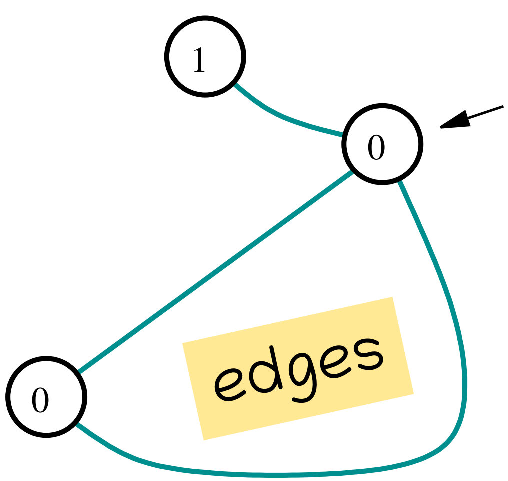

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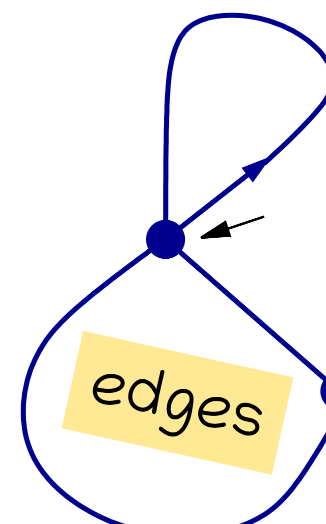
Vertices



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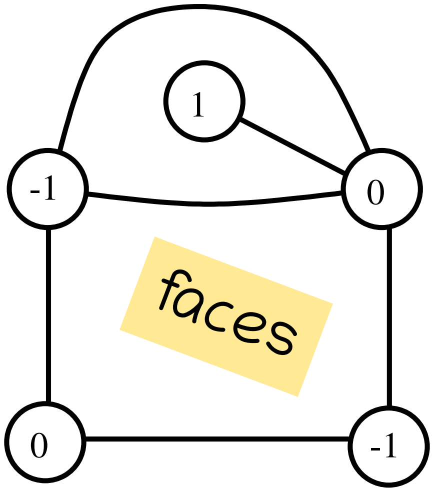
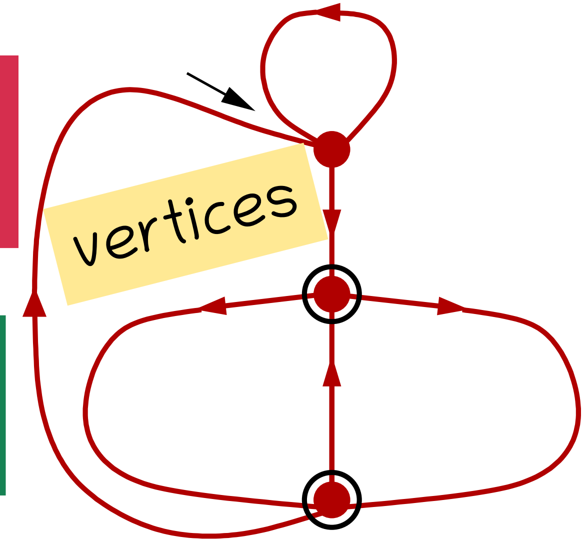


Bicoloured faces

Local minima

Alternating vertices

Clockwise faces



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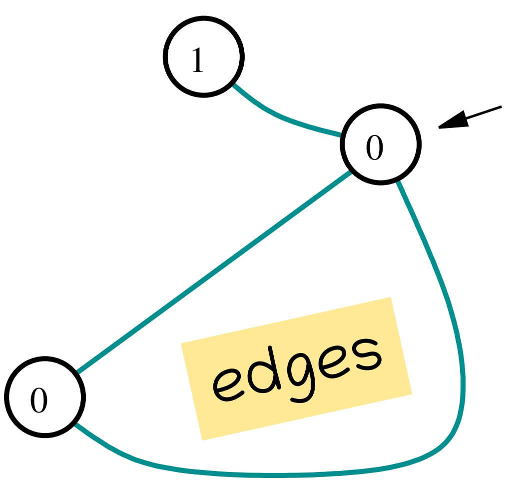
Weakly labelled map ←*duality*→ *Partial* Eulerian orientation

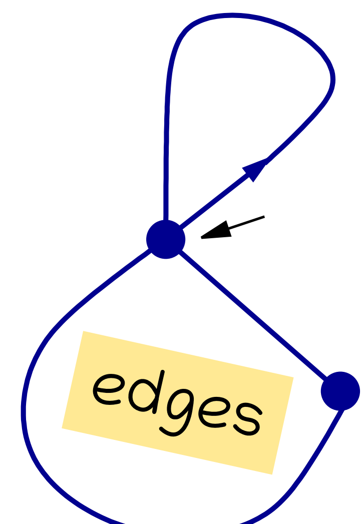


Mono-chromatic edges

Faces

Unoriented edges

Vertices



The generating function of labelled quadrangulations

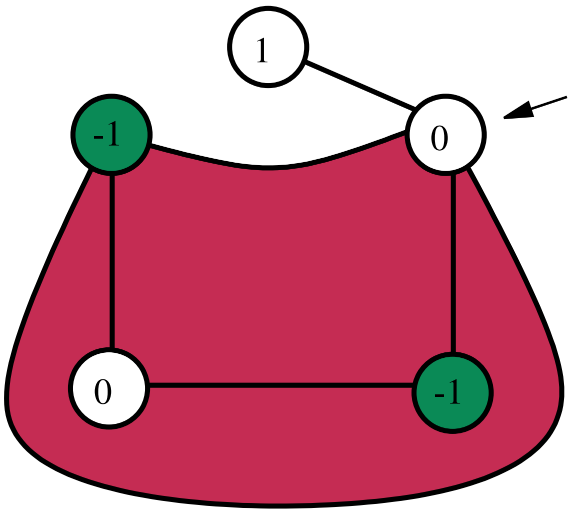
Convention: root edge labelled from 0 to 1

New statistic

Generating function:

$$Q = \sum_{\text{labelled quad.}} t^{\text{faces}} \omega^{\text{bic. faces}} \nu^{\text{local min.}}$$

$t^3 \omega^2 \nu^2$



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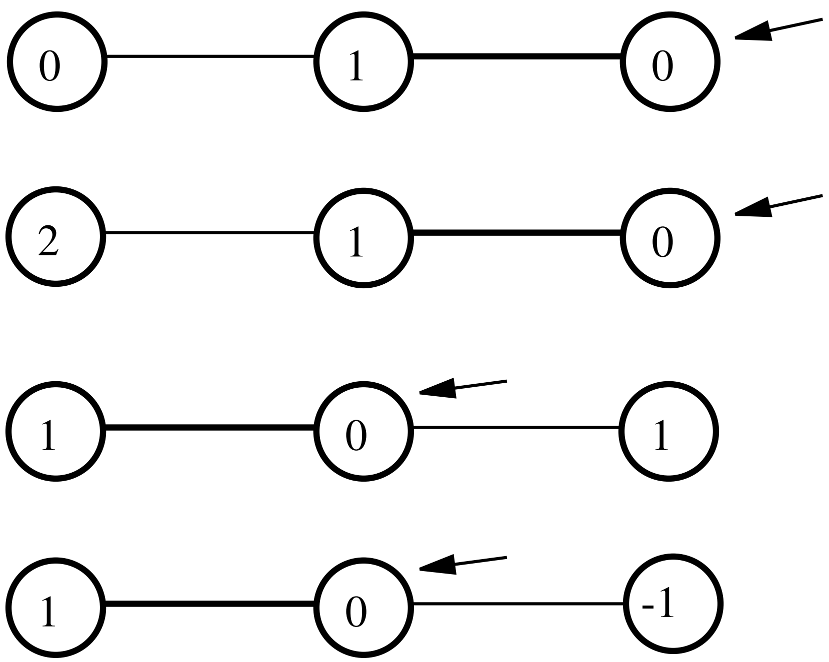

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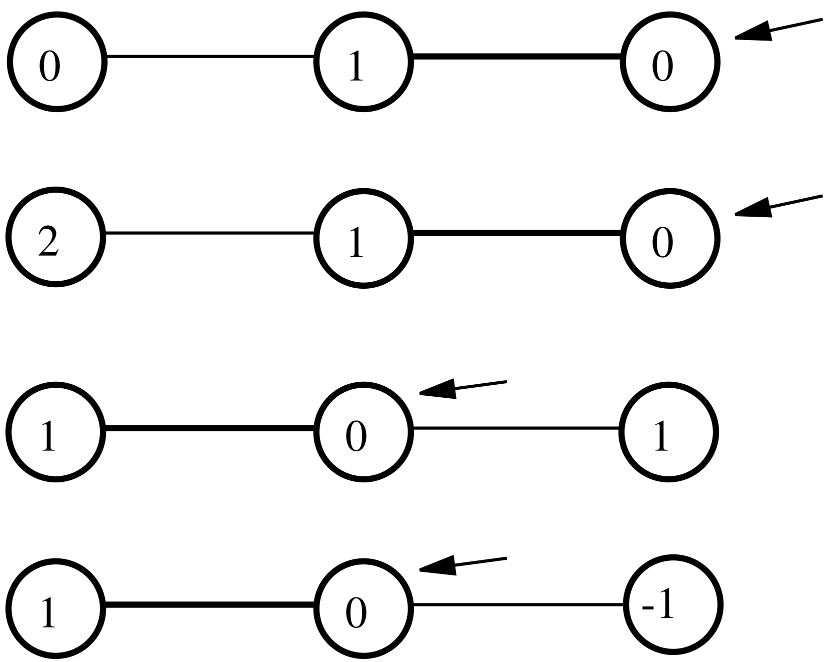

