Generalised Pandya relations for the neutron-proton interaction

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The Pandya relation

The seniority quantum number

A generalised Pandya relation

An application

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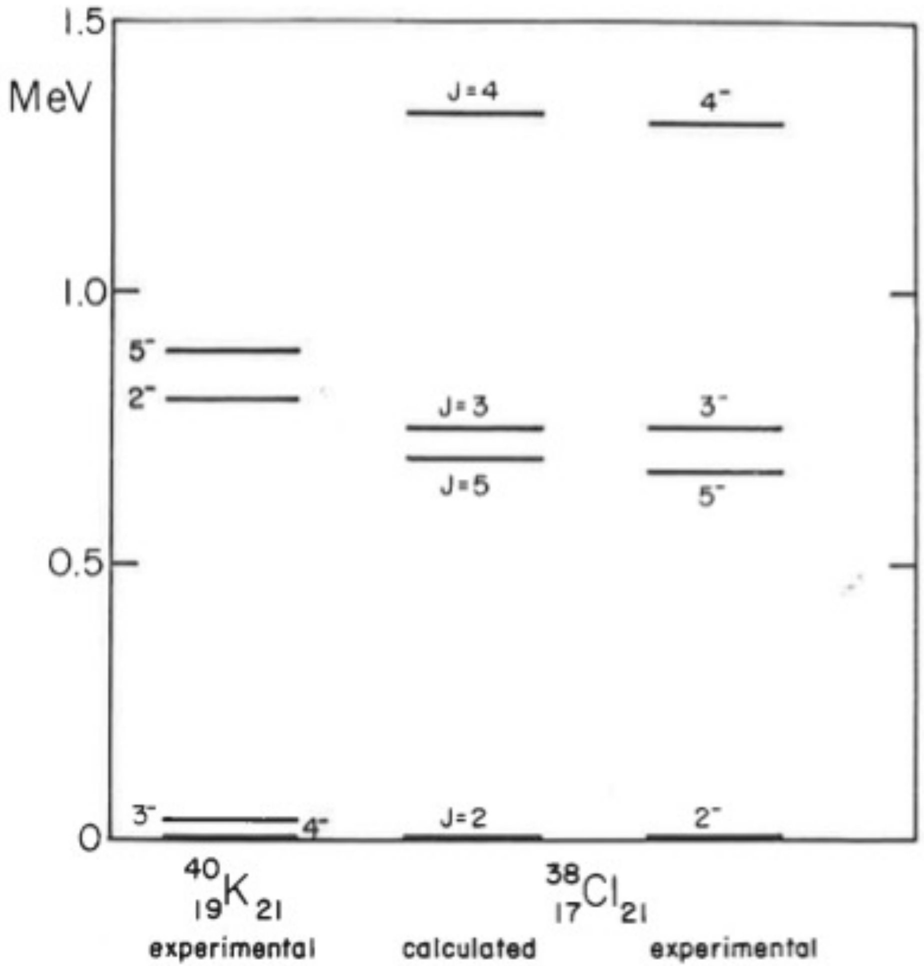
The Pandya relation

- A relation between the particle-particle and the particle-hole interaction.
- For a neutron particle in orbital j_ν and a proton hole in orbital j_π :

$$V^{J}_{j_\nu j_\pi^{-1}} = E_0 - \sum_{J'} (2J'+1) \begin{Bmatrix} j_\nu & j_\pi & J \\ j_\nu & j_\pi & J' \end{Bmatrix} V^{J'}_{j_\nu j_\pi}$$

A historical example



The ^{38}Cl spectrum is derived from ^{40}K.

Pandya relation with $j_\nu = f_{7/2}$ and $j_\pi = d_{3/2}$.

