Particle Emission in a Self–Consistent Field

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Summary

- Nucleonic–clustering: overview and recent developments
- Cluster–Hartree–Fock theory
- Application: $^{216}\mathrm{Rn} \rightarrow\ ^{212}\mathrm{Po} + \alpha$
- Conclusions

α–decay in the early XXth century

Fig. 1.

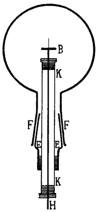
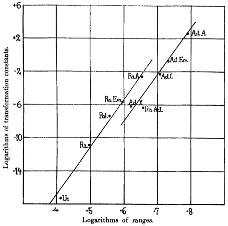


Fig. 3.



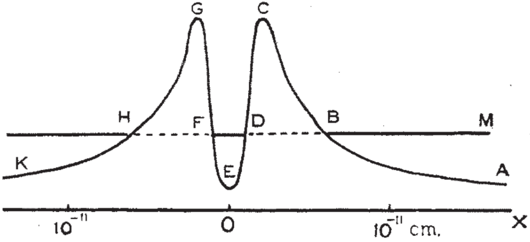
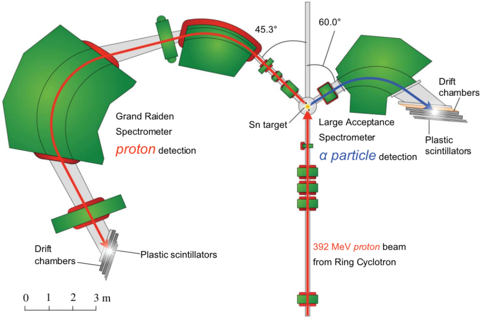


FIG. 1.

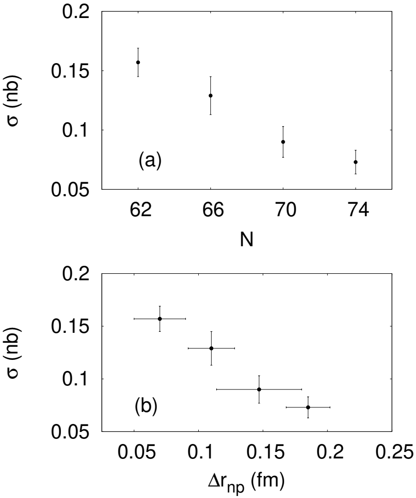
"Much has been written of the explosive violence with which the α–particle is hurled from its place in the nucleus. But from the process pictured above, one would rather say that the α–particle slips away almost unnoticed." – R.W. Gurney and E.U. Condon, Nature **122**, 1928.

H. Geiger and J.M. Nuttall, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **22** (130), 1911.

Formation of α clusters in dilute neutron–rich matter[1]

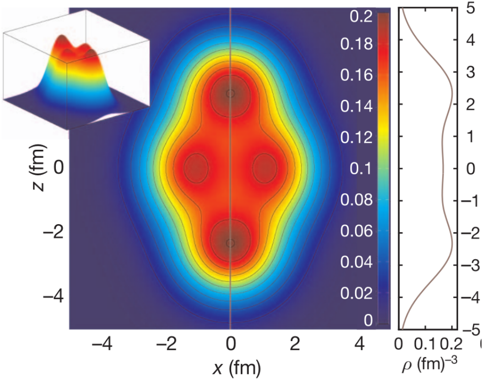


Schematic illustration of the experimental setup used to probe the reaction $^{A}\mathrm{Sn}\,(p, p\alpha)^{A-4}\,\mathrm{Cd}$.



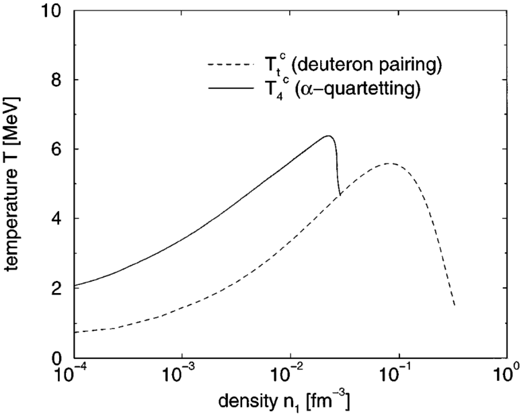
[1] J. Tanaka et al., Science **371**, 6526 (2021).

