Higher Rank Dimers

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based on joint work with

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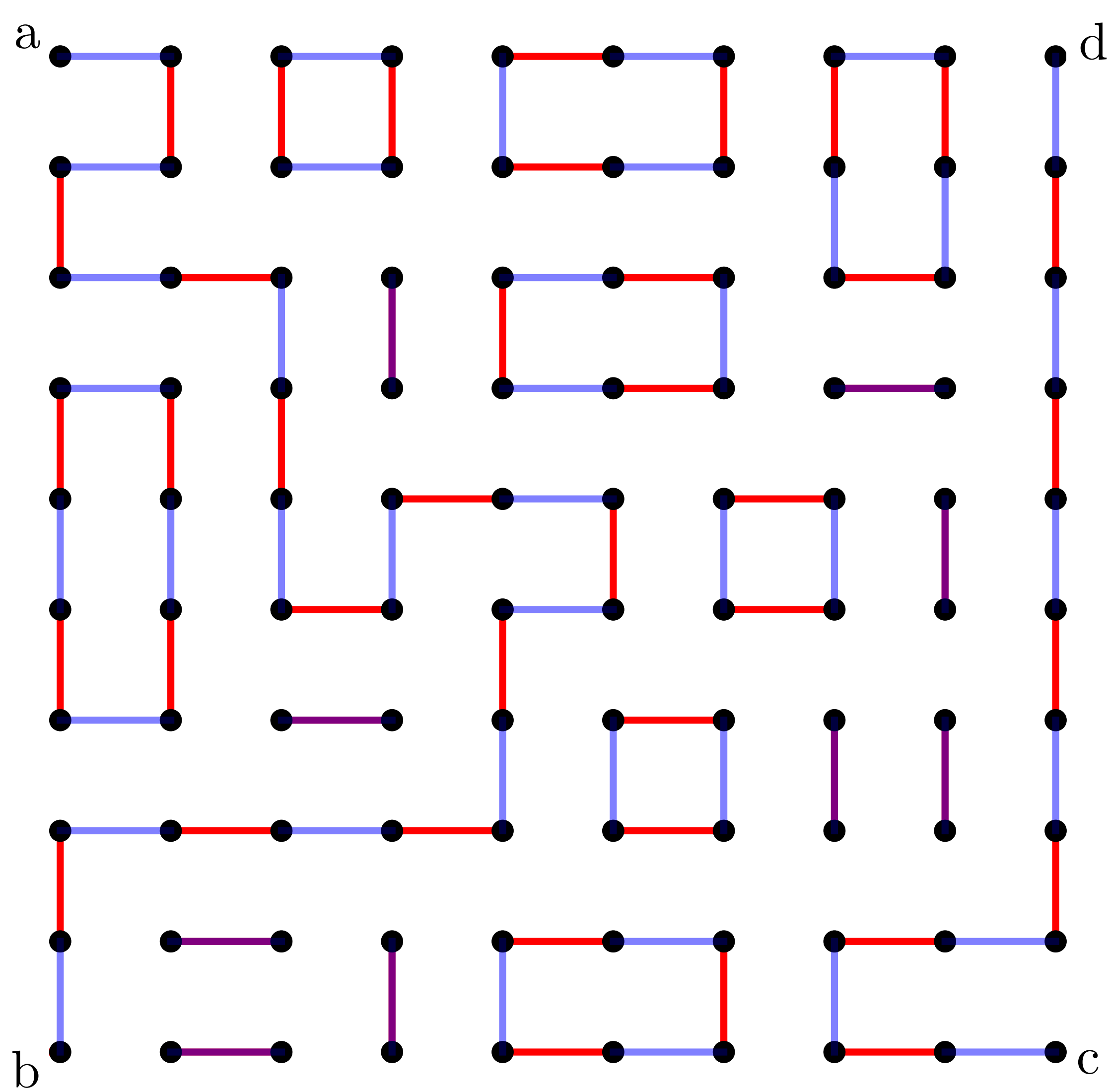
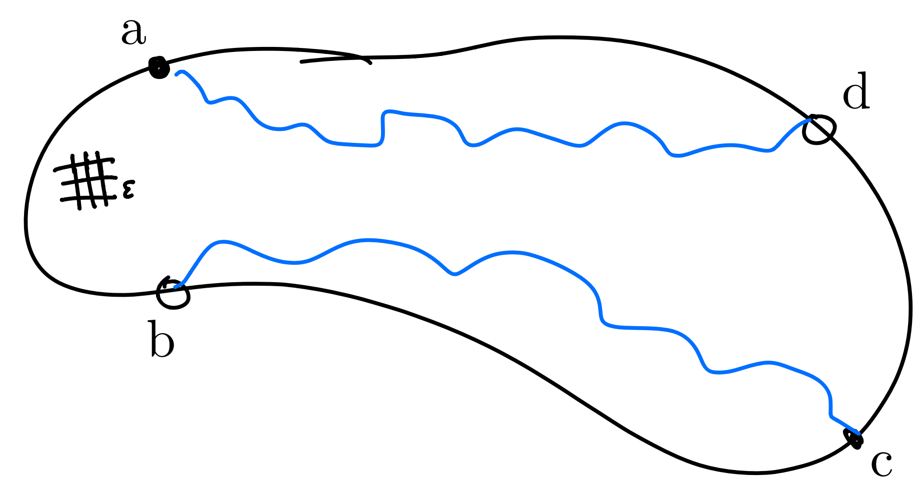
Connection probabilities in double-dimers (**2**-multiwebs)

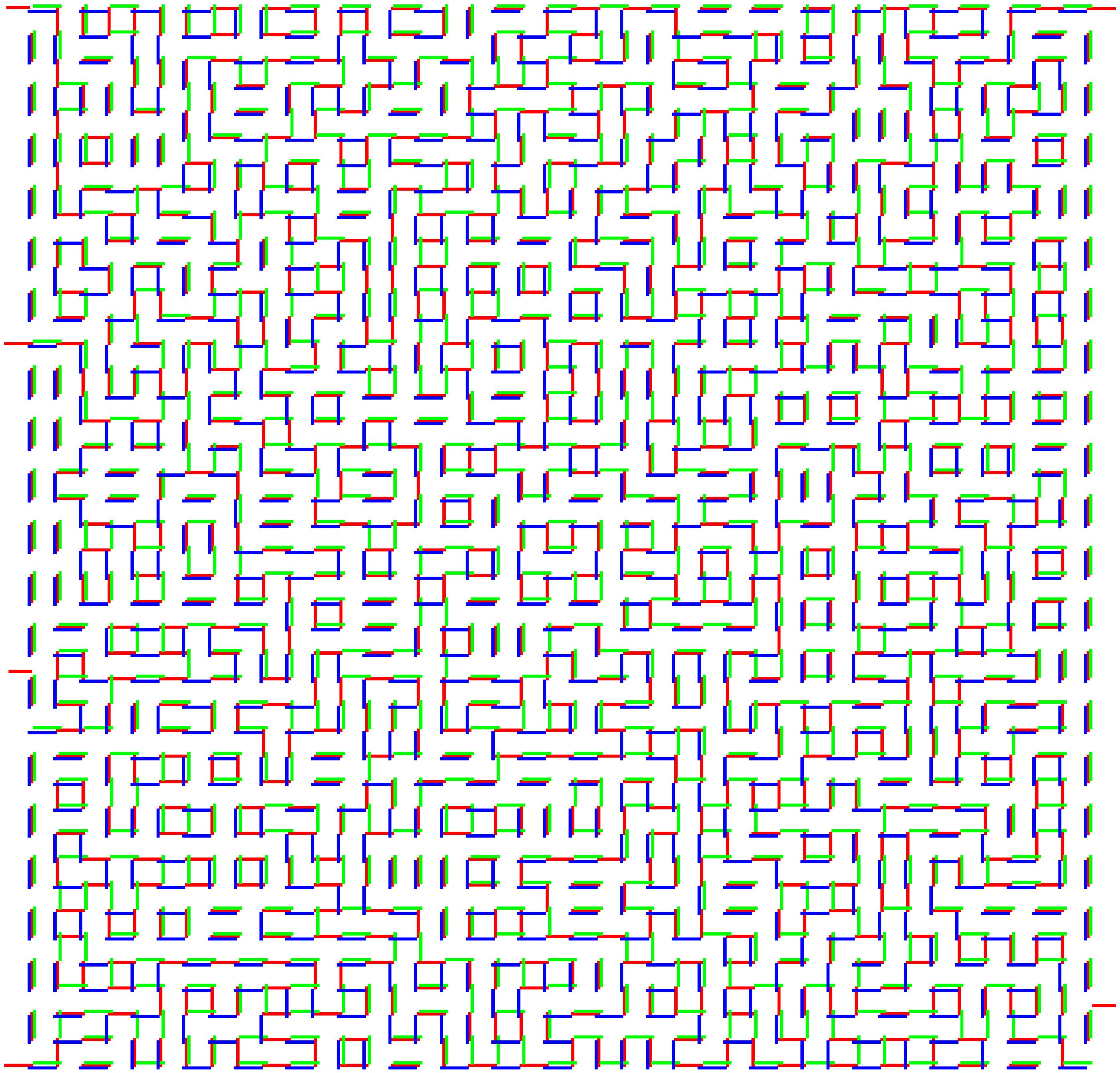




Take two dimer covers of a rectangle, one of which misses the four corners. What is the probability that, in the union, the corner connection goes top-to-bottom?

Thm [K-Wilson '06]: In the scaling limit, for any domain with four boundary points a, b, c, d (with appropriate boundary conditions), the probability is the *cross ratio* of the four image points when the domain is conformally mapped to the upper half plane.

triple dimer model:



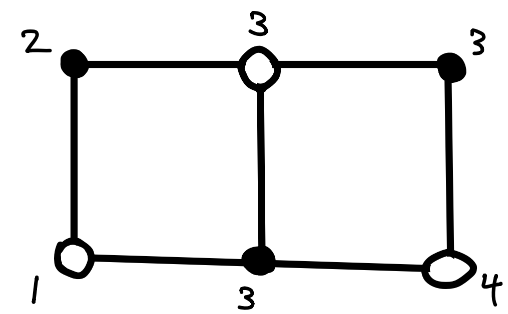
internal structure?

1. Multiwebs
2. Motivation
3. Kasteleyn matrix
4. Traces
5. Theorem
6. Applications

Tensor networks and multiwebs

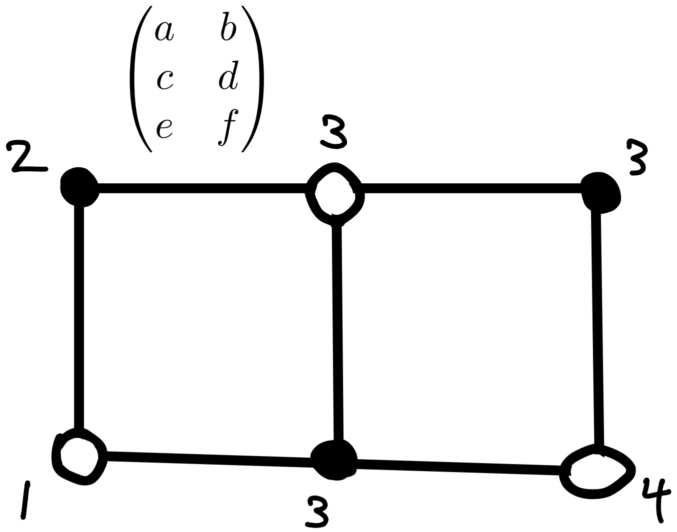
$G = (V, E)$ is a (planar) (bipartite) graph

$\mathbf{n} : V \to \mathbb{N}$ vertex multiplicities



Assign to vertex v a vector space Y_v of dimension n_v.