S. Goldstein & I. Talmi, Phys. Rev. **102** (1956) 589
S.P. Pandya, Phys. Rev. **103** (1956) 956
I. Talmi, *Simple Models of Complex Nuclei*

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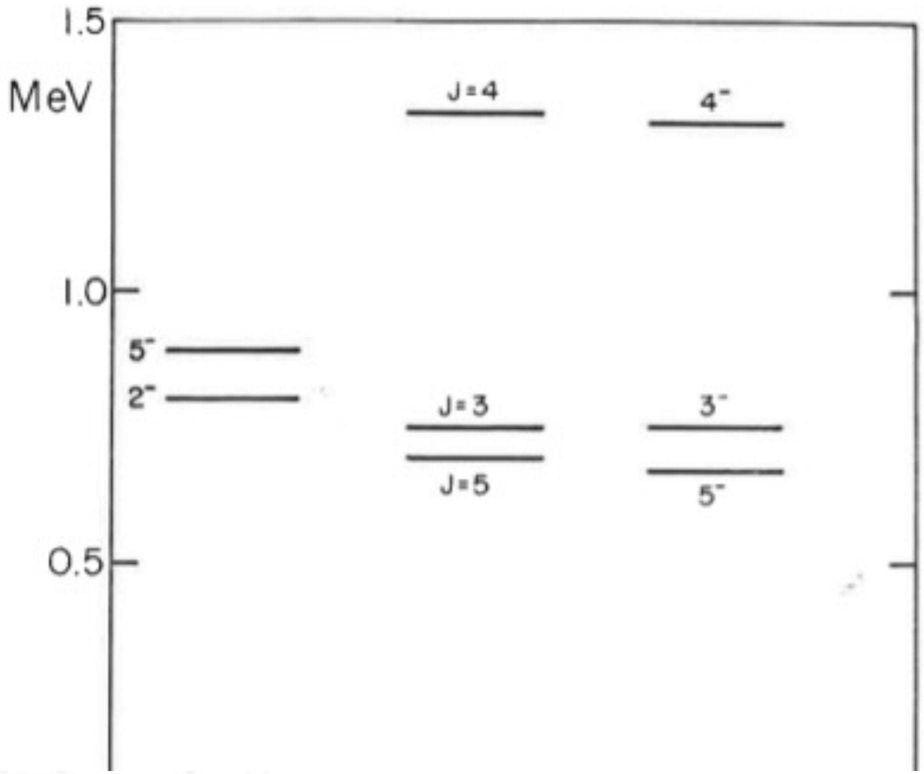

The spectrum of ^{38}Cl can be derived from that of ^{40}K.

Pandya relation with $j_\nu = f_{7/2}$ and $j_\pi = d_{3/2}$.

This relation is due to Pandya (1956) who did not publish his complicated derivation. Simple different proofs were given by Talmi (1960), de Shalit and Talmi (1963) and Bertsch (1972).

S. Goldstein & I. Talmi, Phys. Rev. **102** (1956) 589
S.P. Pandya, Phys. Rev. **103** (1956) 956
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The seniority (quantum) number

Seniority υ is the number of particles not in pairs coupled to $J = 0$ (Racah).

Conditions for the conservation of seniority by a given (two-body) interaction $\hat{V}$ can be derived from the analysis of a 3-particle system.

Any interaction between identical fermions with spin j conserves seniority if $j \leq 7/2$.

Any interaction between identical bosons with spin j conserves seniority if $j \leq 2$.

G. Racah, Phys. Rev. **63** (1943) 367
A. de Shalit & I. Talmi, *Nuclear Shell Theory*
I. Talmi, *Simple Models of Complex Nuclei*

A state with good seniority

Consider n_ν neutrons in orbital j_ν with seniority υ_ν and angular momentum J_ν, n_π protons in orbital j_π with seniority υ_π and angular momentum J_π, coupled to total angular momentum J:

$$|j_\nu^{n_\nu}\upsilon_\nu J_\nu, j_\pi^{n_\pi}\upsilon_\pi J_\pi; J\rangle$$

What is the expectation value of the neutron-proton interaction energy in this state?