Bicoloured faces	Local minima	weight
1	2	$\omega^1 v^2$
0	1	$\omega^0 v^1$
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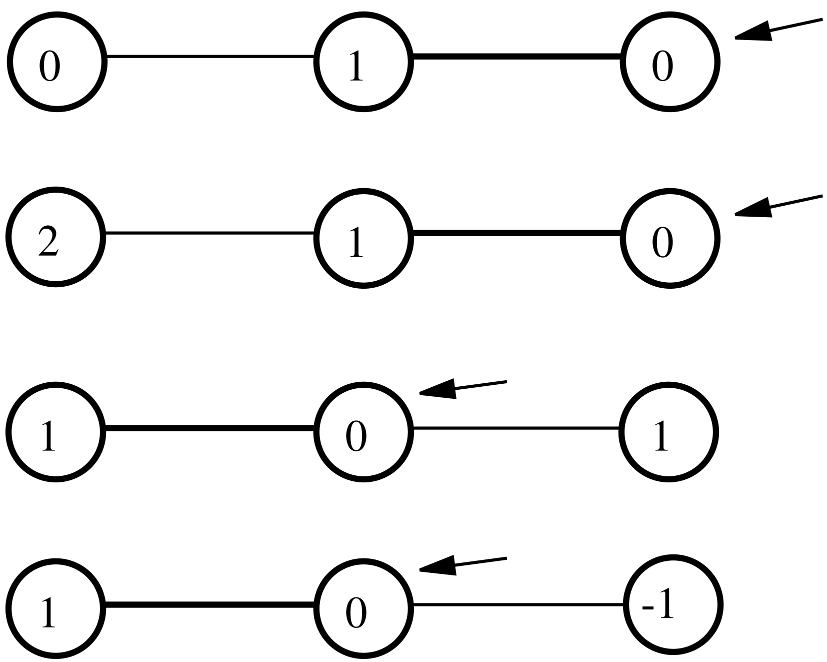

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Earlier work

	ω	ν
• Kostov 00: the 6-vertex model, analytic approach	ω	1
• MBM & Elvey Price 20: orientations on quartic maps and general maps, algebraic approach	1 0	1 1
• Elvey Price & Zinn-Justin (P.) 23: the 6-vertex model, à la Kostov	ω	1
• MBM & Elvey Price 24: arbitrary ν and ω	ω	ν

And also...

[Bonichon et al. 17, Elvey Price & Guttmann 18]

Results: two (very) different forms

The case $\nu = \omega = 1$ [MBM & Elvey Price 20]

Let A be the unique series in t such that:

$$t = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} A^{n+1}.$$

Then the generating function of **quartic Eulerian orientations** is

$$Q = \frac{t - A}{3t^2} - 1$$
$$= 4t + 35t^2 + 402t^3 + \cdots .$$

Results: two (very) different forms

The case v=1 (6 vertex model) [Kostov 00, EP & Zinn-Justin 20]

Jacobi theta function:

$$\theta(q,z) \equiv \theta(z) := \sum_{n\geq 0} (-1)^n q^{n(n+1)/2} \sin(2n+1)z.$$

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Write $\omega = -2\cos(2\alpha)$. Let q be the only series in t such that:

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(\frac{\theta''(\alpha)}{\theta'(\alpha)} - \frac{\theta(\alpha)\theta^{(3)}(\alpha)}{\theta'(\alpha)^2}\right).$$

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Moreover, define

$$A = \frac{\cos^2\alpha}{96\sin^4\alpha}\,\frac{\theta(\alpha)^2}{\theta'(\alpha)^2}\left(\frac{\theta^{(3)}(0)}{\theta'(0)} - \frac{\theta^{(3)}(\alpha)}{\theta'(\alpha)}\right).$$

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Then the generating function of quartic Eulerian orientations, with weight ω per alternating vertex, is

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$$Q = \frac{t - A}{3t^2} - 1$$
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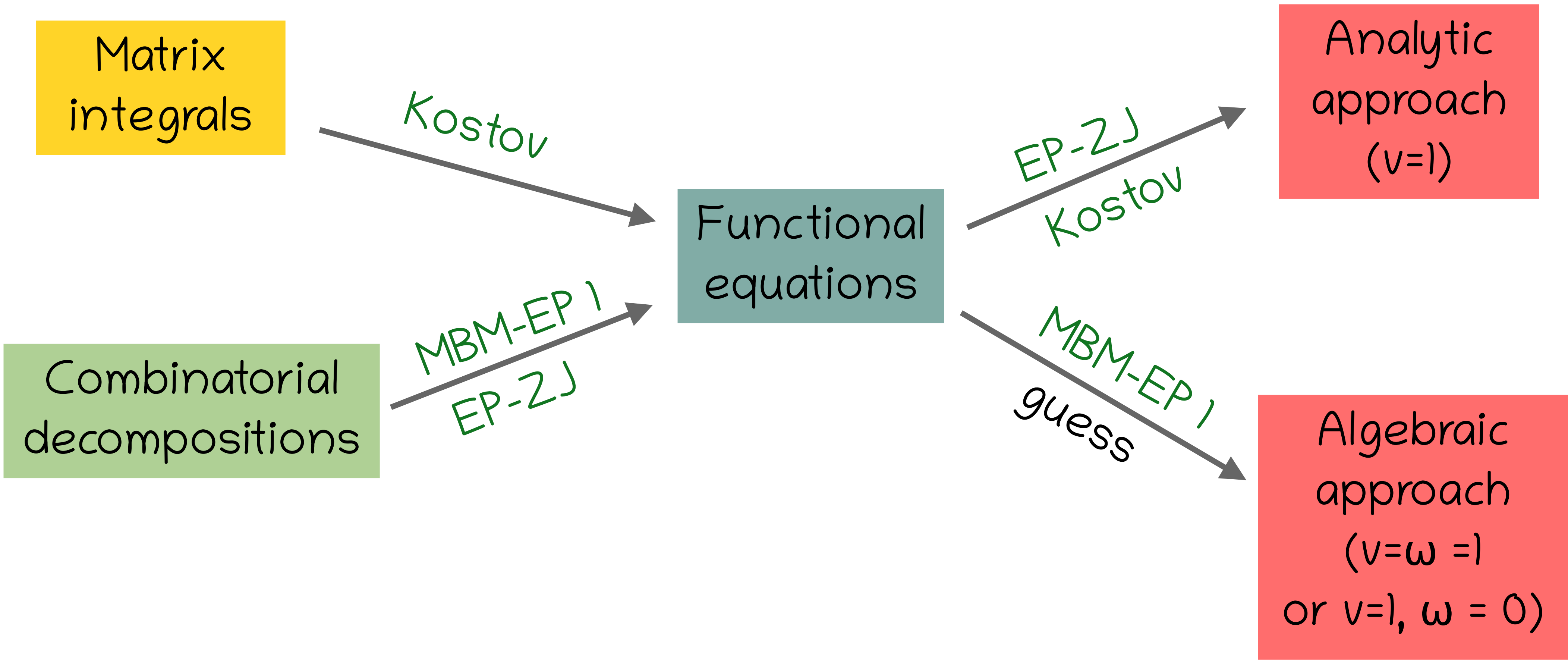
III. Some ingredients, some results

Approaches



Matrix integrals
Combinatorial decompositions
Kostov
MBM-EP 1
EP-2J
Functional equations
EP-2J
Kostov
MBM-EP 1
guess
Analytic approach (v=1)
Algebraic approach (v=ω =1 or v=1, ω = 0)

Map functional equations: some features

- Introduce **more general maps**...

  The outer face has **any degree**

- ... and the corresponding "**catalytic" variables**:

  y for the outer degree

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Example: Uncoloured quadrangulations with any outer degree

$$U(y) = \sum_{\text{near-quadr.}} t^{\text{finite faces}} y^{\frac{\text{outer degree}}{2} - 1}$$