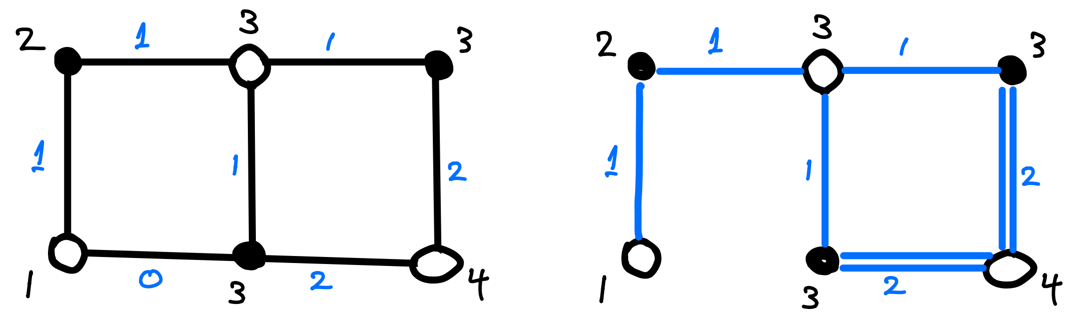
A *quiver representation* is a collection $\Phi = \{\phi_{bw}\}_{bw \in E}$ where $\phi_{bw} : Y_b \to Y_w$.
(connection)



When $\mathbf{n} \equiv n$ and $\phi_{bw} \in \mathrm{GL}_n$, we have a *$GL_n$-local system*. Or *$GL_n$-connection*.

Multiwebs

An **$\mathbf{n}$**-*multiweb* m in G is a function $m : E \to \mathbb{Z}_{\geq 0}$ summing to n_v at each vertex v:



$\Omega_{\mathbf{n}}$ is the set of $\mathbf{n}$-multiwebs.

(We need $\sum_{w \in W} n_w = \sum_{b \in B} n_b$ in order for $\Omega_{\mathbf{n}}$ to be nonempty.)

and some inequalities

Ex: For $\mathbf{n} \equiv 1$, $\Omega_1 = \{\text{dimer covers}\}$

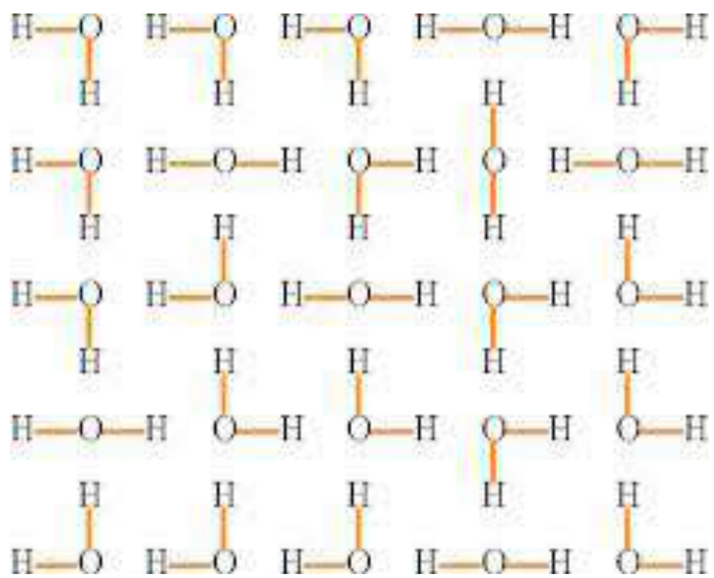
We define a *trace* function $\text{Tr} : \Omega_{\mathbf{n}} \to \mathbb{R}$ (later)

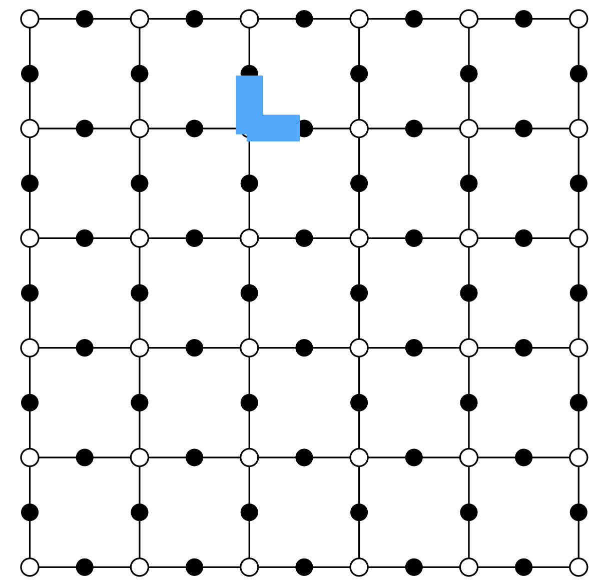
Thm:[Douglas, K, Shi '23], [K, Ovenhouse '23] We have

$$\det K(\Phi) = \pm \sum_{m \in \Omega_{\mathbf{n}}} \text{Tr}(m).$$

Vertex models

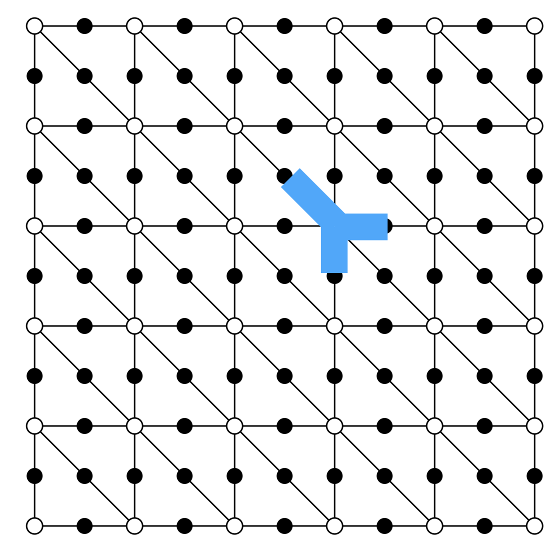
Six vertex model/ Square ice model





Put $n_w = 2$ and $n_b = 1$

20-vertex model



$n_w \equiv 3$

$n_b \equiv 1$

The "free fermionic" points of these models are determinantal tensor networks.

Dimers and Kasteleyn theory

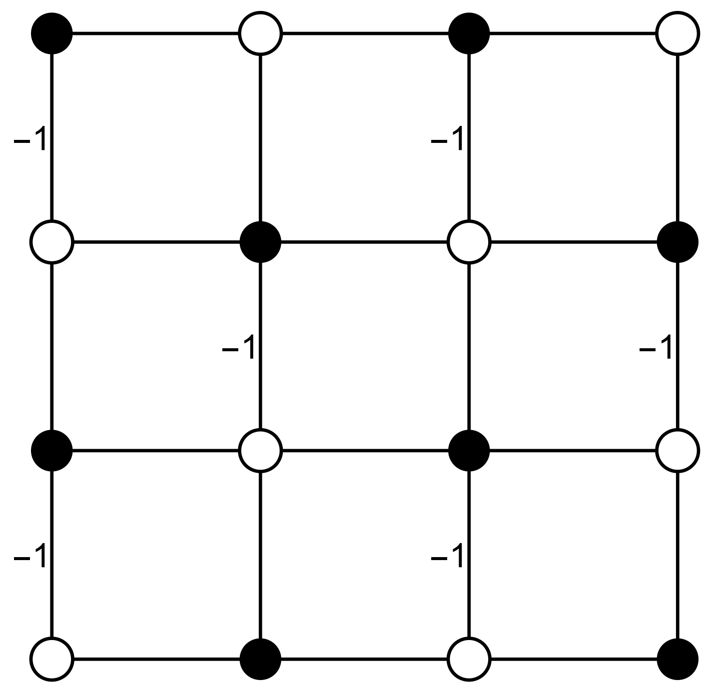
Let G be a planar, bipartite graph.

Let K be the Kasteleyn matrix: $K : \mathbb{C}^B \to \mathbb{C}^W$

"Adjacency matrix with Kasteleyn connection"

$$K_{wb} = \begin{cases} \pm 1 & w \sim b \\ 0 & \text{else.} \end{cases}$$

where a face of length l has monodromy $(-1)^{l/2+1}$.



Kasteleyn, Temperley/Fisher (1963) proved

Thm: $|\det K| = \#\{\text{dimer covers}\}$

For multiwebs:

Thm:[Douglas, K, Shi '23], [K, Ovenhouse '23] We have

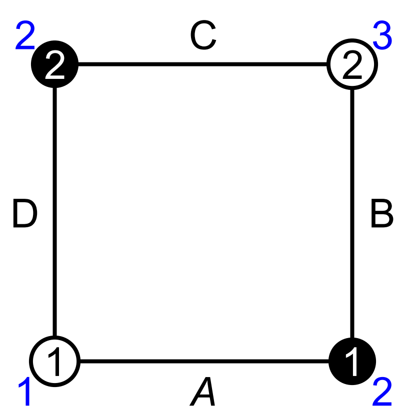
$$\det \tilde{K}(\Phi) = \pm \sum_{m \in \Omega_{\mathbf{n}}} \mathrm{Tr}(m).$$

We define a Kasteleyn matrix $K(\Phi)$ on G:

$$K(w,b) = \begin{cases} \pm\phi_{wb} & w \sim b \\ 0 & \text{else.} \end{cases}$$

"tensor Φ with the Kasteleyn connection."

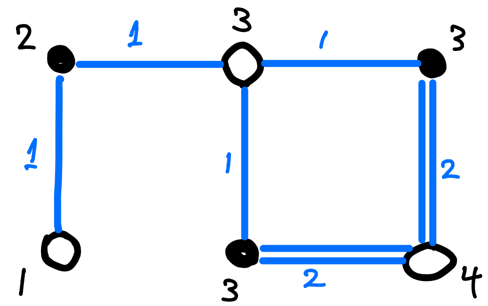
Ex.