Neutron-proton interaction energy

Here it is:

$$\left\langle j_\nu^{n_\nu} \upsilon_\nu J_\nu, j_\pi^{n_\pi} \upsilon_\pi J_\pi; J \middle| \hat{V}_{\nu\pi} \middle| j_\nu^{n_\nu} \upsilon_\nu J_\nu, j_\pi^{n_\pi} \upsilon_\pi J_\pi; J \right\rangle$$
$$= E_0\left(\bar{V}_{j_\nu j_\pi}\right) + \frac{1}{2} \upsilon_\nu \upsilon_\pi \left[\left(W_{\nu\pi} + \widetilde{W}_{\nu\pi}\right) + q_\nu q_\pi \left(W_{\nu\pi} - \widetilde{W}_{\nu\pi}\right)\right]$$

with

$$E_0\left(\bar{V}_{j_\nu j_\pi}\right) = \left(n_\nu n_\pi - q_\nu q_\pi \upsilon_\nu \upsilon_\pi\right) \bar{V}_{j_\nu j_\pi}$$

$$\bar{V}_{j_\nu j_\pi} = \sum_J \frac{2J+1}{(2j_\nu+1)(2j_\pi+1)} V^J_{j_\nu j_\pi} \quad \text{(monopole)}$$

$$q_\rho = \frac{2j_\rho + 1 - 2n_\rho}{2j_\rho + 1 - 2\upsilon_\rho}, \qquad -1 \leq q_\rho \leq +1$$

M. Rejmund *et al.*, Phys. Lett. B **753** (2016) 86

Neutron-proton interaction energy

The W functions are sums over $12j$ symbols:

$$W_{\nu\pi} \propto \sum_{J'J'_\nu J'_\pi} (2J'+1)C^2 \begin{bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ J' & J'_\pi & J & J'_\nu \\ j_\nu & j_\pi & J_\pi & J_\nu \end{bmatrix} V^{J'}_{j_\nu j_\pi}$$

$$\widetilde{W}_{\nu\pi} \propto -\sum_{J'J'_\nu J'_\pi} (2J'+1)C^2 \begin{Bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ J' & J'_\pi & J & J'_\nu \\ j_\nu & j_\pi & J_\pi & J_\nu \end{Bmatrix} V^{J'}_{j_\nu j_\pi}$$

The C coefficients are CFPs:

$$C = \left[j_\nu^{n_\nu-1}(v'_\nu J'_\nu)J_\nu|\}j_\nu^{n_\nu} v_\nu J_\nu\right] \left[j_\pi^{n_\pi-1}(v'_\pi J'_\pi)J_\pi|\}j_\pi^{n_\pi} v_\pi J_\pi\right]$$

One neutron and one proton

For one neutron and one proton we have

$$n_\nu = n_\pi = v_\nu = v_\pi = 1 \Longrightarrow q_\nu = q_\pi = +1$$

Therefore

$$\langle j_\nu, j_\pi; J | \hat{V}_{\nu\pi} | j_\nu, j_\pi; J \rangle = W_{\nu\pi}$$
$$\propto \sum_{J'} (2J'+1) \begin{bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ J' & 0 & J & 0 \\ j_\nu & j_\pi & J_\pi & J_\nu \end{bmatrix} V^{J'}_{j_\nu j_\pi}$$
$$= V^{J}_{j_\nu j_\pi}$$

1 neutron particle and 1 proton hole

For 1 neutron particle and 1 proton hole we have

$$n_\nu = v_\nu = 1, \;\; n_\pi = 2j_\pi, \;\; v_\pi = 1 \Longrightarrow q_\nu = +1, \;\; q_\pi = -1$$

Therefore

$$\langle j_\nu, j_\pi; J|\hat{V}_{\nu\pi}|j_\nu, j_\pi; J\rangle = (2j_\pi + 1)\bar{V}_{j_\nu j_\pi} + \tilde{W}_{\nu\pi}$$

with

$$\tilde{W}_{\nu\pi} \propto -\sum_{J'}(2J' + 1)\begin{Bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ J' & 0 & J & 0 \\ j_\nu & j_\pi & J_\pi & J_\nu \end{Bmatrix} V^{J'}_{j_\nu j_\pi}$$

$$= -\sum_{J'}(2J' + 1)\begin{Bmatrix} j_\nu & j_\pi & J \\ j_\nu & j_\pi & J' \end{Bmatrix} V^{J'}_{j_\nu j_\pi} = V^{J}_{j_\nu j_\pi^{-1}}$$