Clustering in light and infinite systems



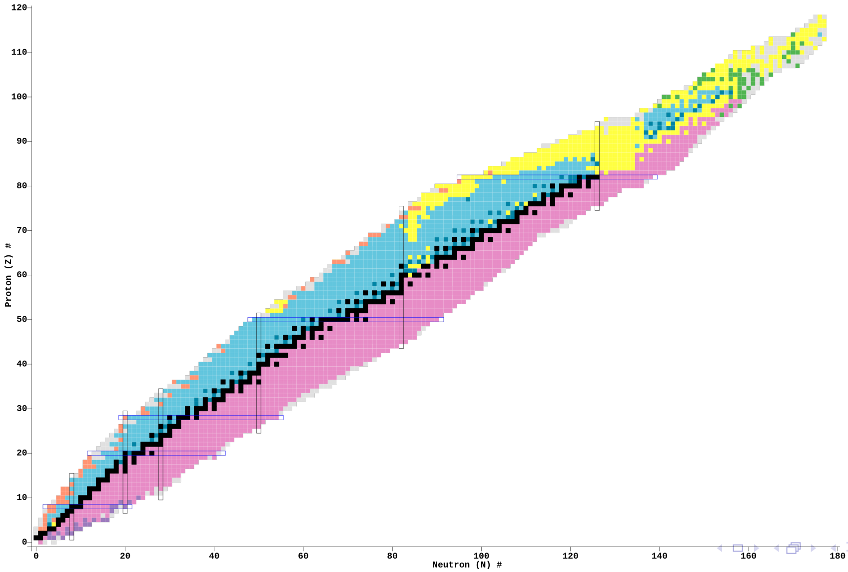
Self–consistent ground–state density of $^{20}\mathrm{Ne}$.

[J.-P. Ebran et al., Nature **487**, 341-344 (2012).]



Critical temperature for the onset of quantum condensation in symmetric nuclear matter. [G. Röpke et al., Phys. Rev. Lett. **80** (15), 1998.]

What about medium and heavy nuclei?

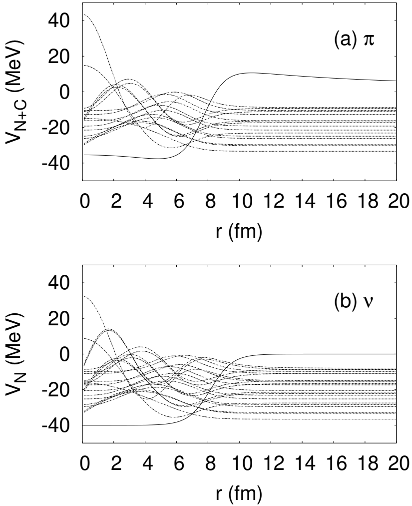


Woods–Saxon plus spin–orbit mean–field

$$
\begin{aligned}
V(\vec{r},\vec{s}) &= V_N(\vec{r}) + V_C(\vec{r}) + V_{so}(\vec{r},\vec{s}) \\
V_N(\vec{r}) &= -V_N^{(0)} f(\vec{r}, r_{0c}, a_c) \\
V_{so}(\vec{r},\vec{s}) &= -V_{so}^{(0)} \frac{1}{r} \frac{\mathrm{d} f(\vec{r}, r_{0so}, a_{so})}{\mathrm{d}r} 2\vec{L}\cdot\vec{s} \\
V_C(\vec{r}) &= \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \mathrm{d}\vec{r}' \\
f(\vec{r}, r_0, a) &= \frac{1}{1+e^{\frac{r-R}{a}}}
\end{aligned}
$$

Parameters are typically set from single–particle energy levels, charge radii etc. A mean–field wave-function, where $\mathbf{x} = (\vec{r},\vec{s})$, is of the form

$$
\begin{aligned}
\psi_{am}(\mathbf{x}) &= \mathcal{R}_a(r)\, \mathcal{Y}_{\jmath_a m}^{\left(\ell_a \frac{1}{2}\right)}(\Omega) = \\
&= \frac{1}{r} u_a(r) \left[i^{\ell_a} Y_{\ell_a} \otimes \chi_{\frac{1}{2}} \right]_{\jmath_a m}
\end{aligned}
$$