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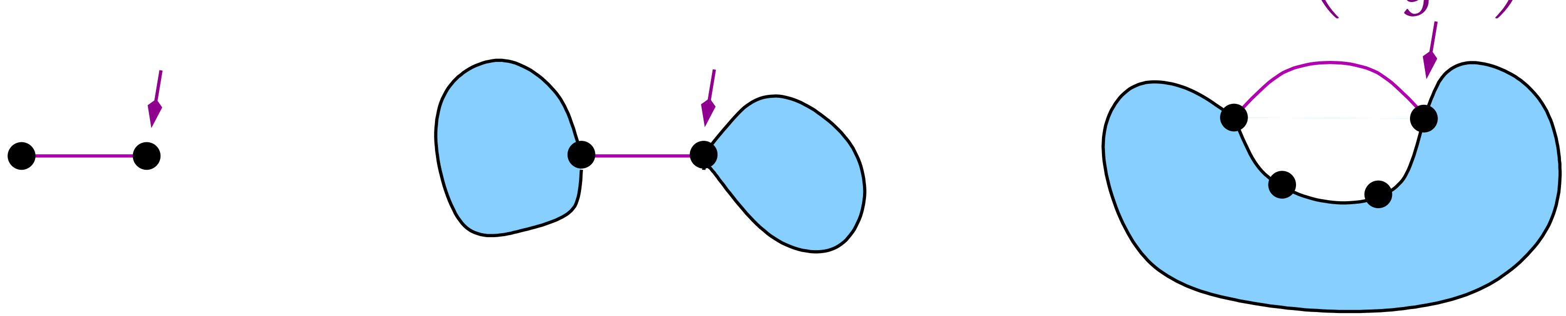
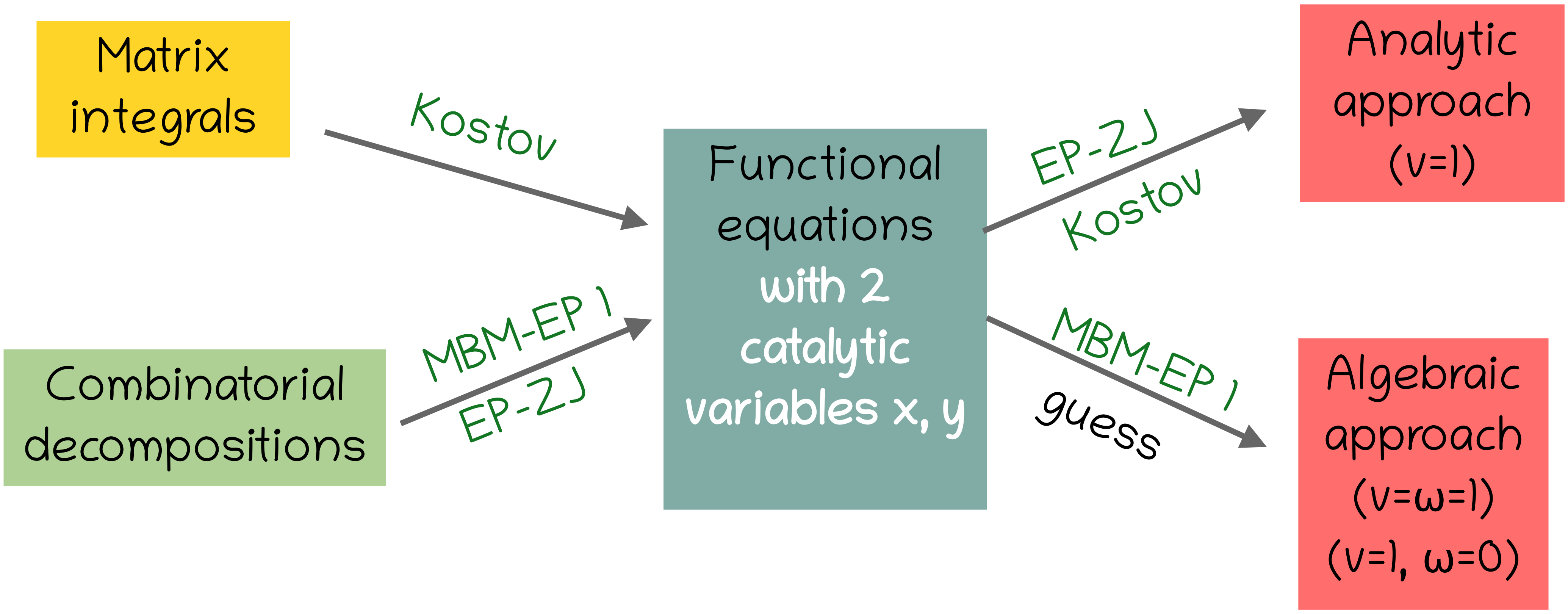
$$U(y) = \sum_{\text{near-quadr.}} t^{\text{finite faces}} \, y^{\frac{\text{outer degree}}{2} - 1}$$

$$U(y) = t^0 y^0 \quad + \quad y U(y)^2 \quad + \quad t \, [y^{\geq 0}] \left(\frac{U(y)}{y} \right).$$



Labelled quadrangulations: approaches



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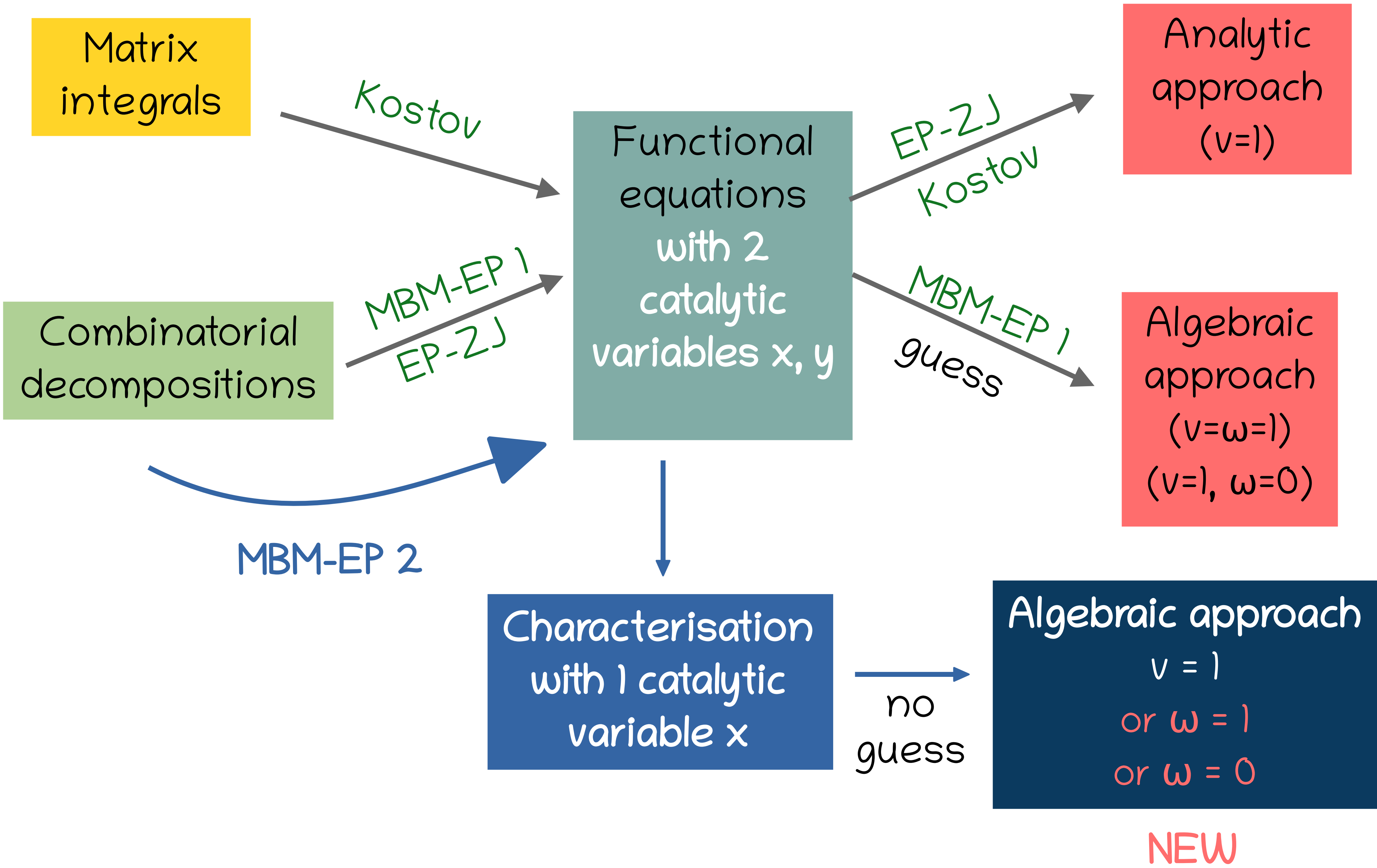
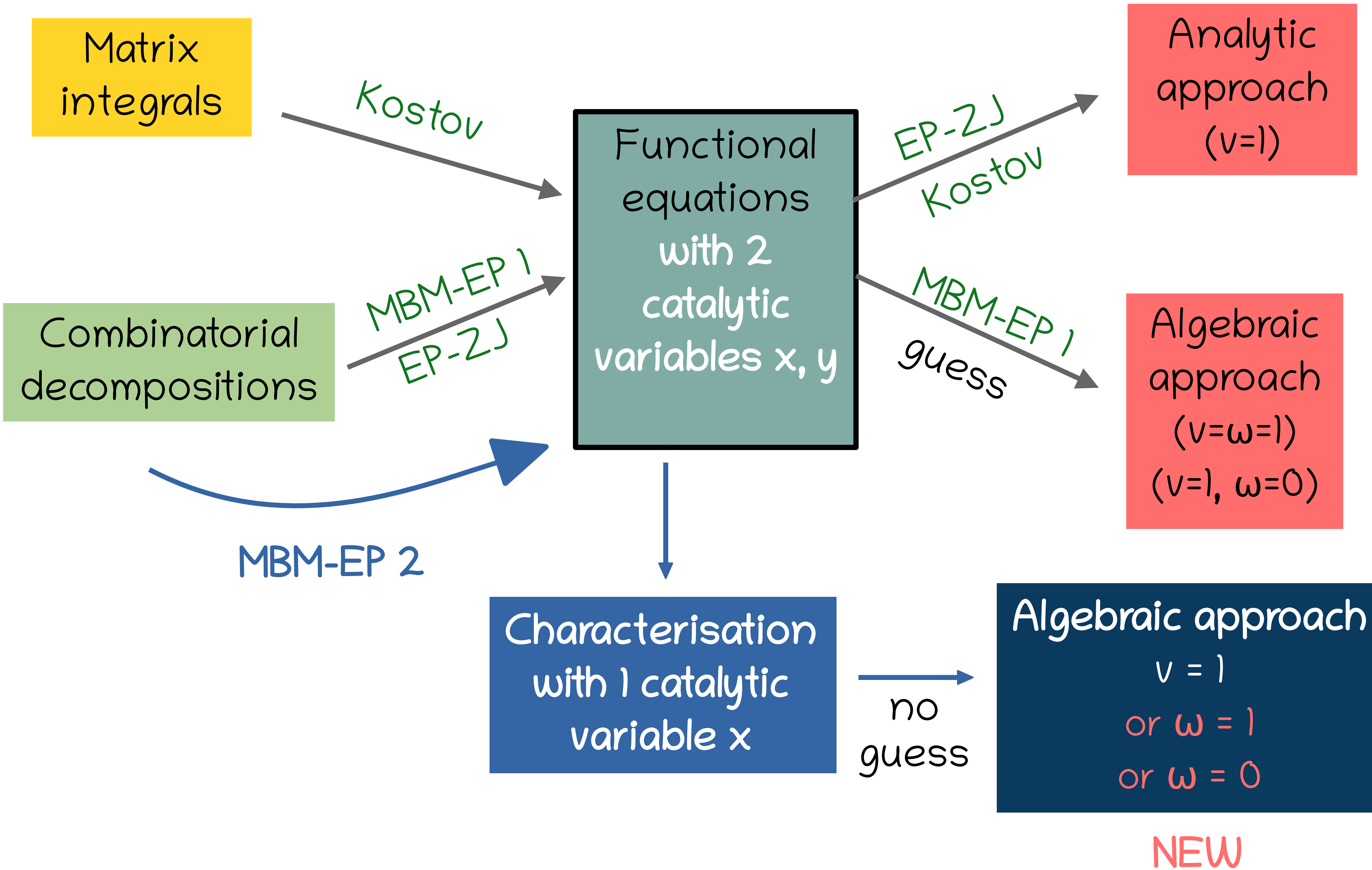