Particle-hole transformation

Interaction energy between particles:

$$\langle j_\nu^{n_\nu} \upsilon_\nu J_\nu, j_\pi^{n_\pi} \upsilon_\pi J_\pi; J | \hat{V}_{\nu\pi} | j_\nu^{n_\nu} \upsilon_\nu J_\nu, j_\pi^{n_\pi} \upsilon_\pi J_\pi; J \rangle$$
$$= E_0(\bar{V}_{j_\nu j_\pi}) + \frac{1}{2} \upsilon_\nu \upsilon_\pi \left[(W_{\nu\pi} + \widetilde{W}_{\nu\pi}) + q_\nu q_\pi (W_{\nu\pi} - \widetilde{W}_{\nu\pi}) \right]$$

Particle-hole transformation $n_\rho \to 2j_\rho + 1 - n_\rho$ and $q_\rho \to -q_\rho$.

Interaction energy between particles and holes:

$$\langle j_\nu^{n_\nu} \upsilon_\nu J_\nu, j_\pi^{-n_\pi} \upsilon_\pi J_\pi; J | \hat{V}_{\nu\pi} | j_\nu^{n_\nu} \upsilon_\nu J_\nu, j_\pi^{-n_\pi} \upsilon_\pi J_\pi; J \rangle$$
$$= E_0'(\bar{V}_{j_\nu j_\pi}) + \frac{1}{2} \upsilon_\nu \upsilon_\pi \left[(W_{\nu\pi} + \widetilde{W}_{\nu\pi}) - q_\nu q_\pi (W_{\nu\pi} - \widetilde{W}_{\nu\pi}) \right]$$

Generalised Pandya relation

The particle-hole transformation is obtained by replacing the $12j$ symbol of the first kind by the negative of the $12j$ symbol of the second kind (or *vice versa*).

Generalised Pandya relation

The particle-hole transformation is obtained by replacing the $12j$ symbol of the first kind by the negative of the $12j$ symbol of the second kind (or *vice versa*).

Further generalisation: The particle-hole transformation is obtained by replacing $3nj$ symbols of the first kind by the negative of $3nj$ symbols of the second kind (or *vice versa*).

Two neutron & two proton orbitals

Consider a state with neutrons in orbitals $j_{1\nu}$ and $j_{2\nu}$ and protons in orbitals $j_{1\pi}$ and $j_{2\pi}$:

$$\left|\left(j_{1\nu}^{n_{1\nu}} \upsilon_{1\nu} J_{1\nu}, j_{2\nu}^{n_{2\nu}} \upsilon_{2\nu} J_{2\nu}\right) J_\nu, \left(j_{1\pi}^{n_{1\pi}} \upsilon_{1\pi} J_{1\pi}, j_{2\pi}^{n_{2\pi}} \upsilon_{2\pi} J_{2\pi}\right) J_\pi; J\right\rangle$$
$$\equiv \left|J_\nu, J_\pi; J\right\rangle$$

What is the expectation value of the neutron-proton interaction energy in this state?

Neutron-proton interaction energy

Here it is:

$$\langle J_\nu, J_\pi; J | \hat{V}_{\nu\pi} | J_\nu, J_\pi; J \rangle = \sum_{kl} E_0\left(\bar{V}_{j_{k\nu} j_{l\pi}}\right)$$

$$+ \sum_{kl} \frac{1}{2} v_{k\nu} v_{l\pi} \left[\left(W_{k\nu l\pi} + \widetilde{W}_{k\nu l\pi}\right) + q_{k\nu} q_{l\pi} \left(W_{k\nu l\pi} - \widetilde{W}_{k\nu l\pi}\right)\right]$$

with

$$E_0\left(\bar{V}_{j_{k\nu} j_{l\pi}}\right) = \left(n_{k\nu} n_{l\pi} - q_{k\nu} q_{l\pi} v_{k\nu} v_{l\pi}\right) \bar{V}_{j_{k\nu} j_{l\pi}}$$

$$\bar{V}_{j_{k\nu} j_{l\pi}} = \sum_J \frac{2J+1}{(2j_{k\nu}+1)(2j_{l\pi}+1)} V^J_{j_{k\nu} j_{l\pi}} \quad \text{(monopole)}$$

$$q_{k\rho} = \frac{2j_{k\rho} + 1 - 2n_{k\rho}}{2j_{k\rho} + 1 - 2v_{k\rho}}, \qquad -1 \le q_{k\rho} \le +1$$