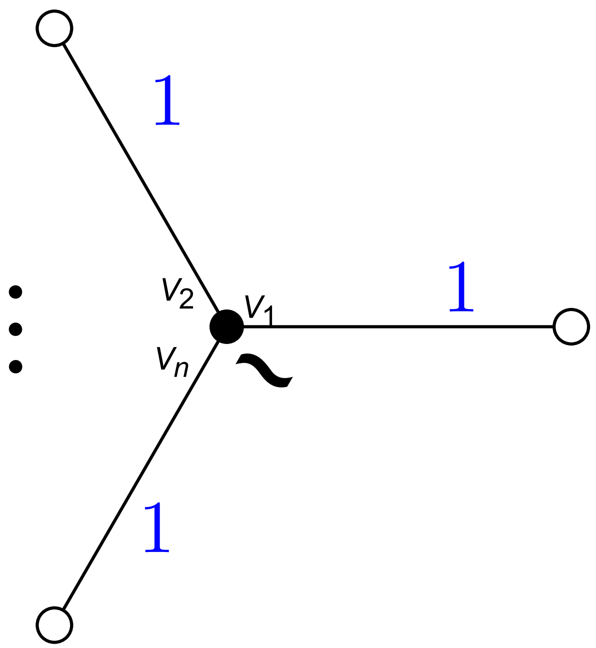
$$K(\Phi) = \begin{pmatrix} A & -D \\ B & C \end{pmatrix}$$

$$\tilde{K}(\Phi) = \begin{pmatrix} a_1 & a_2 & -d_1 & -d_2 \\ b_{11} & b_{12} & c_{11} & c_{12} \\ b_{21} & b_{22} & c_{21} & c_{22} \\ b_{31} & b_{32} & c_{31} & c_{32} \end{pmatrix}$$

Trace of an **n**-multiweb



First assume $m_e = 0$ or 1 for all edges



$V_i \cong \mathbb{R}^n$ with basis $e_1, \ldots, e_n$

Define $v_b \in V_1 \otimes \cdots \otimes V_n$ by

$$v_b = \sum_{\sigma \in S_n} (-1)^\sigma e^1_{\sigma(1)} \otimes \cdots \otimes e^n_{\sigma(n)}$$

the "codeterminant"

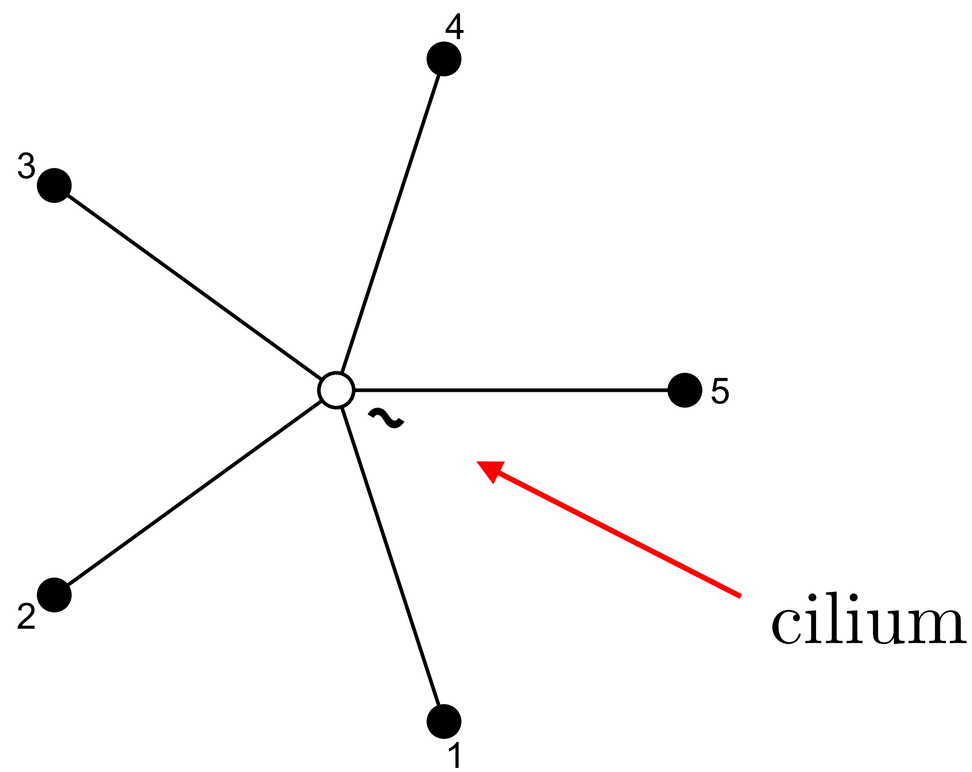
invariant under SL_n-base change

Similarly define v_w using $(\mathbb{R}^{n_w})^*$.

Then define

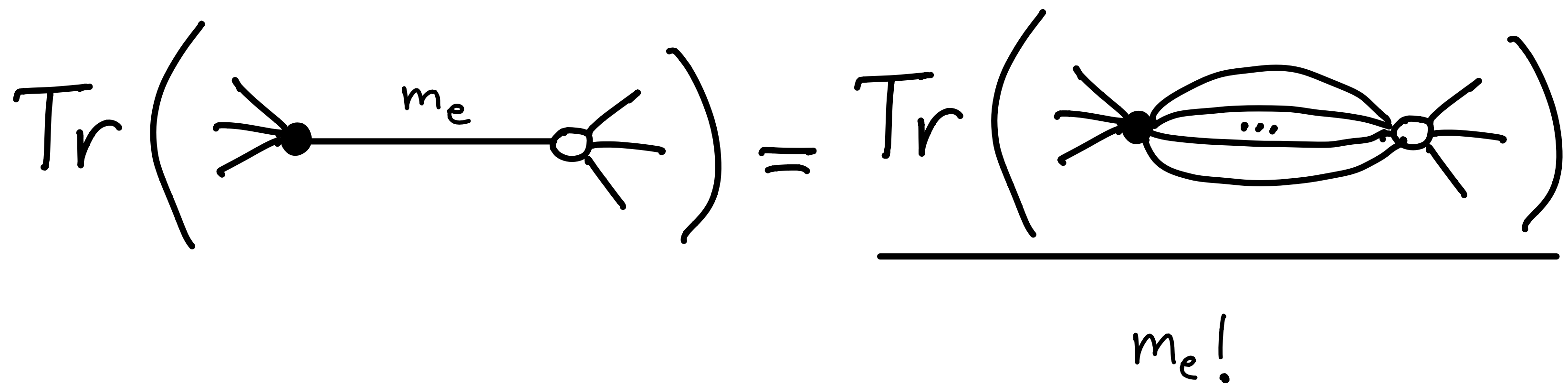
$$Tr(m) = \left\langle \bigotimes_{w \in W} v_w \middle| \bigotimes_{e=wb} \phi_{wb} \middle| \bigotimes_{b \in B} v_b \right\rangle$$

We need a linear order of the edges out of each vertex: use the circular order, plus a starting edge, at black vertices, and the anticircular order, plus starting edge, at white vertices.



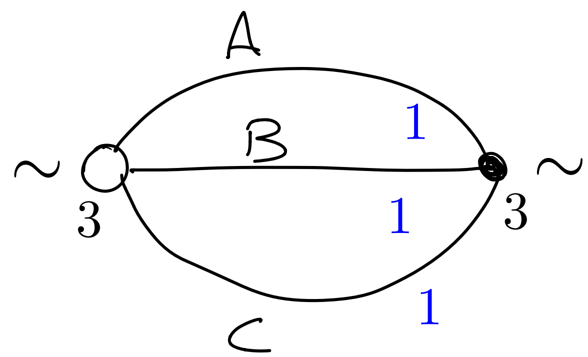
The **sign** of the trace will depend on this choice of linear order.

If edges have multiplicity > 1:



Trace example

$n \equiv 3$



V basis e_1, e_2, e_3

V^* basis f_1, f_2, f_3

$$v_b = e_1 \otimes e_2 \otimes e_3 - e_1 \otimes e_3 \otimes e_2 + \cdots - e_3 \otimes e_2 \otimes e_1$$

$A_{11}B_{22}C_{33}$ $\qquad$ $-A_{11}B_{23}C_{32}$

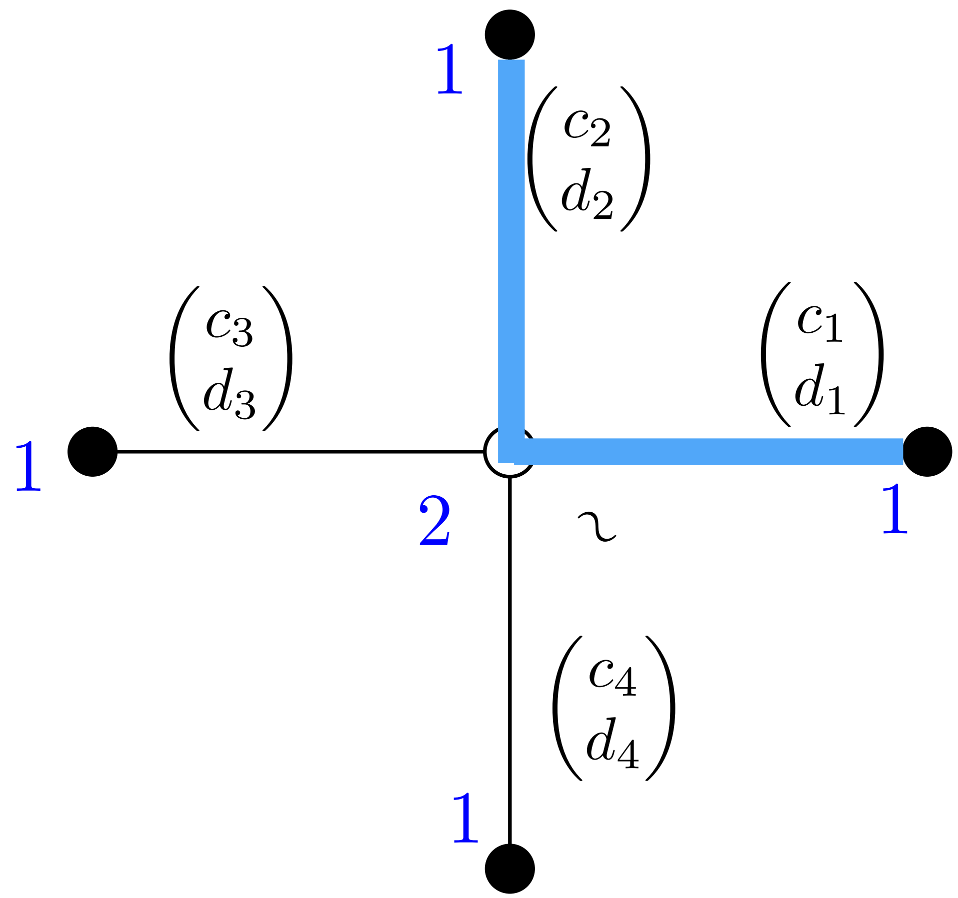
$$v_w = f_1 \otimes f_2 \otimes f_3 - f_1 \otimes f_3 \otimes f_2 + \cdots - f_3 \otimes f_2 \otimes f_1$$

