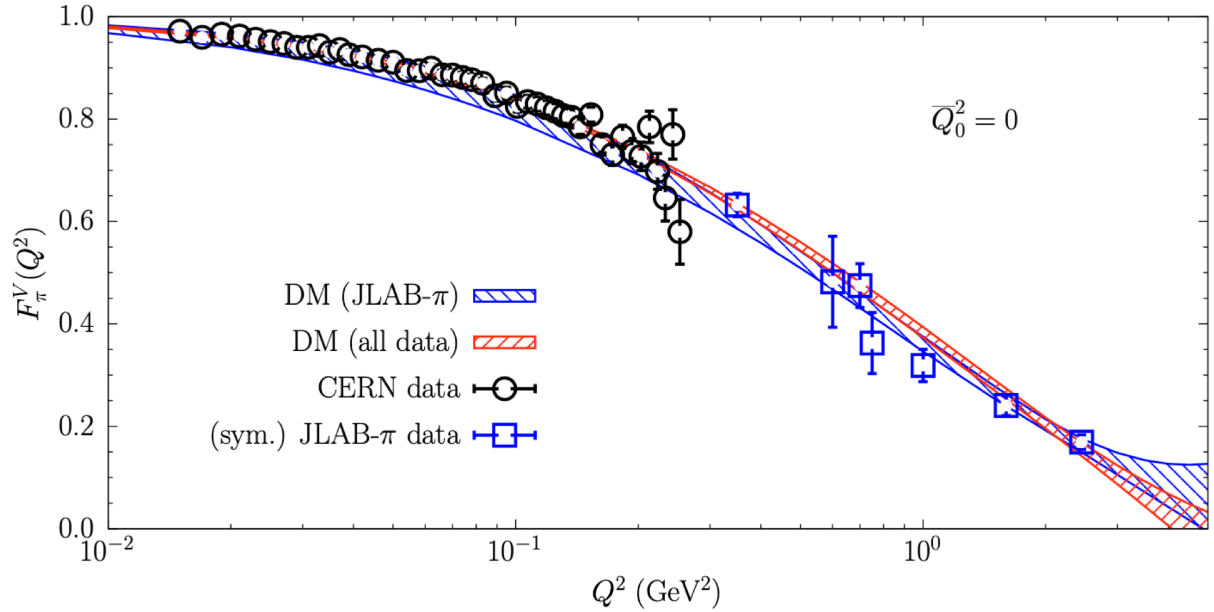
Unitarity constraints and hadronic form factors within the Dispersive Matrix approach: the pion form factor case

Work in collaboration with Silvano Simula (PRD '23 [arXiv:2309.02135])

Ludovico Vittorio (LAPTh & CNRS, Annecy, France)