Neutron-proton interaction energy

The W functions are sums over $18j$ symbols:

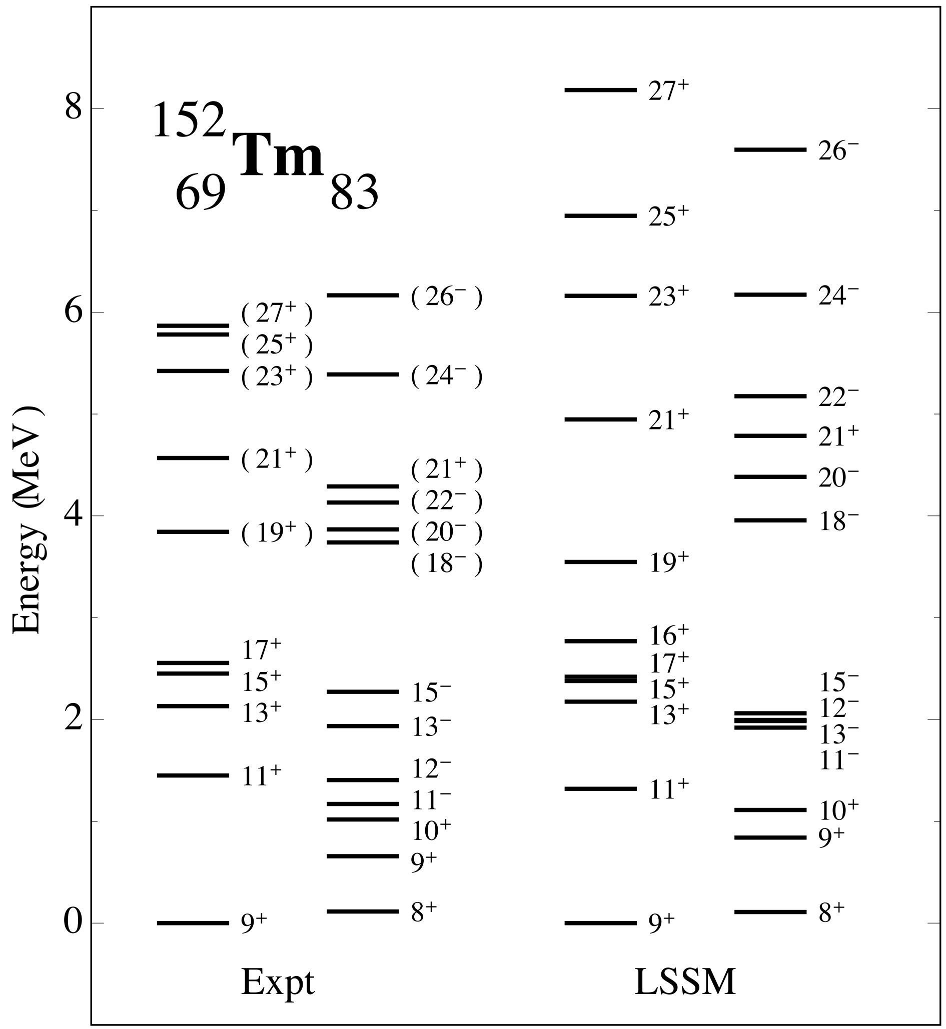
$$W_{k\nu l\pi} \propto \sum_{J' J'_{k\nu} J'_{l\pi}} (2J'+1) C_{kl}^2 \begin{bmatrix} j_{k\nu} & j_{l\pi} & J_{l\pi} & J_{\pi} & J_{\nu} & J_{k\nu} \\ J' & J'_{l\pi} & J_{\bar{l}\pi} & J & J_{\bar{k}\nu} & J'_{k\nu} \\ j_{k\nu} & j_{l\pi} & J_{l\pi} & J_{\pi} & J_{\nu} & J_{k\nu} \end{bmatrix} V^{J'}_{j_{k\nu} j_{l\pi}}$$