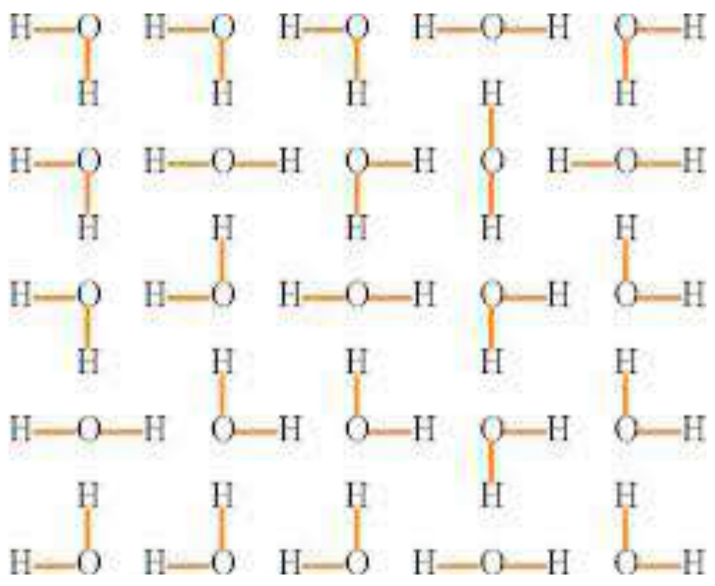
$$Tr(m) = A_{11}\underbrace{B_{22}C_{33} + \cdots + A_{33}}_{\text{36 terms}}B_{22}C_{11}$$

$$Tr(m) = Tr(AB^{-1})Tr(CB^{-1}) - Tr(AB^{-1}CB^{-1}) \qquad \text{if } A, B, C \in SL_3$$

$$= [xyz]\det(xA + yB + zC) \qquad \text{for general } A, B, C$$

Ex 2. square ice model

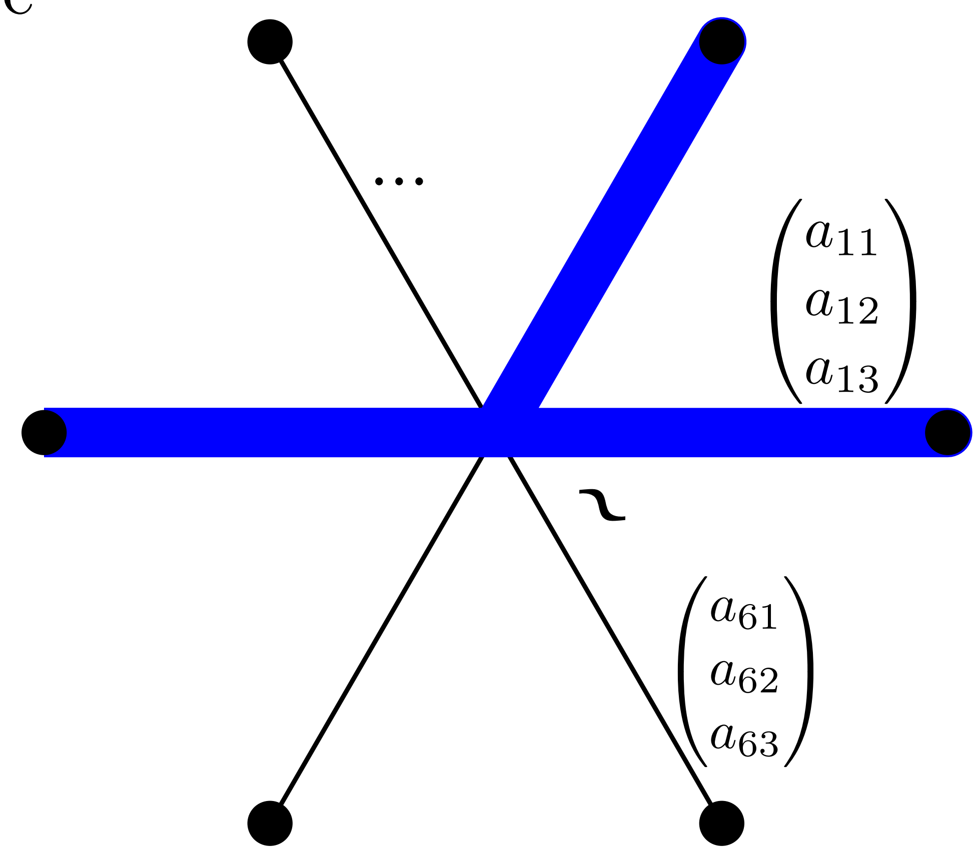




$$Tr = \det \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = c_1 d_2 - c_2 d_1$$

Note: all traces > 0 iff $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \in Gr_{2,4}^{+}$.

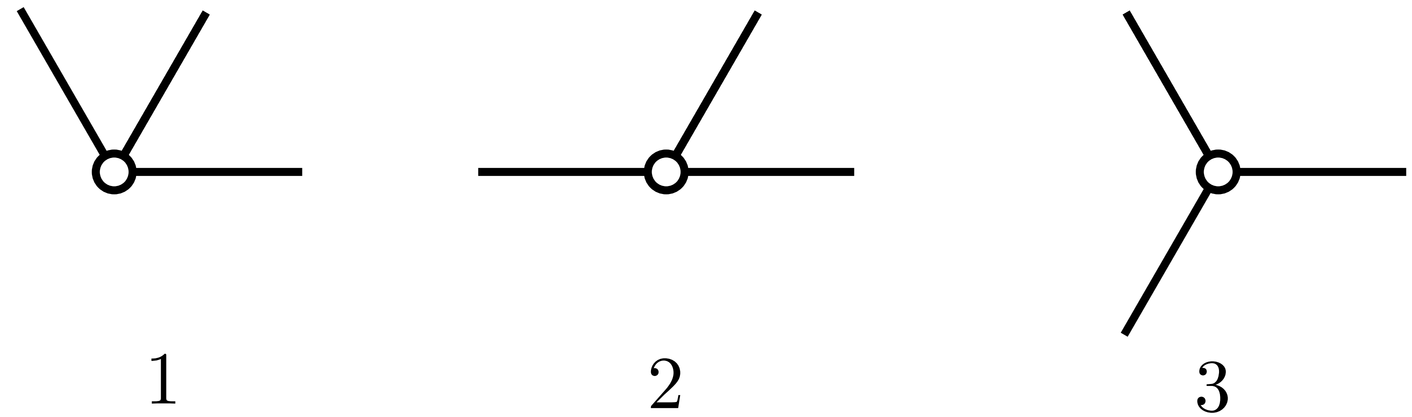
Ex 3. Triangular ice

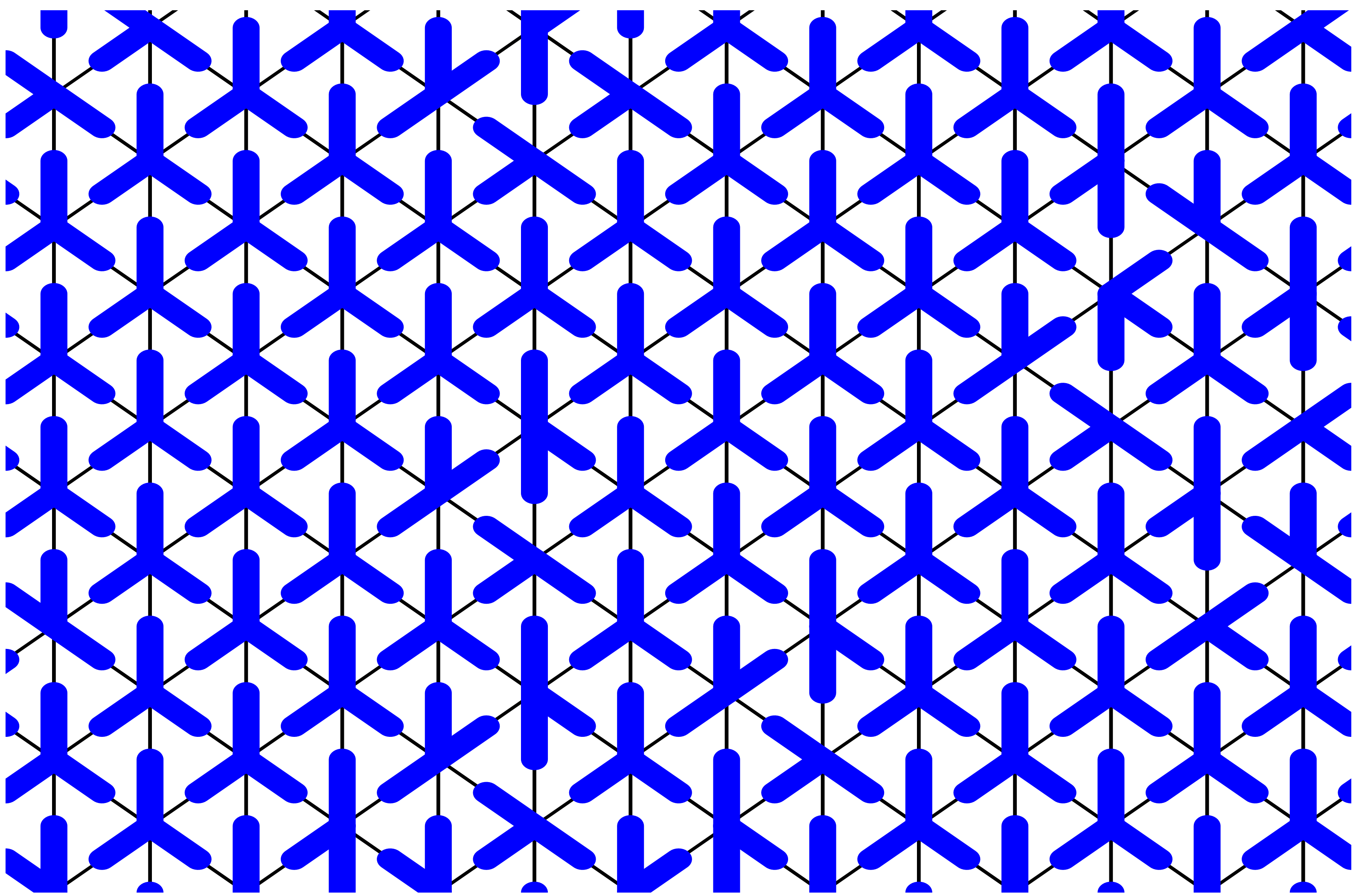


$$Tr = \det \begin{pmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{13} & a_{23} & a_{43} \end{pmatrix} = I_{124}$$

$$\text{all traces positive if } \begin{pmatrix} a_{11} & & a_{61} \\ a_{12} & \dots & a_{62} \\ a_{13} & & a_{63} \end{pmatrix} \in Gr^+_{3,6}.$$

symmetric free fermionic weights





Non-bipartite graphs

Thm [K', Wu(24+)] Let G be a (not-necessarily bipartite) planar graph. Let $\mathbf{n} \equiv 2n$ and let Φ be an $\mathrm{Sp}(2n)$ local system.