GdR InF Workshop - November 7th, 2023



Many thanks to Guido Martinelli for discussions and suggestions

State-of-the-art of muon g-2

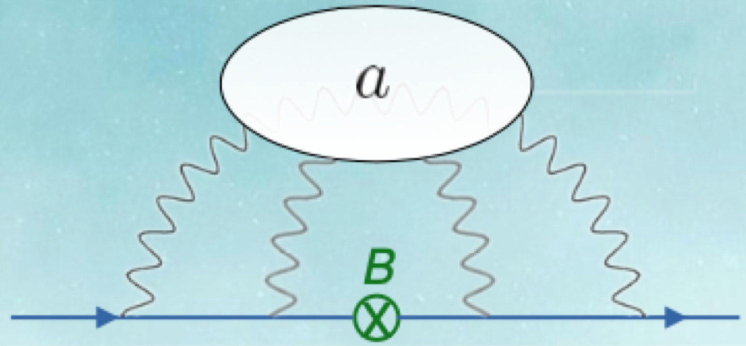
$$\mu = g\frac{e}{2m}S$$

Loop correction to $g=2$

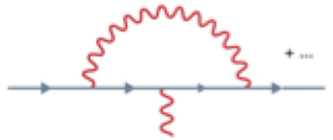
Vacuum polarization renormalizes g

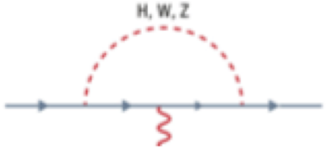
$$a = \frac{g-2}{2}$$

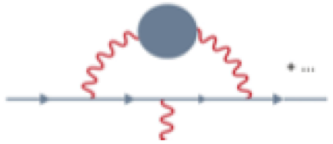
Anomalous magnetic moment

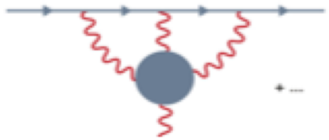


$$a = a^{QED} + a^{weak} + a^{HVP} + a^{LBL}$$









… and possibly, any sort of unknown particle

Measure precisely $a \longrightarrow$ probe completeness of the Standard Model

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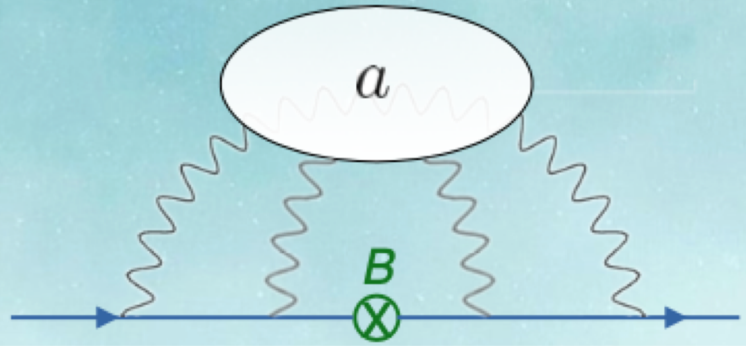

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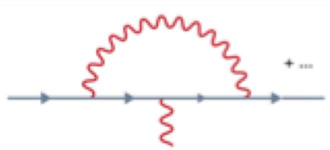

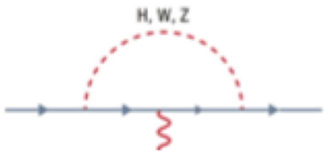

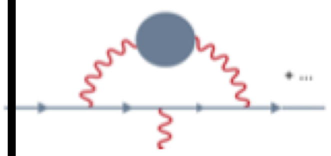

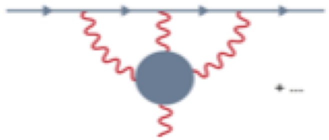

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aHVP from experimental data

The Hadronic Vacuum Polarization (HVP) tensor is defined as

$$\Pi_{\mu\nu}(Q^2) = \int d^4x e^{iQ\cdot x} \langle j_\mu(x) j_\nu(0) \rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

$$j_\mu(x) \equiv \sum_f q_f \bar{\psi}_f(x)\gamma_\mu \psi_f(x)$$

Through the optical theorem and unitarity:

$$\mathrm{Im}\,\Pi(s) = \frac{s}{4\pi\alpha}\sigma_{\mathrm{tot}}(e^+e^- \to \mathrm{hadrons}) = \frac{\alpha}{3}R_{\mathrm{had}}(s)$$

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Master formula for HVP contribution to a_μ

$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\mathrm{thr}}}^{\infty} ds \frac{\hat{K}(s)}{s^2} R_{\mathrm{had}}(s)$$

$$\left\{ R_{\mathrm{had}}(s) = \frac{3s}{4\pi\alpha^2}\frac{s\sigma_e(s)}{s+2m_e^2}\sigma(e^+e^- \to \mathrm{hadrons}). \right\}$$

Hoferichter's talk @ LF(U)V Workshop '22
(see also for instance JHEP '19 [arXiv:1810.00007])