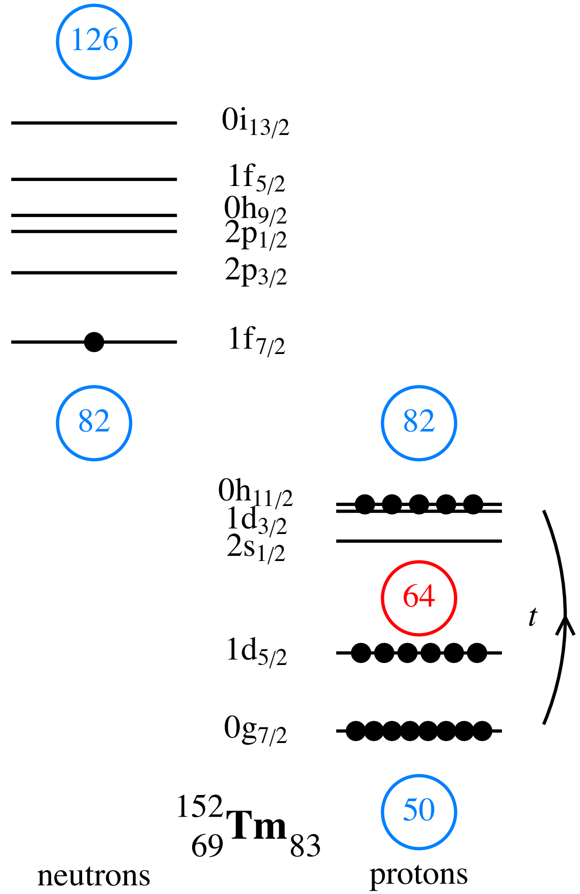
$$\widetilde{W}_{k\nu l\pi} \propto - \sum_{J' J'_{k\nu} J'_{l\pi}} (2J'+1) C_{kl}^2 \begin{Bmatrix} j_{k\nu} & j_{l\pi} & J_{l\pi} & J_{\pi} & J_{\nu} & J_{k\nu} \\ J' & J'_{l\pi} & J_{\bar{l}\pi} & J & J_{\bar{k}\nu} & J'_{k\nu} \\ j_{k\nu} & j_{l\pi} & J_{l\pi} & J_{\pi} & J_{\nu} & J_{k\nu} \end{Bmatrix} V^{J'}_{j_{k\nu} j_{l\pi}}$$

The C coefficients are CFPs:

$$C_{kl} = \left[j_{k\nu}^{n_{k\nu}-1} (v'_{k\nu} J'_{k\nu}) J_{k\nu} | \} j_{k\nu}^{n_{k\nu}} v_{k\nu} J_{k\nu} \right] \left[j_{l\pi}^{n_{l\pi}-1} (v'_{l\pi} J'_{l\pi}) J_{l\pi} | \} j_{l\pi}^{n_{l\pi}} v_{l\pi} J_{l\pi} \right]$$