Approaches



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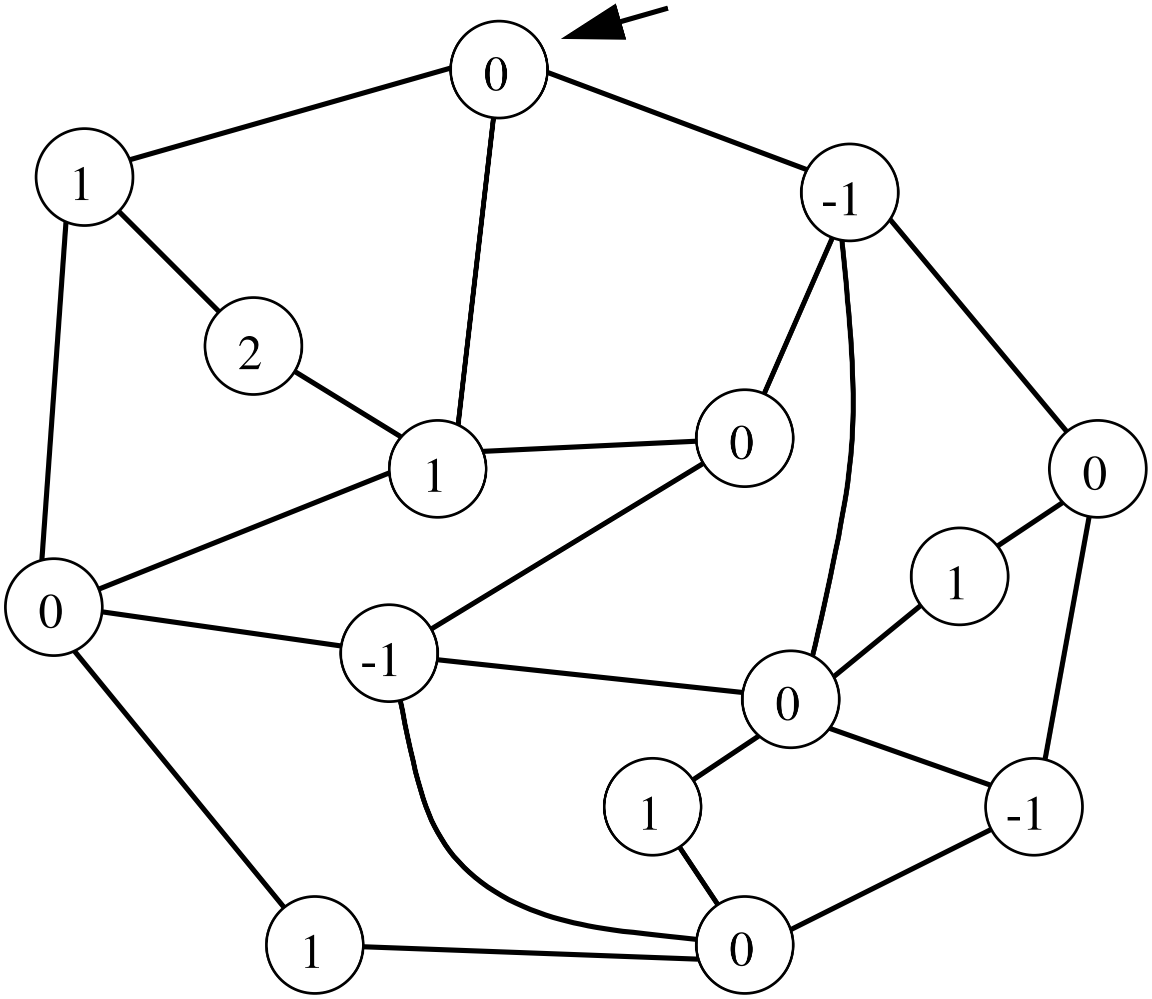
An interesting class of labelled maps (à la Dobrushin)



Boundary: 0 -1 0 -1 0 ... -1 0 1 0 1 ... 1

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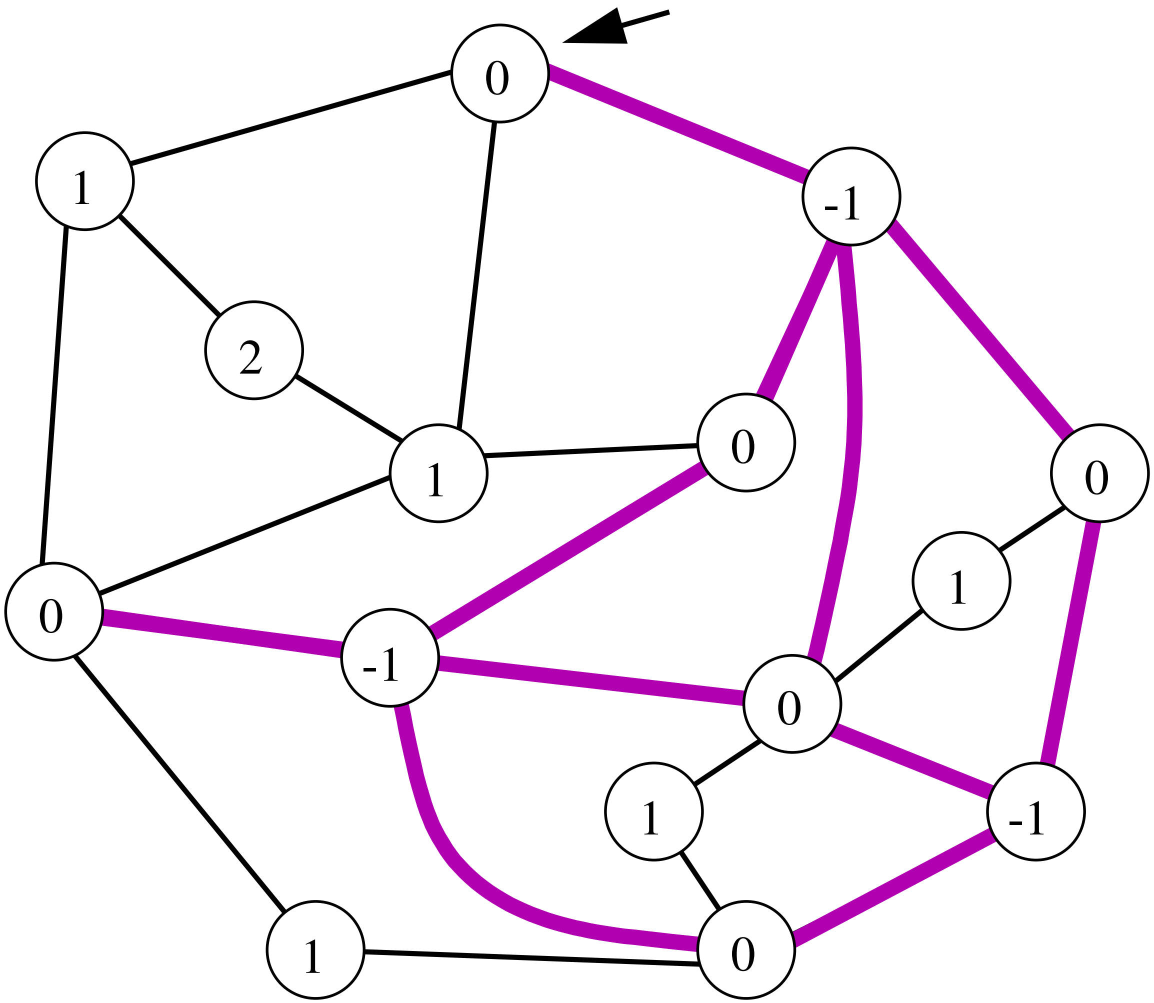

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Non-positive submap attached at the root