Surface Gaussian interaction

The two–body interaction in relative r and c.o.m. R coordinates is

$$v\left(\vec{r},\vec{r}'\right) = -\overline{v}_0 \kappa\left(r'\right) e^{-\frac{|\vec{r}-\vec{r}'|^2}{b^2}} \left[1 + x_c e^{-\frac{(R-R_0)^2}{B^2}}\right]$$

- $\overline{v}_0$ and b are determined from a fit of the starting mean field.
- $B = 1\ \mathrm{fm}$ so as to not perturb the low–level energy spectrum.
- R_0 and x_c are determined from decay data.
- $\kappa\left(r'\right) = \frac{\rho^{(0)}\left(r'\right)}{\langle\rho(r')\rangle}$ is a convergence factor that accounts for the screening of the interaction and prevents the collapse of the nucleus.

Cluster–Hartree-Fock theory I

The main equations read

$$\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 + \Gamma^{(\mathrm{dir})}(\vec{r})\right]\psi_{am}(\mathbf{x}) + \int d^3\vec{r}'\,\Gamma^{(\mathrm{exc})}(\vec{r},\vec{r}')\,\psi_{am}(\mathbf{x}') = \varepsilon_a\psi_{am}(\mathbf{x})$$

depending on the direct and exchange terms and densities of the form

$$\Gamma^{(\mathrm{dir})}(\vec{r}) = \int d^3\vec{r}'\,v(\vec{r},\vec{r}')\,\rho(\vec{r}')$$

$$\Gamma^{(\mathrm{exc})}(\vec{r},\vec{r}') = -\int d^3\vec{r}'\,v(\vec{r},\vec{r}')\,\rho(\vec{r},\vec{r}')$$

$$\rho(\vec{r}') = \sum_c v_c^2 \sum_s \psi_{cs}^\dagger(\mathbf{x}')\,\psi_{cs}(\mathbf{x}')$$

$$\rho(\vec{r},\vec{r}') = \sum_c v_c^2 \sum_s \psi_{cs}^\dagger(\mathbf{x}')\,\psi_{cs}(\mathbf{x})$$

Cluster–Hartree-Fock theory II

One can derive a set of coupled second order radial differential equations

$$-u_a''(r) + \frac{\ell_a(\ell_a+1)}{r^2}u_a(r) + \frac{2\mu}{\hbar^2}\left[V_a(r) - \varepsilon_a\right]u_a(r) = 0$$

or equivalently an eigenvalue problem

$$\sum_{n'} H^{(\beta)}_{na,n'a} d_a^{(n')} = \varepsilon_a d_a^{(n)}$$

determined by the Hamiltonian matrix

$$H^{(\beta)}_{na,n'a} = \hbar\omega\left(2n + \ell_a + \frac{3}{2}\right) + \langle\beta n\ell_a|V_a(r)|\beta n'\ell_a\rangle - \frac{\hbar\omega}{2}\langle\beta n\ell_a|\beta r^2|\beta n'\ell_a\rangle$$

in terms of a local equivalent potential and the multipole expansion of $v\left(\vec{r},\vec{r}'\right)$

$$\begin{aligned} V_a(r) \quad = \quad & -\overline{v}_0\sqrt{4\pi}\int_0^\infty dr' r'^2 \rho\left(r'\right) v_0\left(r,r'\right) + \overline{v}_0 \sum_c v_c^2 \frac{u_c(r)}{u_a(r)} (i)^{\ell_c-\ell_a} \\ \times \quad & \sum_L C^{\jmath_c\, L\, \jmath_a}_{\frac{1}{2}\, 0\, \frac{1}{2}} C^{\jmath_a\, L\, \jmath_c}_{\frac{1}{2}\, 0\, \frac{1}{2}} \int_0^\infty dr' u_a\left(r'\right) u_c\left(r'\right) v_L\left(r,r'\right) \end{aligned}$$

The HF equations are solved according to the iteration scheme

$$V_a^{(\mathrm{new})}(r) = (1-\xi)\,V_a^{(\mathrm{old})}(r) + \xi V_a^{(\mathrm{calc})}(r)\,,\;\; \xi \to 1 \text{ at convergence}$$