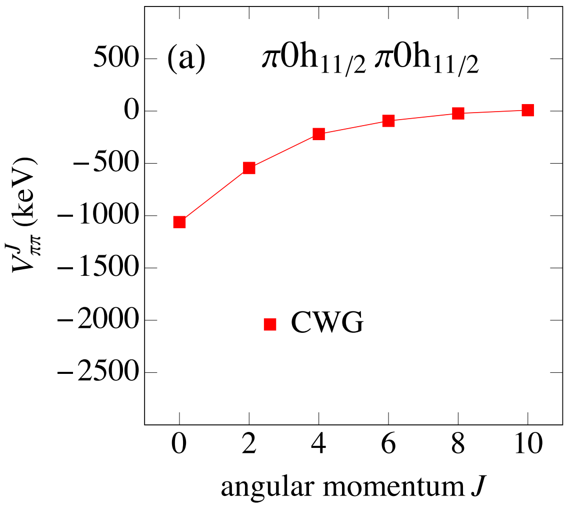
An application: ^{152}Tm

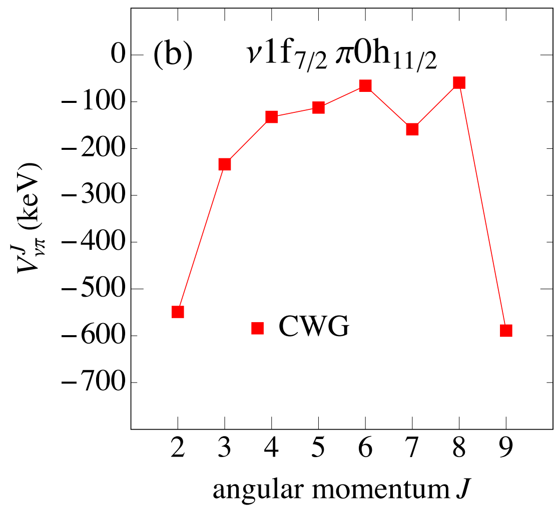




B.S. Nara Singh *et al.*, Phys. Rev. C **98** (2018) 024319

Simple shell-model calculation





One neutron in $1f_{7/2}$.

Five protons $0h_{11/2}$.

Two-body matrix elements from the CWG interaction.

Simple shell-model calculation

Why are the 9^{+} and 8^{+} nearly degenerate?

Assume 1 neutron in $1f_{7/2}$ ($\upsilon_{\nu} = 1$) and five protons in $0h_{11/2}$ with $\upsilon_{\pi} = 1$ and apply the formula:

$$E(8_1^+) - E(9_1^+) =$$
$$\frac{3}{11}V_{\nu\pi}^2 + \frac{21}{715}V_{\nu\pi}^3 - \frac{81}{715}V_{\nu\pi}^4 - \frac{51}{455}V_{\nu\pi}^5 -$$
$$\frac{87}{1540}V_{\nu\pi}^6 - \frac{1161}{68068}V_{\nu\pi}^7 + \frac{11953}{20020}V_{\nu\pi}^8 - \frac{2653}{4420}V_{\nu\pi}^9$$

Anti-aligned and aligned matrix elements cancel!

Conclusion

Generic expressions for the neutron-proton interaction in a state with good seniority for neutrons and protons.