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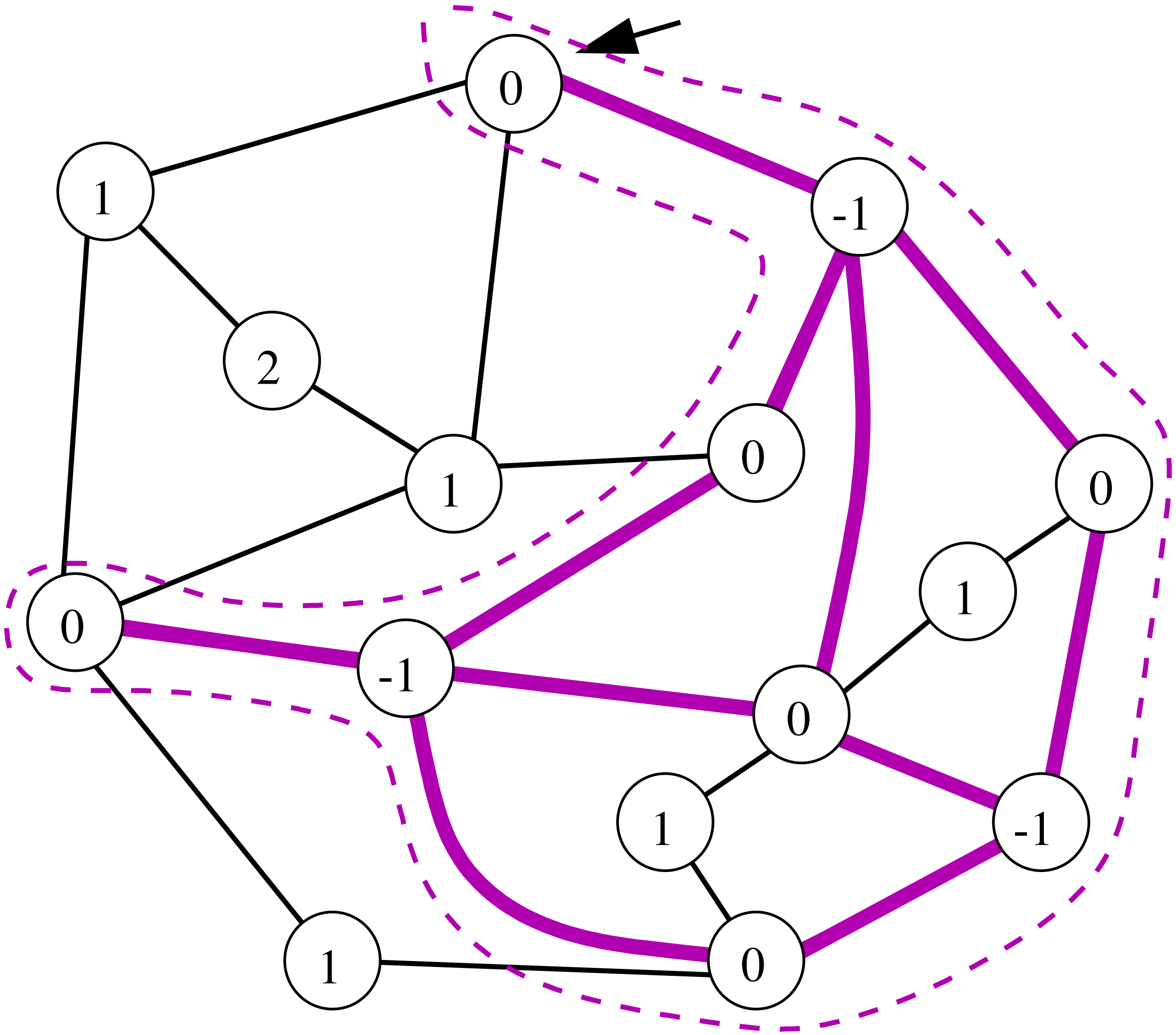

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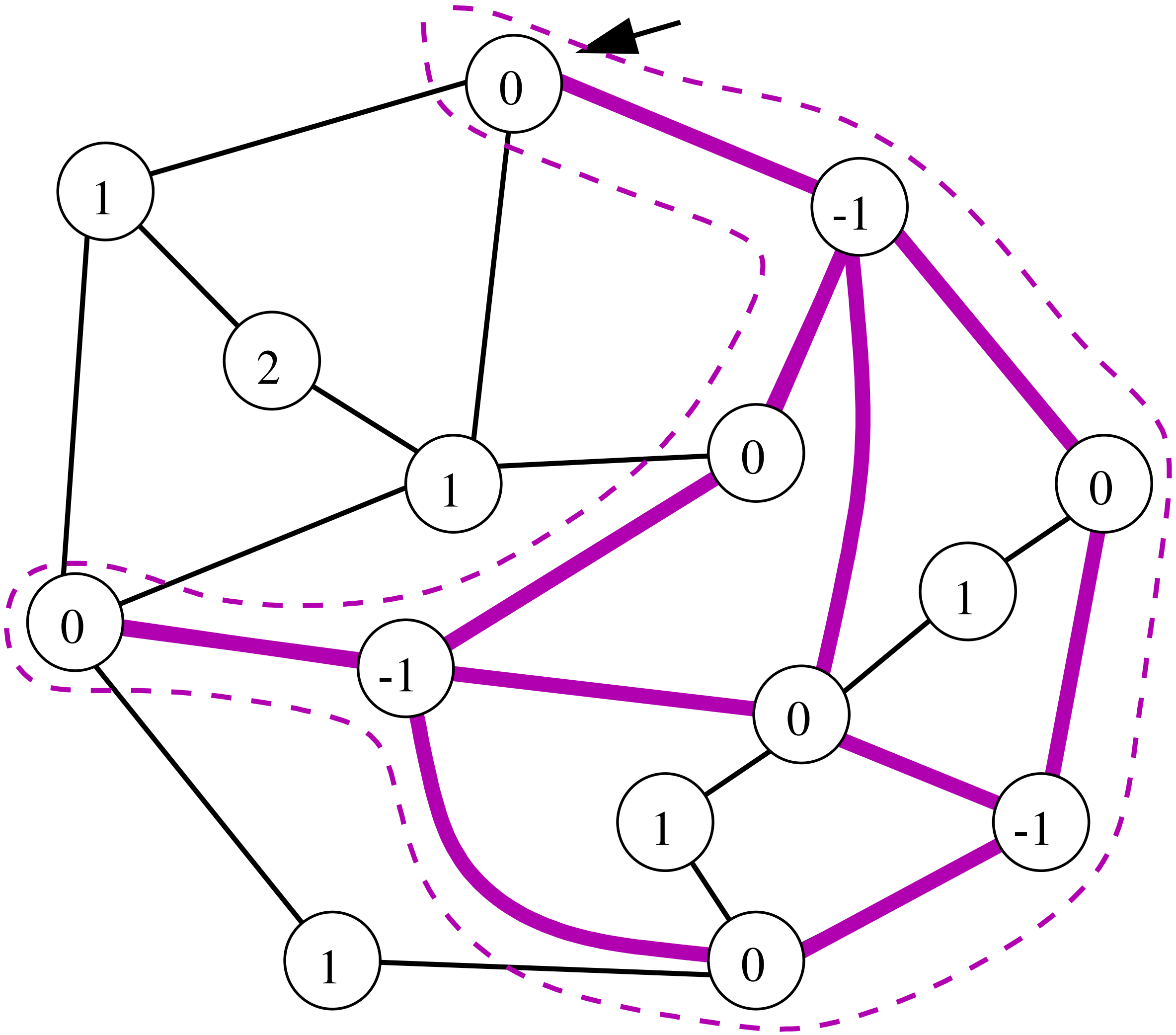

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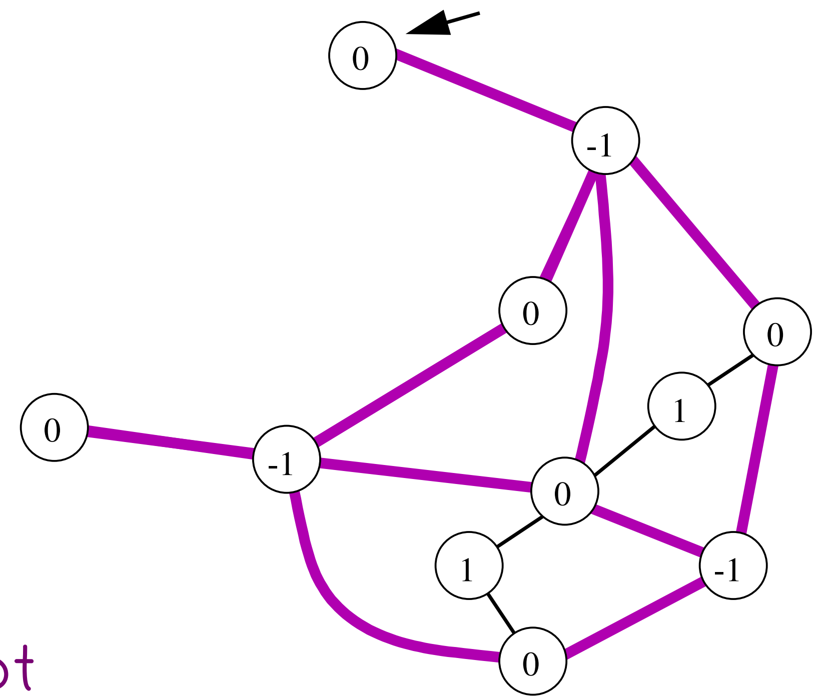