$$\mathrm{Pf}\tilde{H} = \pm \sum_{m \in \Omega_{2n}} Tr(m).$$

$Sp(2n)$ is the group of $2n \times 2n$ matrices M such that $M^t J M = J$ where

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

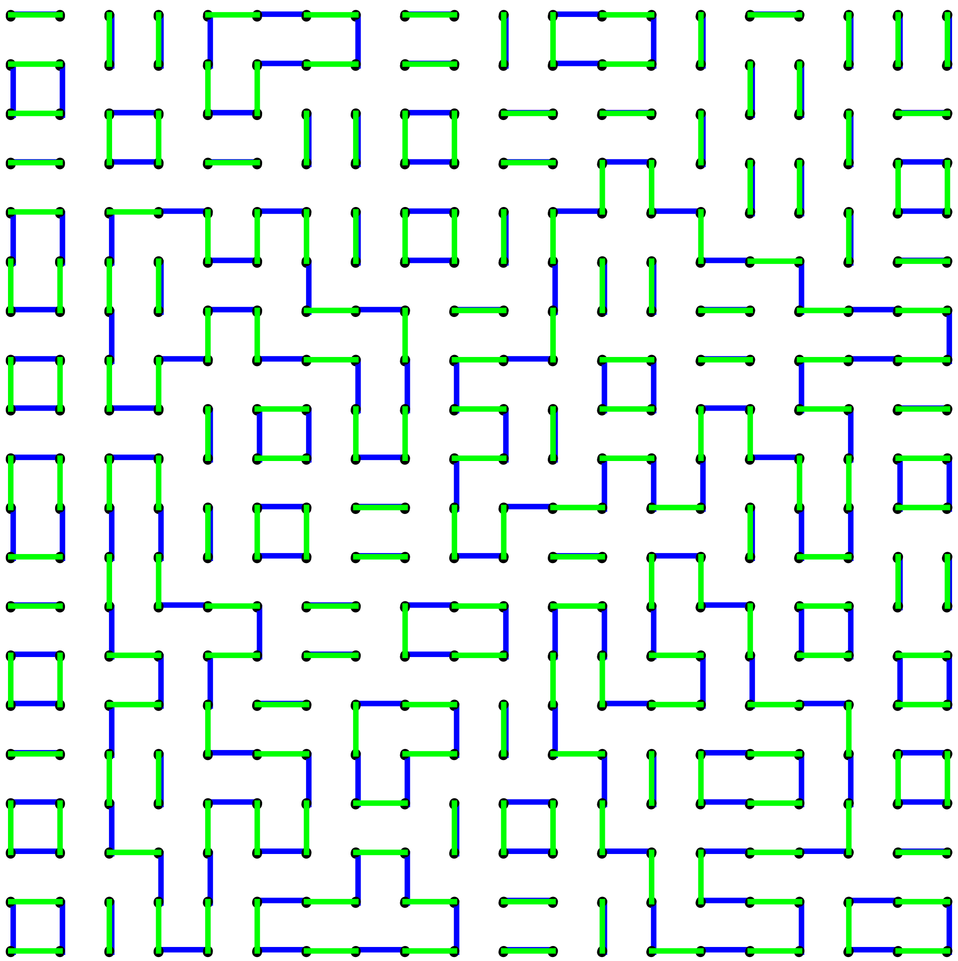
Here $H_{uv} = J\phi_{uv}$.

Note $\quad H_{uv} = J\phi_{uv} = J\phi_{vu}^{-1} = \phi_{vu}^t J = -(J\phi_{vu})^t = -(H_{vu})^t.$

We can tensor with the $Sp(2n)$-*Kasteleyn-connection* to count webs "positively".

(monodromy J^{l-2} around faces of degree l)

Application **2**-multiwebs with SL_2 connection



For a **2**-multiweb $m \in \Omega_{\mathbf{2}}$, we have

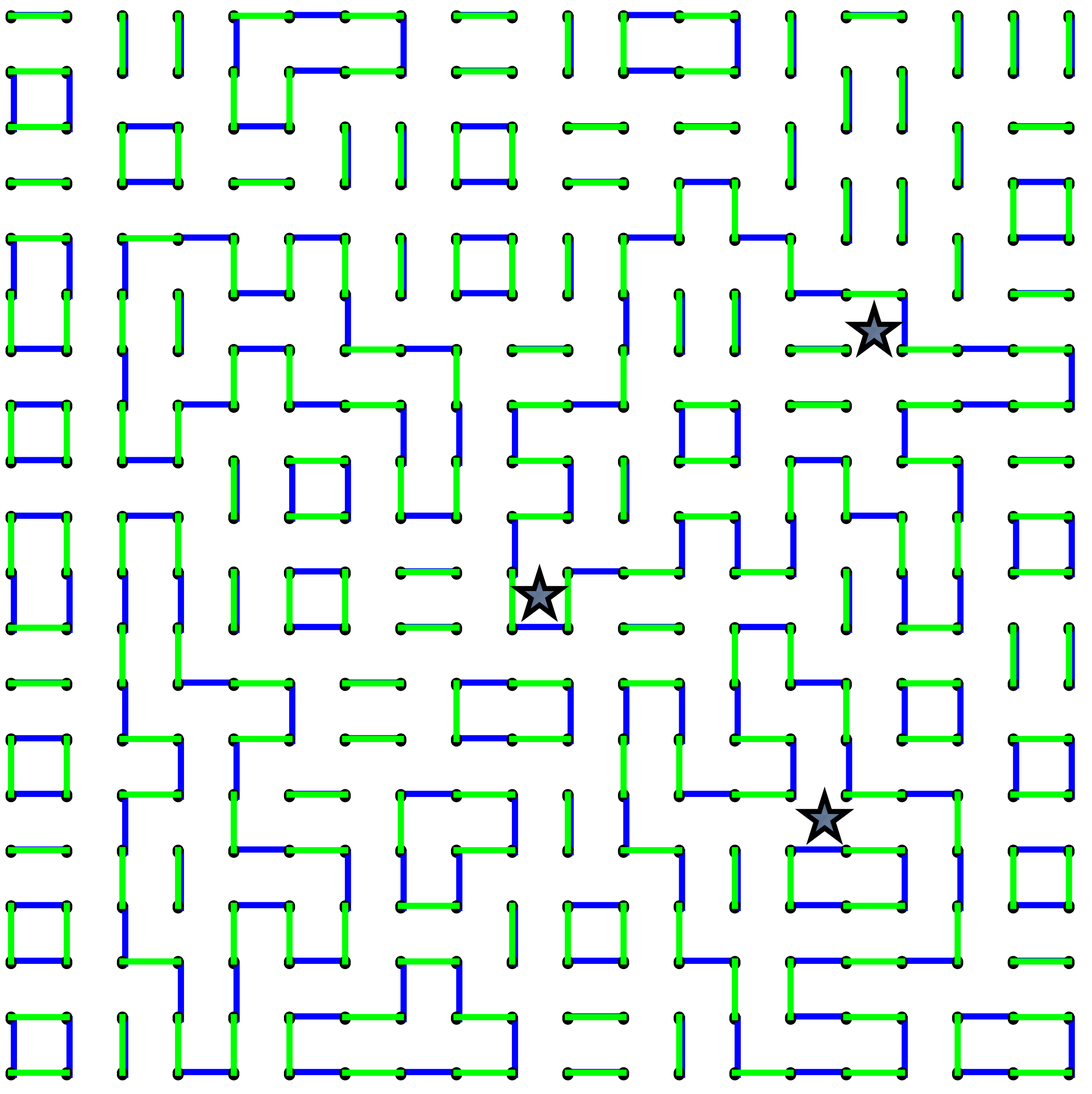
$$Tr(m) = \prod_{\text{loops } \gamma \text{ of } m} Tr(\phi_\gamma)$$

monodromy of the connection around γ

When $\Phi \equiv I$ we define

$$Pr(m) := \frac{Tr_I(m)}{Z} = \frac{2^{\#\text{ loops}}}{Z}$$

Now puncture some faces:



Q. What is the probability that a 2-multiweb has a given isotopy class?

Let $\Phi = \{\phi_e\}$ be a *flat* SL_2-connection on G.

trivial monodromy around contractible cycles

For a **2**-multiweb $m \in \Omega_{\mathbf{2}}$, we have

$$Tr(m) = \prod_{\text{loops } \gamma \text{ of } m} Tr(\phi_\gamma) \quad = 2^{\#\text{ loops}} \prod_{\substack{\text{noncontractible} \\ \text{loops } \gamma}} \frac{1}{2} Tr(\phi_\gamma).$$

The trace "detects" the homotopy type of the loops

$$\pm \det \tilde{K}(\Phi) = \sum_{m \in \Omega_{\mathbf{2}}} Tr(m) = \sum_{\lambda \in \Lambda_2} C_\lambda Tr(\lambda)$$

where λ runs over isotopy classes of simple closed curve systems.

$$\frac{C_\lambda}{Z} = Pr(m \text{ has isotopy class } \lambda).$$

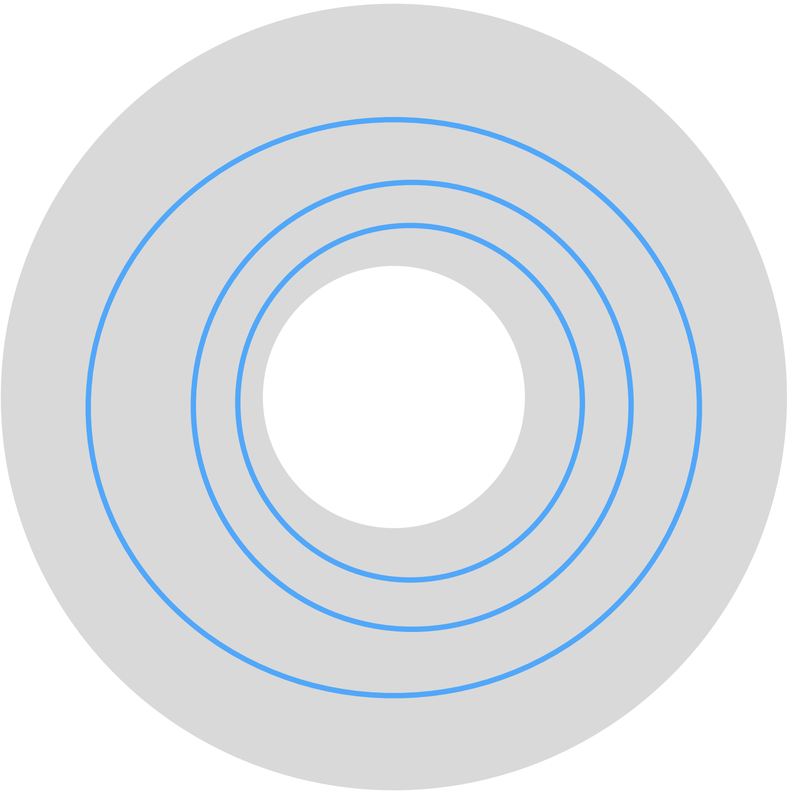
$\det \tilde{K}(I)$

Thm:[Fock-Goncharov '13]: Traces of simple closed curve systems $\lambda \in \Lambda_2$ form a basis for regular functions on the SL_2-character variety.

Cor: C_λ is determined by $K(\Phi)$.

Open question: How to extract C_λ?