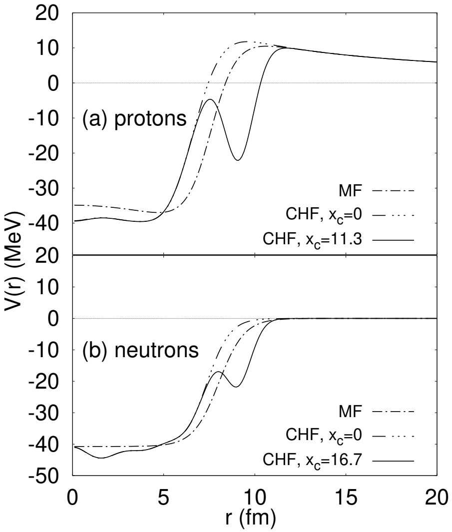
α–particle formation amplitude & decay width

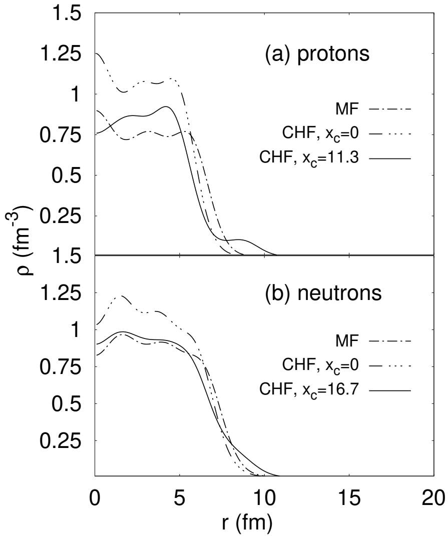
$$\mathcal{F}_0(R) = \langle \Psi_P | \Psi_D \Psi_\alpha \rangle = \sum_{N_\alpha} \mathcal{W}_{N_\alpha} \mathcal{R}^{(4\beta)}_{N_\alpha 0}(R)$$

The expansion coefficients describe in nucleonic degrees of freedom the geometry of the problem (angular momentum recoupling, Talmi–Moshinsky brackets) and nuclear structure details (occupation amplitudes, h.o. radial expansion coefficients). For superfluid nuclei only states near the Fermi level contribute significantly to $\mathcal{W}_{N_\alpha}$. The decay width follows from the matching of $\mathcal{F}_0(R)$ with an outgoing Coulomb wave expressed in terms of the usual reduced radius ρ and Coulomb parameter χ

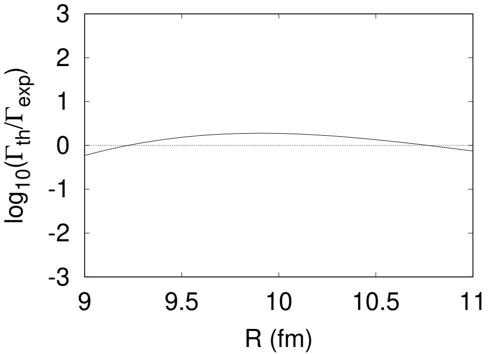
$$\Gamma_{\mathrm{th}}(R) = \hbar v \left[\frac{R\mathcal{F}_0(R)}{G_0(\chi, \rho)} \right]^2$$

Example: $^{216}\mathrm{Rn} \rightarrow {}^{212}\mathrm{Po} + \alpha$ (I)





Example: $^{216}\mathrm{Rn} \to {}^{212}\mathrm{Po} + \alpha$ (II)



$$\Gamma_{\mathrm{exp}} = 1.0 \cdot 10^{-17}\ \mathrm{MeV}\ \mathit{1}$$

$$\langle \log_{10} \frac{\Gamma_{\mathrm{th}(R)}}{\Gamma_{\mathrm{exp}}} \rangle = 0$$

$$R_0 = r_0 \left(A_D^{\frac{1}{3}} + 4^{\frac{1}{3}} \right),\ r_0 \approx 1.3\ \mathrm{fm}$$

$$s_\alpha = \int_0^\infty |R\mathcal{F}_0(R)|^2 \,\mathrm{d}R \approx 3\%$$

Conclusions[2]

- We have developed a procedure to calculate a self–consistent field starting from a two–body interaction parametrized from structure and decay data.
- The use of a surface term restoring the nuclear radius simplifies standard approaches and offers a good simultaneous description of ground state properties and of the decay width.
- Moving the radius of the α–cluster closer to the geometric contact radius and an extension of the method to axially deformed systems are currently under investigation.

[2] A.D. & D.S. Delion, J. Phys. G 52 (6), 055107 (2025).