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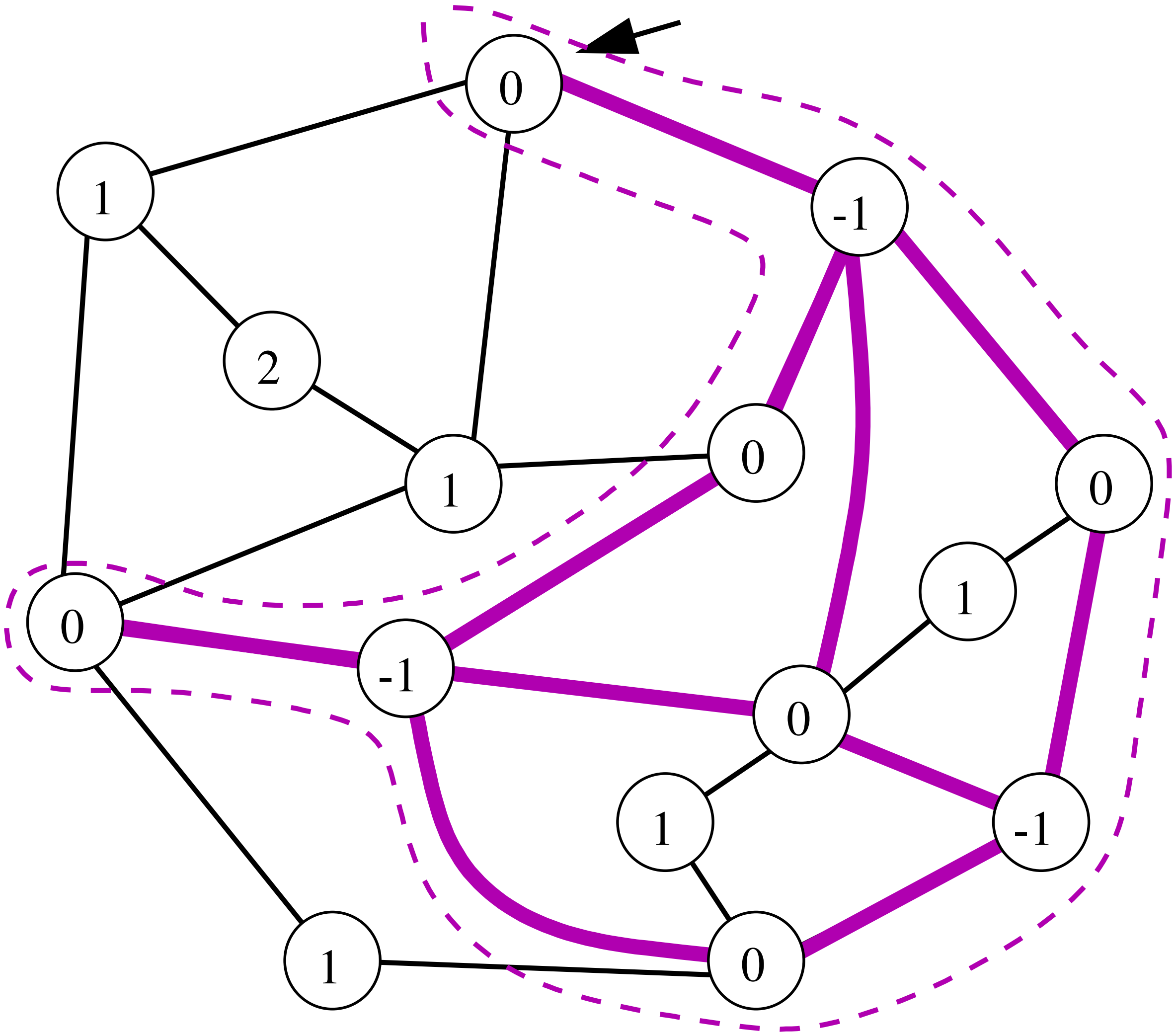

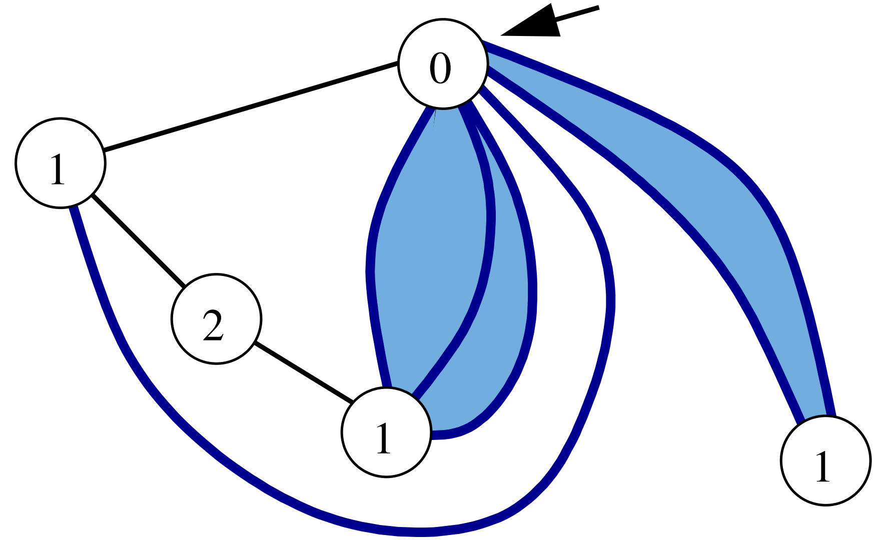

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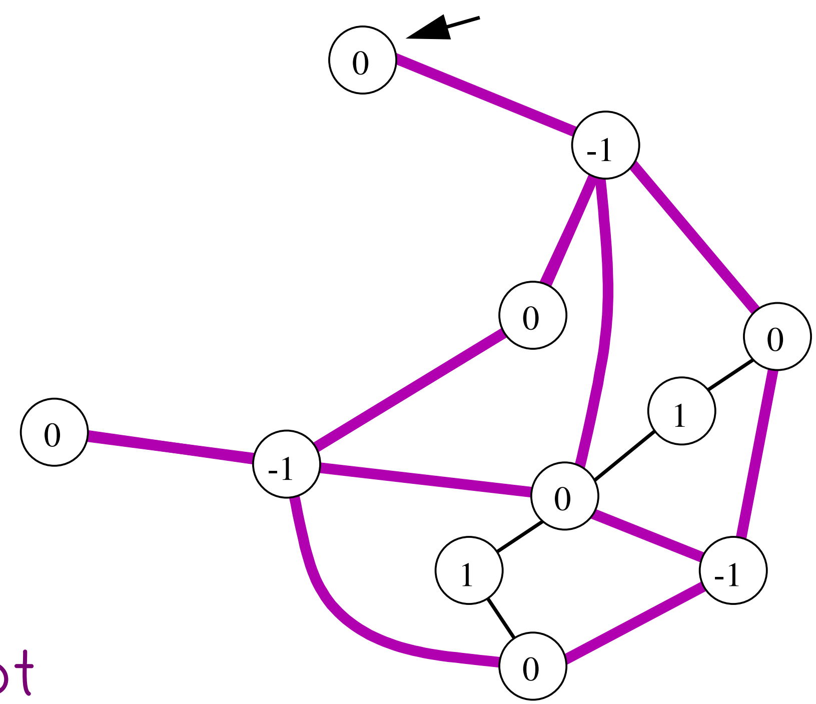

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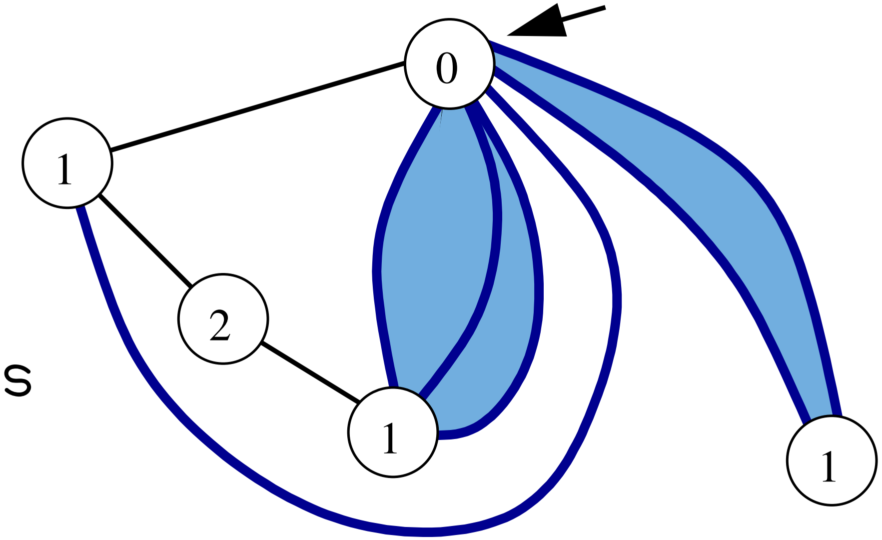
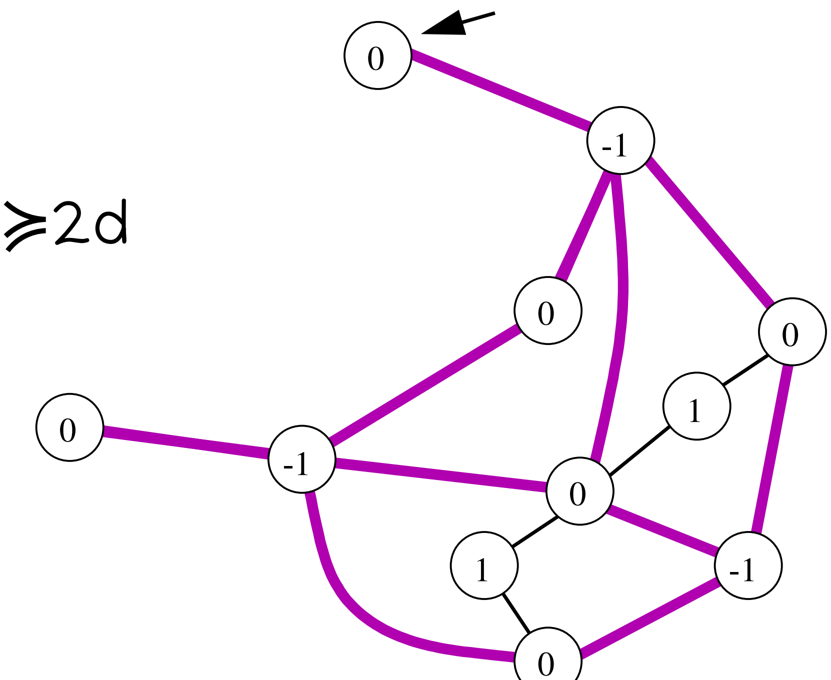
An interesting class of labelled maps (à la Dobrushin)