Example: annulus



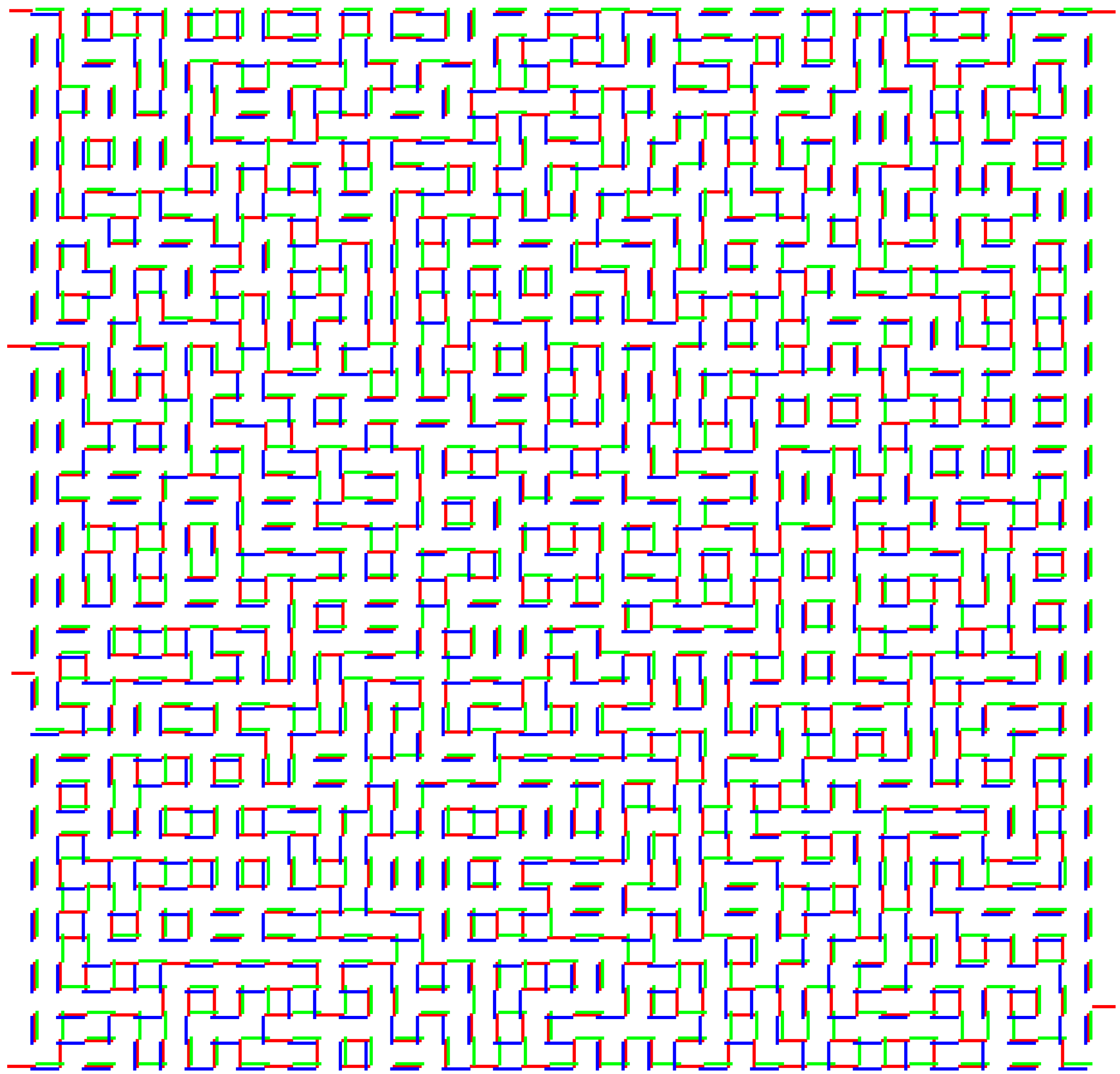
with appropriate boundary conditions,
in limit of mesh $\to 0$,
the distribution only depends
on the conformal modulus.

$\tau =$ conformal modulus

$q = e^{-\pi\tau}$

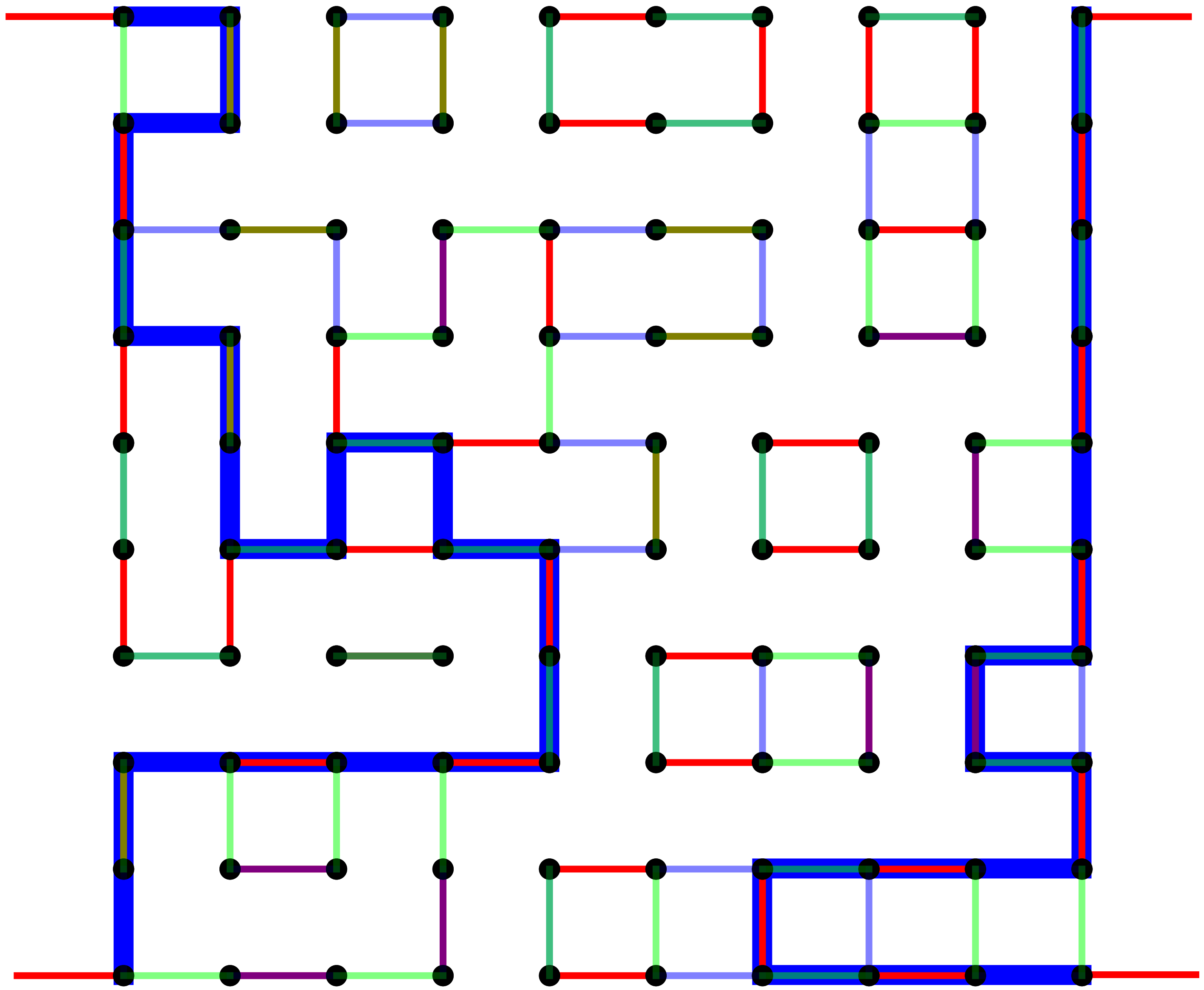
$$\sum_{j=0}^{\infty} Pr(j \text{ curves}) u^j = \prod_{k=0}^{\infty} \frac{(1 + 2uq^{2k+1} + q^{4k+2})^2}{(1 + q^{2k+1})^4}$$

SL_3 application: $\mathbf{n} \equiv 3$



internal structure?

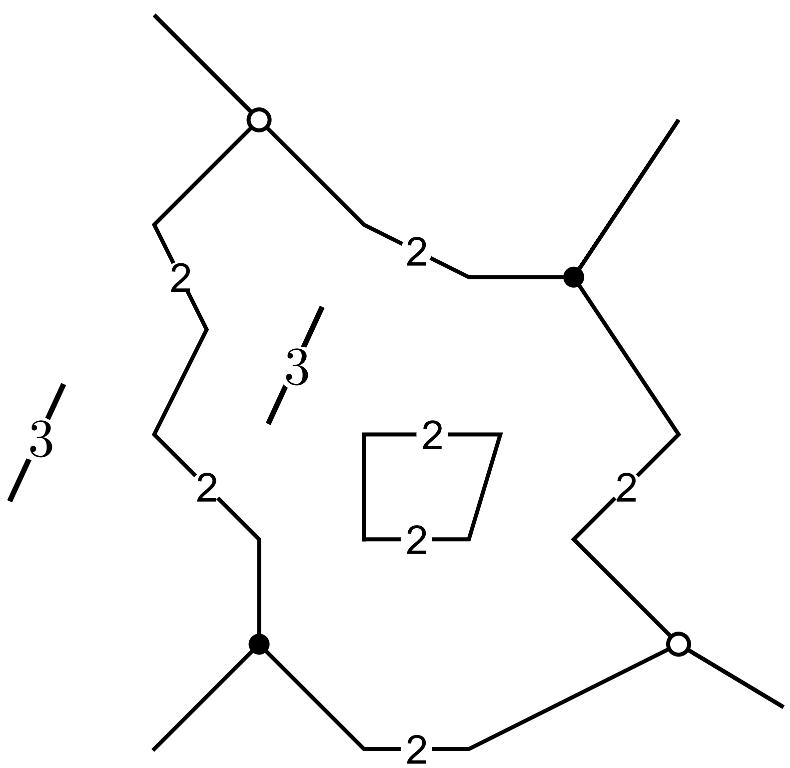
Idea: a colored 3-multiweb has a *spine*: a reduced web “inside” it.



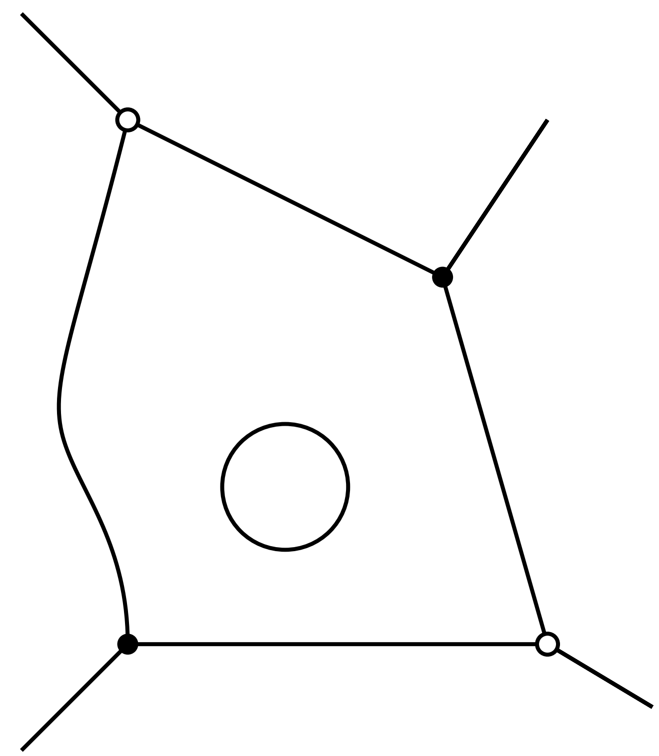
The spine is not canonical, but its topological type *is* canonical.