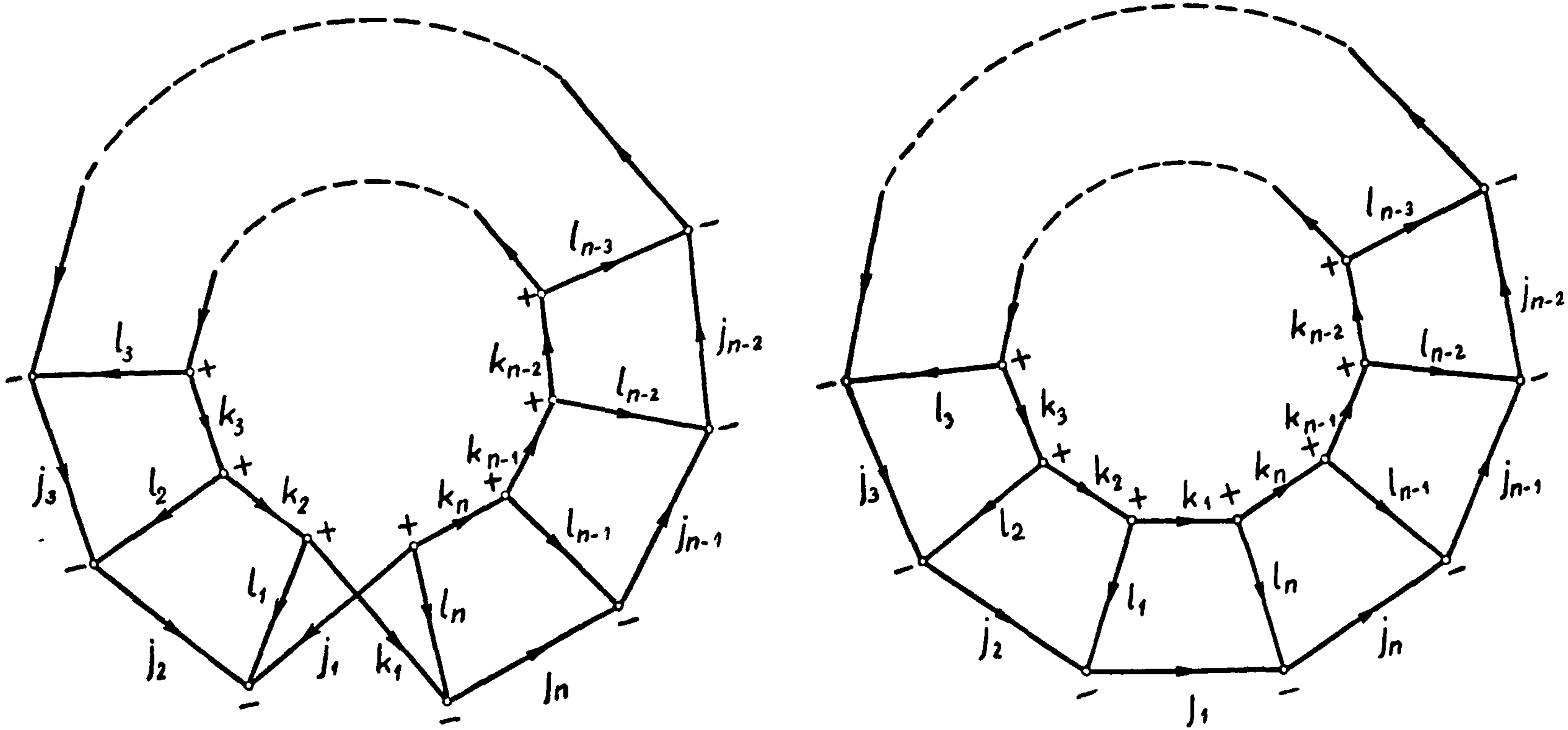
The particle-hole transformation is obtained by replacing $3nj$ symbols of the first kind by the negative of $3nj$ symbols of the second kind (or *vice versa*).

Caveat: the neutron-proton interaction breaks seniority.

Why do I do this?

Because it’s fun!

$3nj$ symbols of the 1st & 2nd kind



$3nj$ symbols of the 1st & 2nd kind

$$\begin{Bmatrix} j_1 & & j_2 & & \cdots & j_n & \\ & l_1 & & l_2 & \cdots & & l_n \\ k_1 & & k_2 & & \cdots & k_n & \end{Bmatrix} = \sum_\lambda (-)^{\Sigma_n + (n-1)\lambda} (2\lambda + 1)$$

$$\times \begin{Bmatrix} j_1 & k_1 & \lambda \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & \lambda \\ k_3 & j_3 & l_2 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & \lambda \\ k_n & j_n & l_{n-1} \end{Bmatrix} \begin{Bmatrix} j_n & k_n & \lambda \\ j_1 & k_1 & l_n \end{Bmatrix}$$

$$\begin{bmatrix} j_1 & & j_2 & & \cdots & j_n & \\ & l_1 & & l_2 & \cdots & & l_n \\ k_1 & & k_2 & & \cdots & k_n & \end{bmatrix} = \sum_\lambda (-)^{\Sigma_n + n\lambda} (2\lambda + 1)$$

$$\times \begin{Bmatrix} j_1 & k_1 & \lambda \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & \lambda \\ k_3 & j_3 & l_2 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & \lambda \\ k_n & j_n & l_{n-1} \end{Bmatrix} \begin{Bmatrix} j_n & k_n & \lambda \\ k_1 & j_1 & l_n \end{Bmatrix}$$

Simple shell-model calculation