d digons



Boundary: 0 1 0 1 ... 1 0

Outer degree ⩾2d

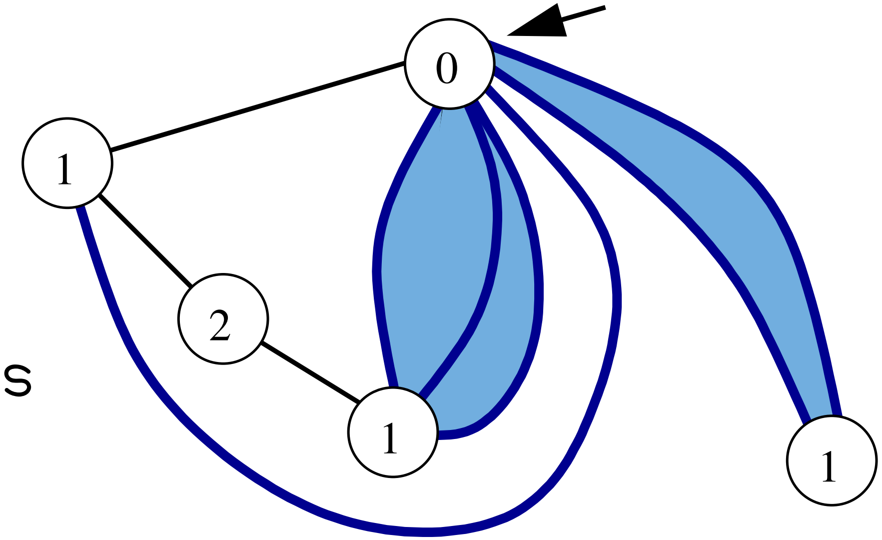


Boundary: 0 -1 0 -1 ... -1 0

An interesting class of labelled maps (à la Dobrushin)

Map with boundary
0 -1 0 -1 0 ... 0 1 0 ... 1 0

d digons



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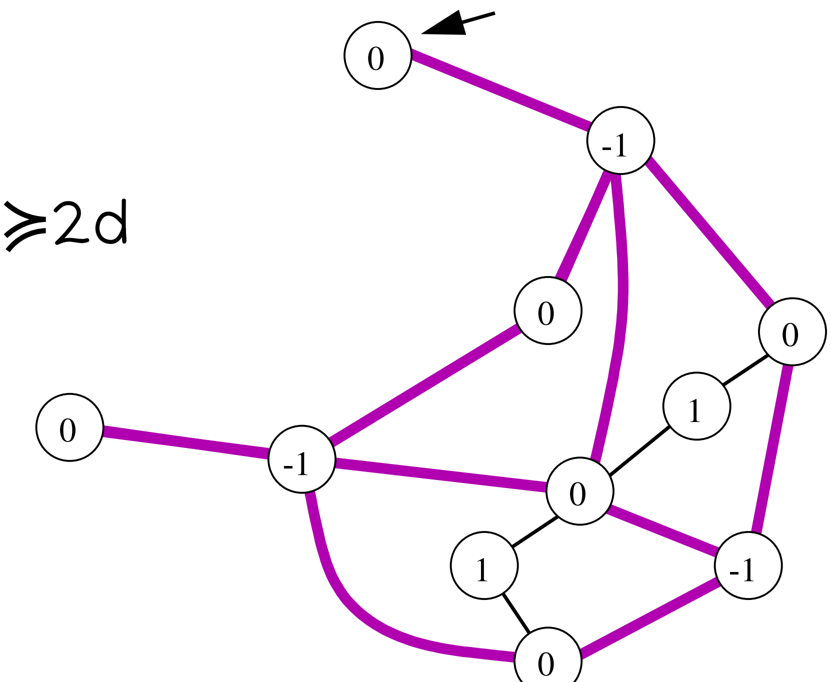
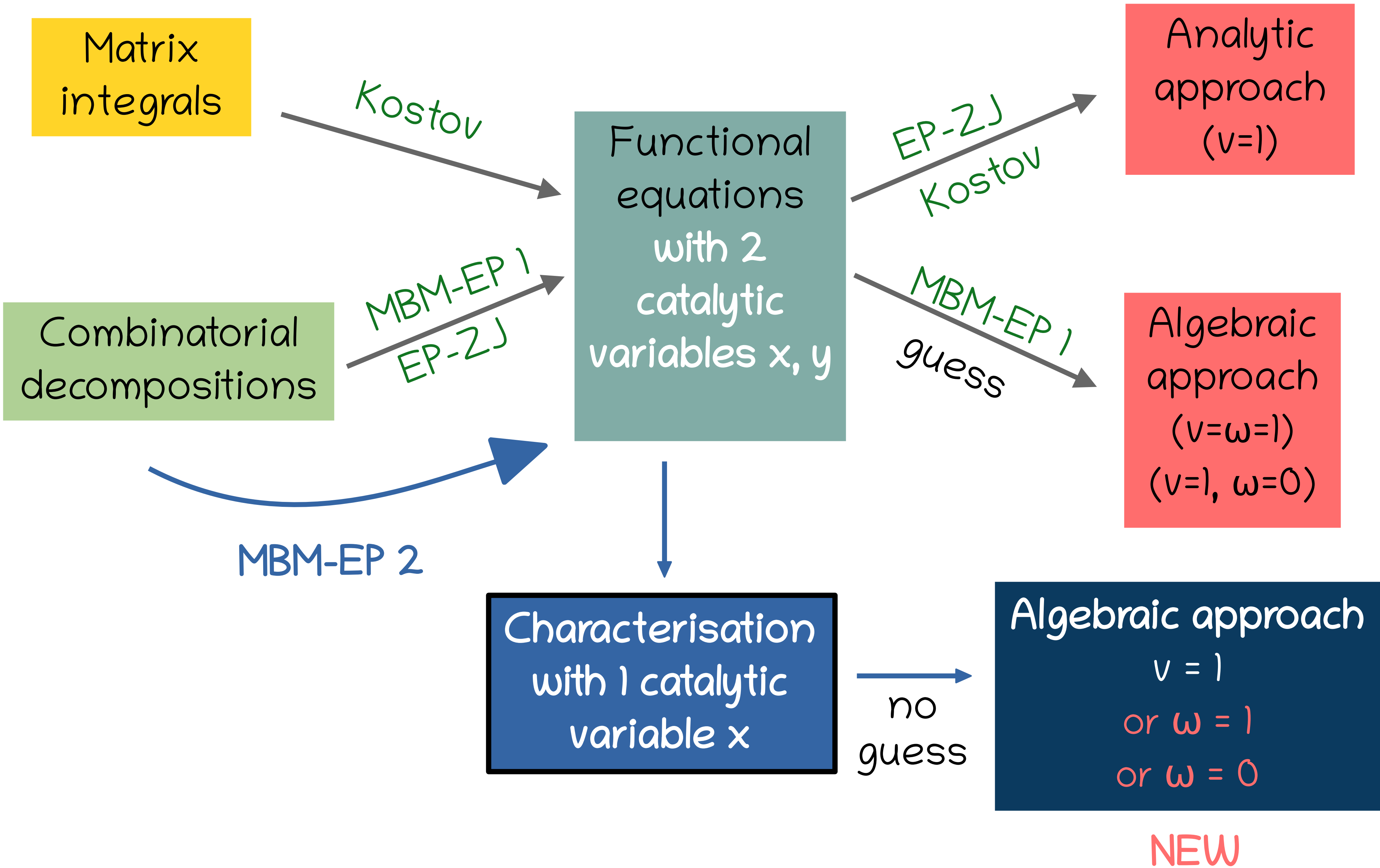


Boundary: 0 -1 0 -1 ... -1 0

$[x^{\geqslant 0}]\, B(1/x)\, P(x)$

Approaches



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A characterisation of the series Q [MBM & EP 24]

There exists a **unique series in t**, with coefficients that are **Laurent series in x** (and polynomials in ω and v), denoted $\mathcal{M}(x)$, such that:

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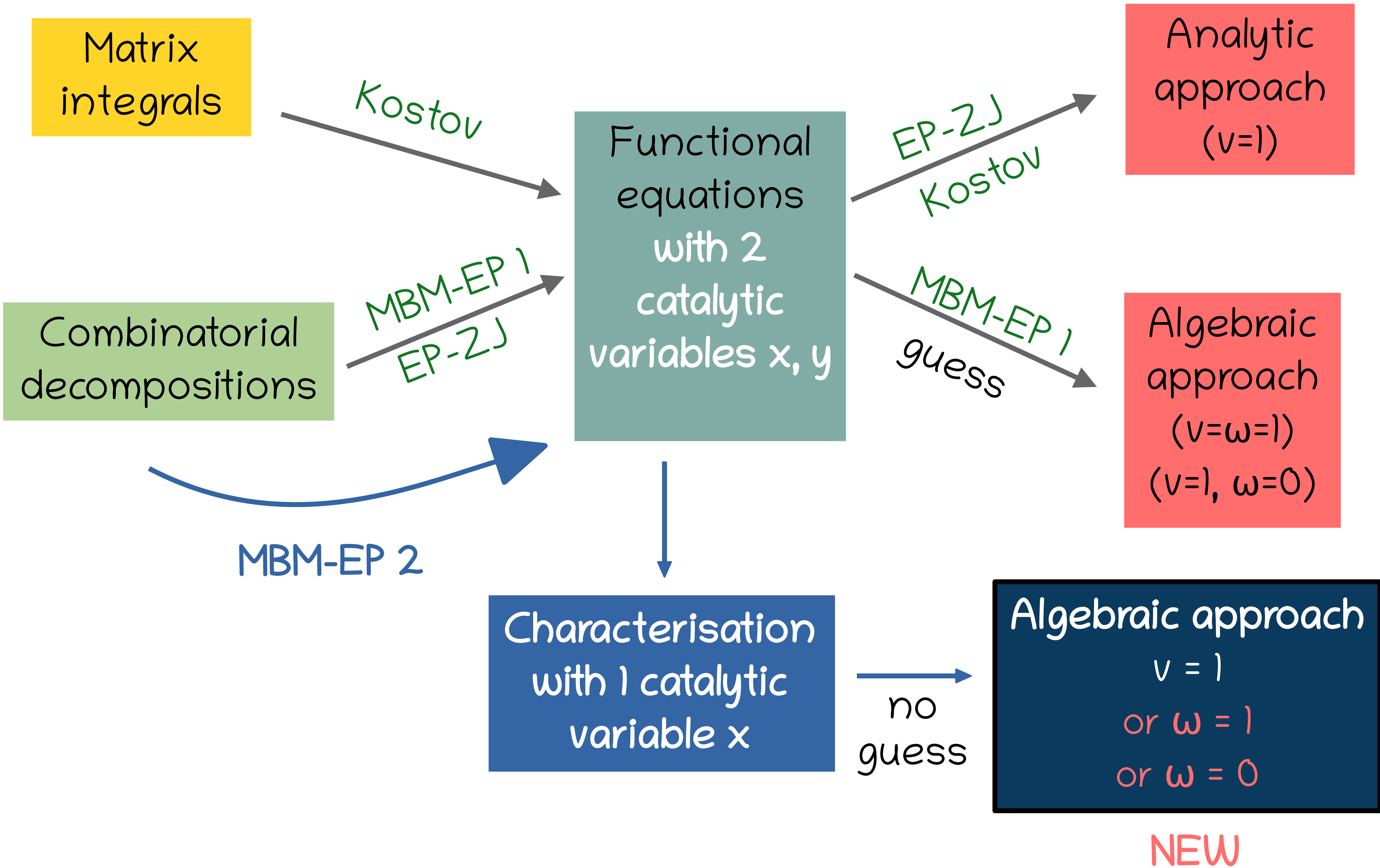
- Then $\mathcal{M}(x)$ has a combinatorial description in terms of labelled maps, and the series counting **labelled quadrangulations** is

$$Q = [x^{-2}]\,\mathcal{M}(x)/t^2 - v.$$

Approaches



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- In fact the above expression **does not depend on x**...

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- In fact the above expression **does not depend on x**...
  and A is then determined by Condition 2.

A new result

The case ω = 1 [MBM & Elvey Price 24]

Let A be the unique series in t such that:

$$t = \sum_{n,k\geq 0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{3n+2k}{n+k} t^k (v-1)^k A^{n+1}.$$

Then the generating function of labelled quadrangulations, counted by faces and local minima, is

$$Q = -v + \frac{1}{t^2} \sum_{n,k\geq 0,\ n+k>0} \frac{1}{n+1}\binom{2n}{n}\binom{2n+k}{k}\binom{3n+2k-1}{2n+k} t^k (v-1)^k A^{n+1}$$

$$= v\,(v+3)\,t + v\,(v+6)\,(2v+3)\,t^2 + v\,(v+1)\left(5v^2+61v+135\right) t^3 + \cdots.$$

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+ similar expression for the case ω = 0 : **Eulerian orientations of general maps**, counted by **edges and vertices**

More results

- **Direct combinatorial proof of the 1-variable characterisation** when $\omega = 0$ and $\nu=1$ → random generation
- For $\omega=0$ and $\omega=1$, a family of trees with the same GF → **bijections ?**
- Simpler solution when $\omega = 2\cos(k\pi/m)$ and $\nu=1$
- Some ingredients of the solution for **general ν and ω.**

- Limit behaviour of the **height of a random vertex** ($\log n$) [Elvey Price]
- Record the number of **vertices of each height j** (and more) [EP]

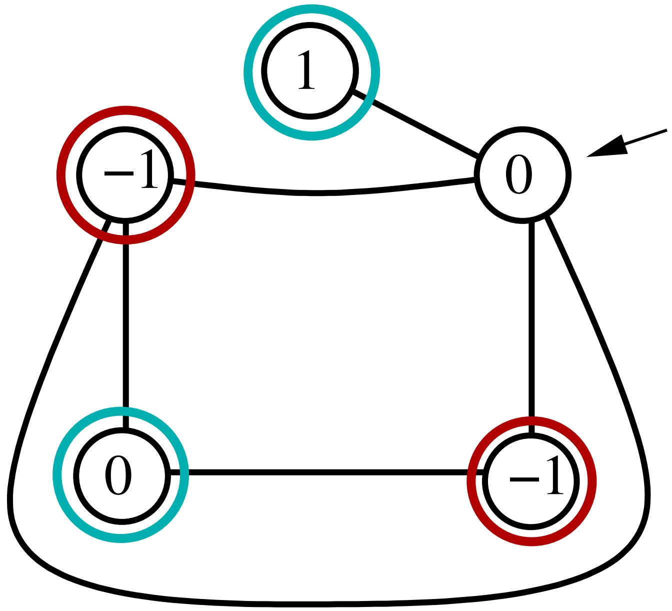
What's the bijection?

Labelled quadrangulations

n faces

m local minima

(M local maxima ?)



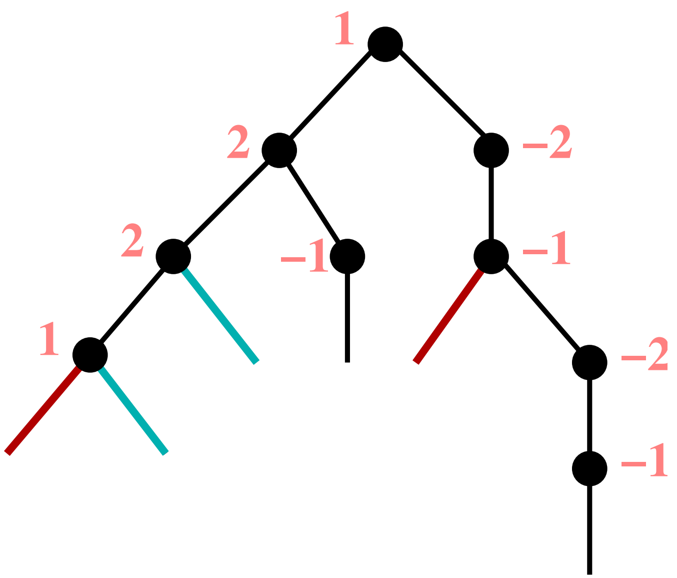
Unary-binary trees of charge 1

No subtree of charge 0

n+2 leaves

m left leaves

(M right leaves ?)



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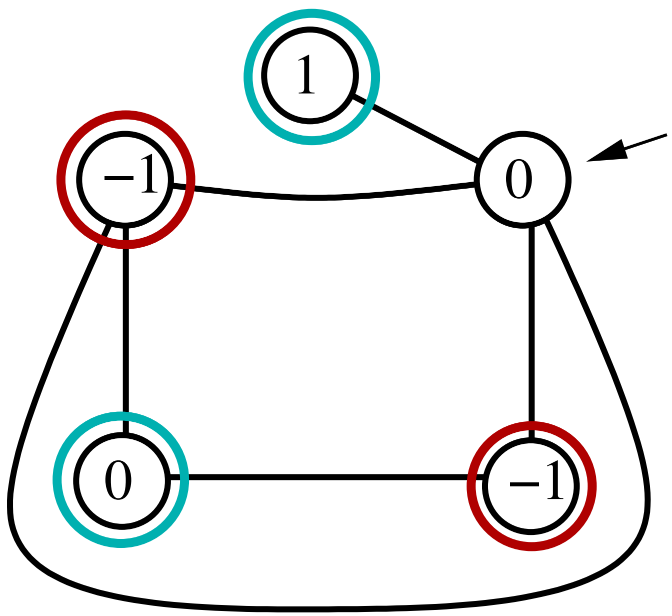
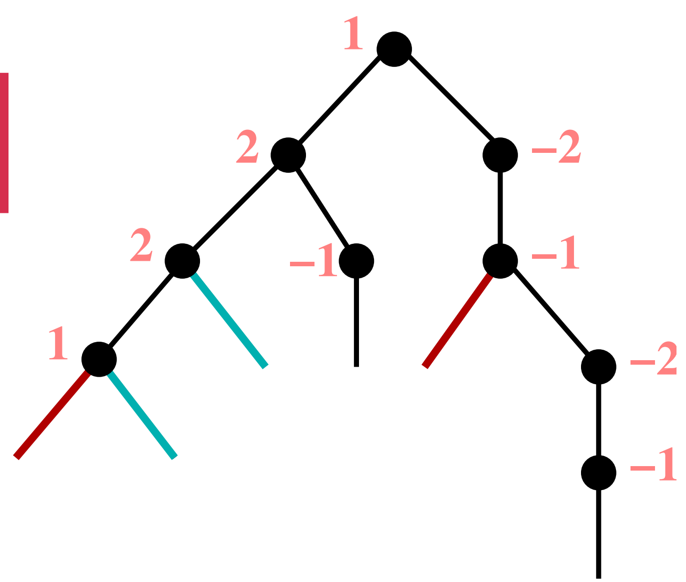


Merci !



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