SL_3 application: $\mathbf{n} \equiv 3$

from 3-multiwebs to 3-webs



multiweb

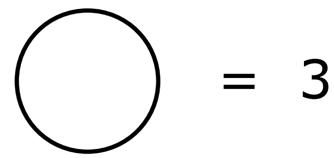


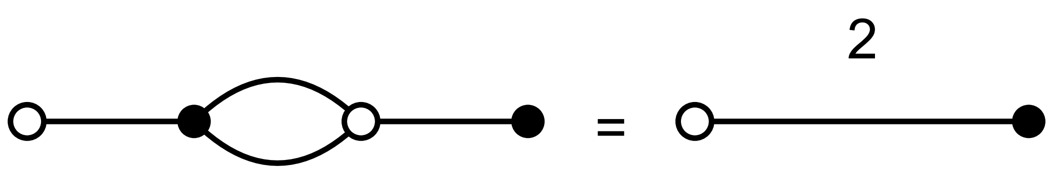
web

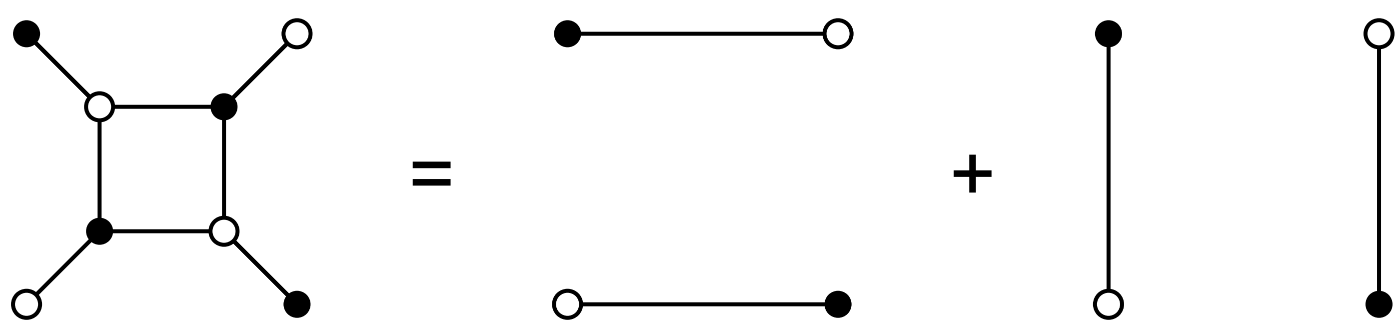
A 3-multiweb or web is *reduced* (nonelliptic) if there are no contractible faces of degree < 6.

Thm[Sikora-Westbury] Traces of reduced (i.e. nonelliptic) webs form a basis for regular functions on the SL_3-character variety.

Web reductions (skein relations) $n = 3$:







Lemma: For a 3-multiweb m on a graph on a surface with a flat SL_3-connection

$$Tr(m) = \sum_{\lambda \in \Lambda_3} C_{\lambda,m} Tr(\lambda)$$

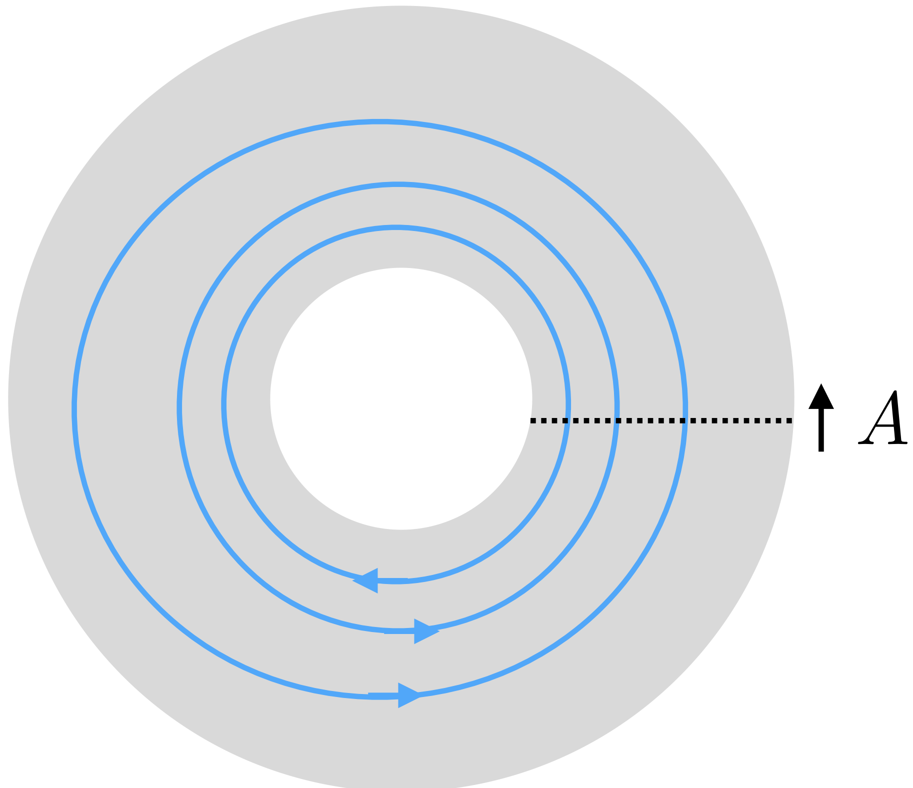
where the sum is over (isotopy classes of) reduced webs λ.

Consequently

Thm: $\det(\tilde{K}(\Phi)) = \sum_{\lambda \in \Lambda_3} C_\lambda Tr(\lambda)$ where the C_λ are functions of $\det \tilde{K}(\Phi)$.

isotopy classes of reduced webs

Example. On an annnulus, every reduced 3-multiweb is a union of noncontractible "oriented" loops.



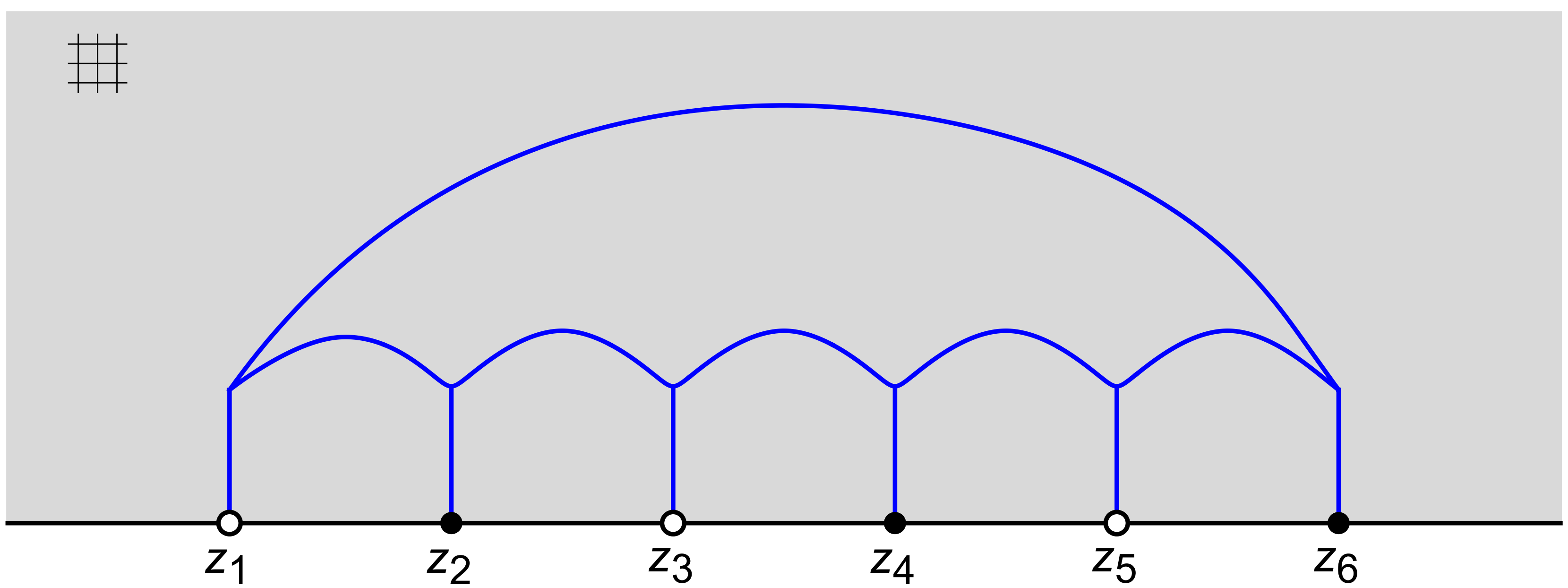
$$\det \tilde{K}(A) = \sum_{\lambda\in\Lambda_3} C_\lambda Tr(\lambda) = \sum_{i,j\geq 0} C_{i,j}(TrA)^i(TrA^{-1})^j.$$

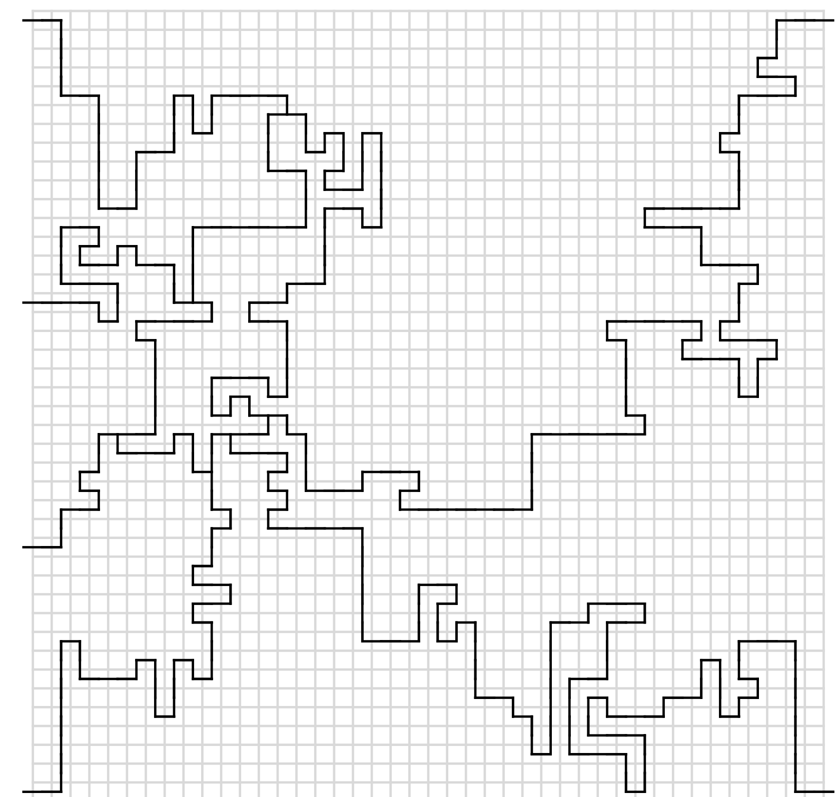
Prop: In the scaling limit on the annulus, $q = e^{-\pi\tau}$

$$\sum_{i,j\geq 0} C_{i,j}u^iv^j = C' \prod_{j=1}^{\infty}(1+uq^j+vq^{2j}+q^{3j})(1+vq^j+uq^{2j}+q^{3j})$$

$u = Tr(A), v = Tr(A^{-1})$

On a planar graph with boundary, one can compute probabilities of various reduced webs:

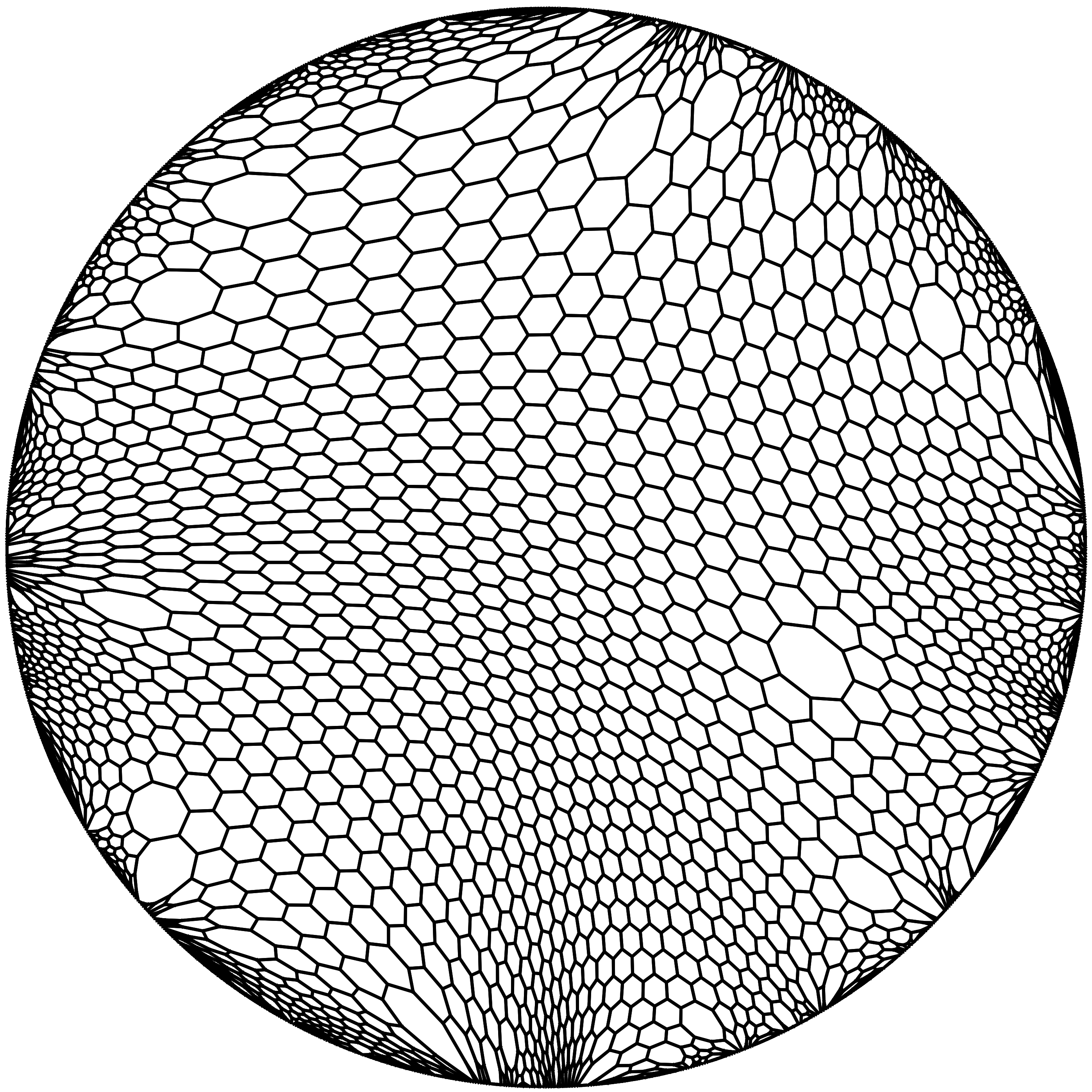




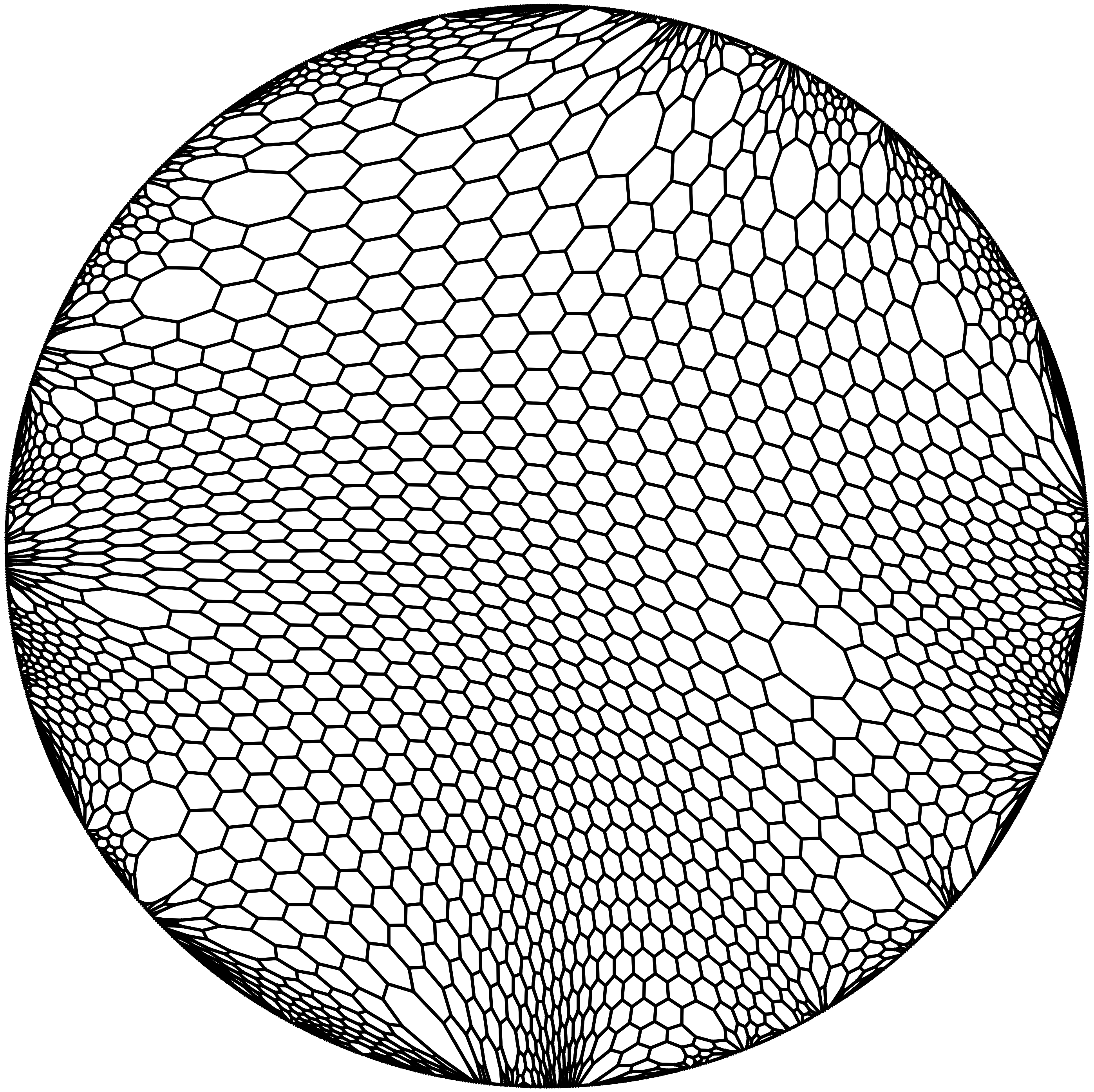
In scaling limit,

$$\mathrm{Pr} = \frac{2(z_2-z_1)(z_3-z_2)(z_4-z_3)(z_5-z_4)(z_6-z_5)(z_6-z_1)}{(z_3-z_1)(z_4-z_2)(z_5-z_3)(z_6-z_4)(z_5-z_1)(z_6-z_2)}$$

Happy birthday Philippe!



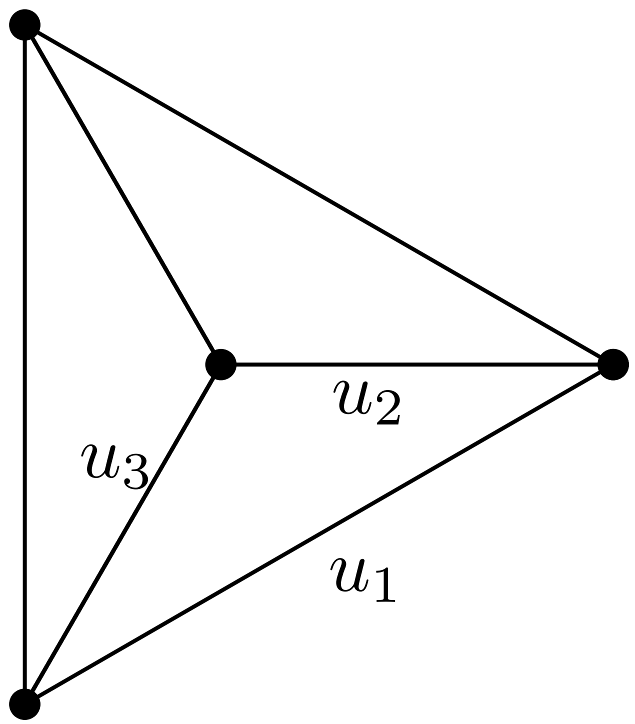
THANK YOU



uniform random reduced 3-web with 1200 bdy vertices, and Tutte embedding

Appetizer: 4-color theorem (SL_3)

Is every planar triangulation 4-colorable?



Thm: Choose for each edge a random unit vector u in $\mathbb{R}^3$.

$$(\text{Number of 4-colorings}) = (-1)^{F/2} 3^E \ \mathbb{E}[\prod_F \det(u_1, u_2, u_3)].$$