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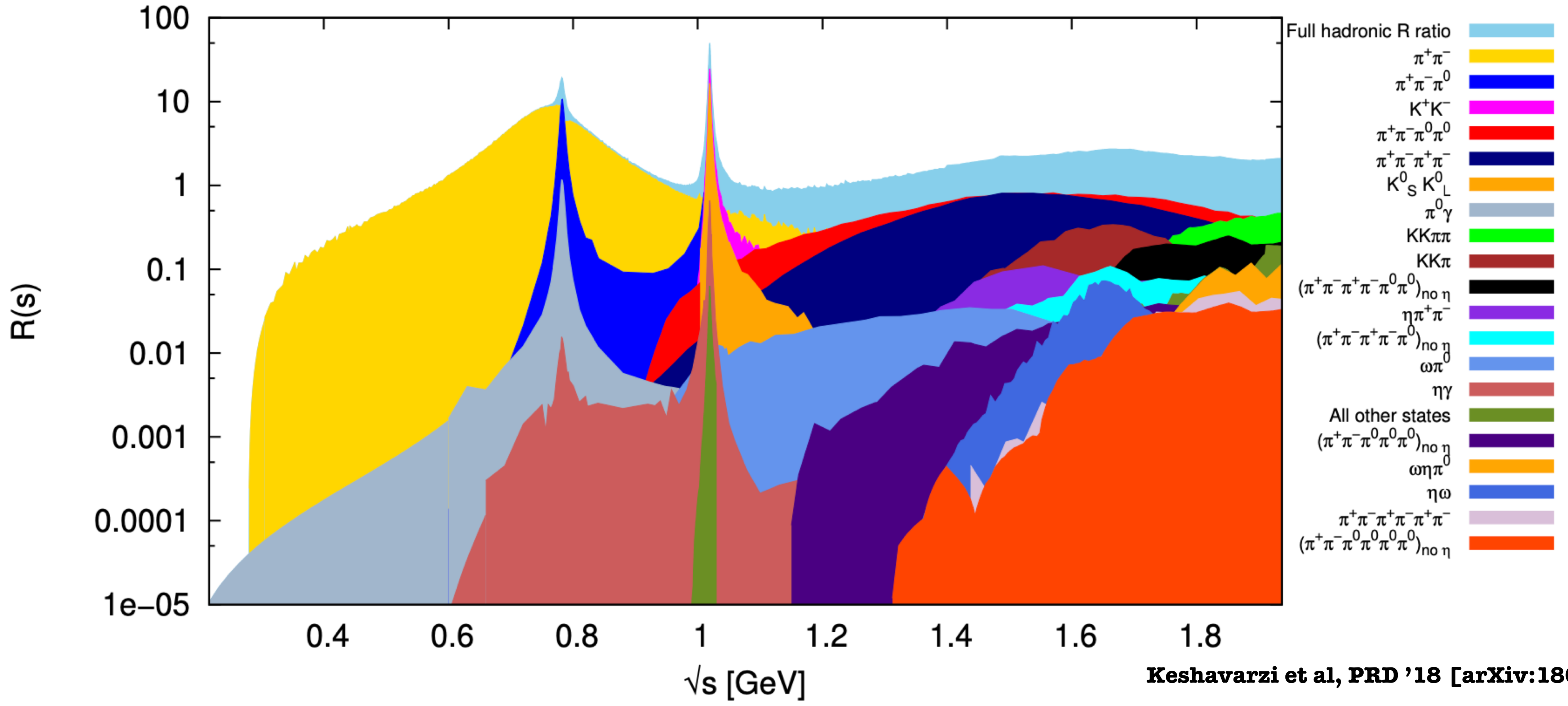
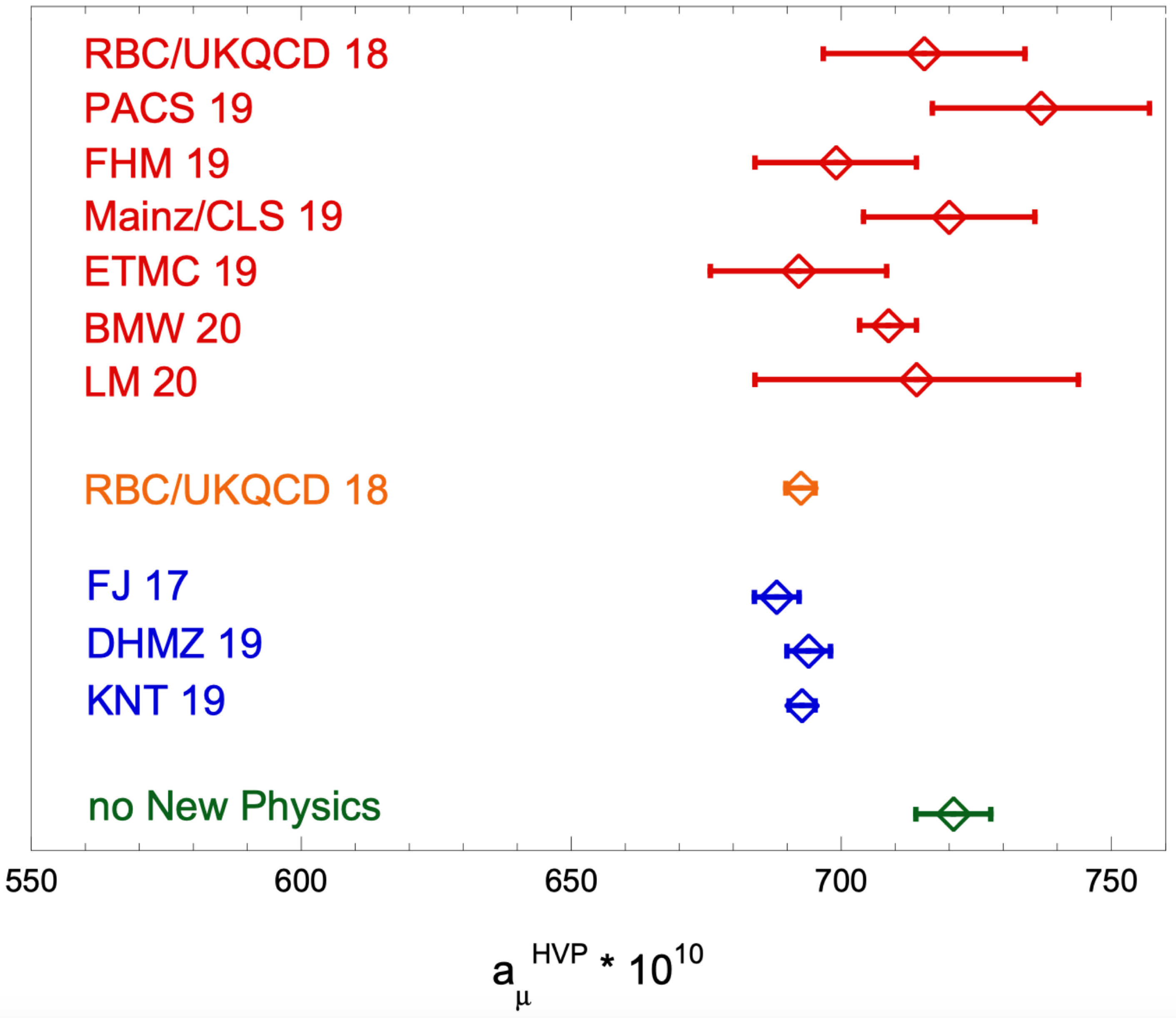


Keshavarzi et al, PRD '18 [arXiv:1802.02995]

Among all the hadronic contributions, for our purposes the most important case is had = $\pi\pi$!

The *new* g-2 puzzle

Comparison among the results of the lattice computations, the experiments («no New Physics») and the results of the dispersive analyses:



Giusti & Simula @ Lattice 2021

How can the pion charge radius help?

$$\langle \pi^+(p')|J_\mu^{em}|\pi^+(p)\rangle = (p+p')_\mu F_\pi^V(Q^2)$$

EM pion form factor (FF)

Its modulus *is a crucial quantity governing the 2π contribution to the HVP of the muon anomalous magnetic moment, since*

$$R_{2\pi}(\omega) = \frac{1}{4}\left(1-\frac{4M_\pi^2}{\omega^2}\right)^{3/2} |F_\pi^V(\omega)|^2 \qquad \left\{ \omega^2 \geq 4M_\pi^2 \right\}$$

Finally, the ***pion charge radius*** *is defined as*

$$\langle r_\pi^2\rangle \equiv -6\frac{dF_\pi^V(Q^2)}{dQ^2}\bigg|_{Q^2=0} \qquad \left\{ Q^2 \equiv -q^2 \right\}$$

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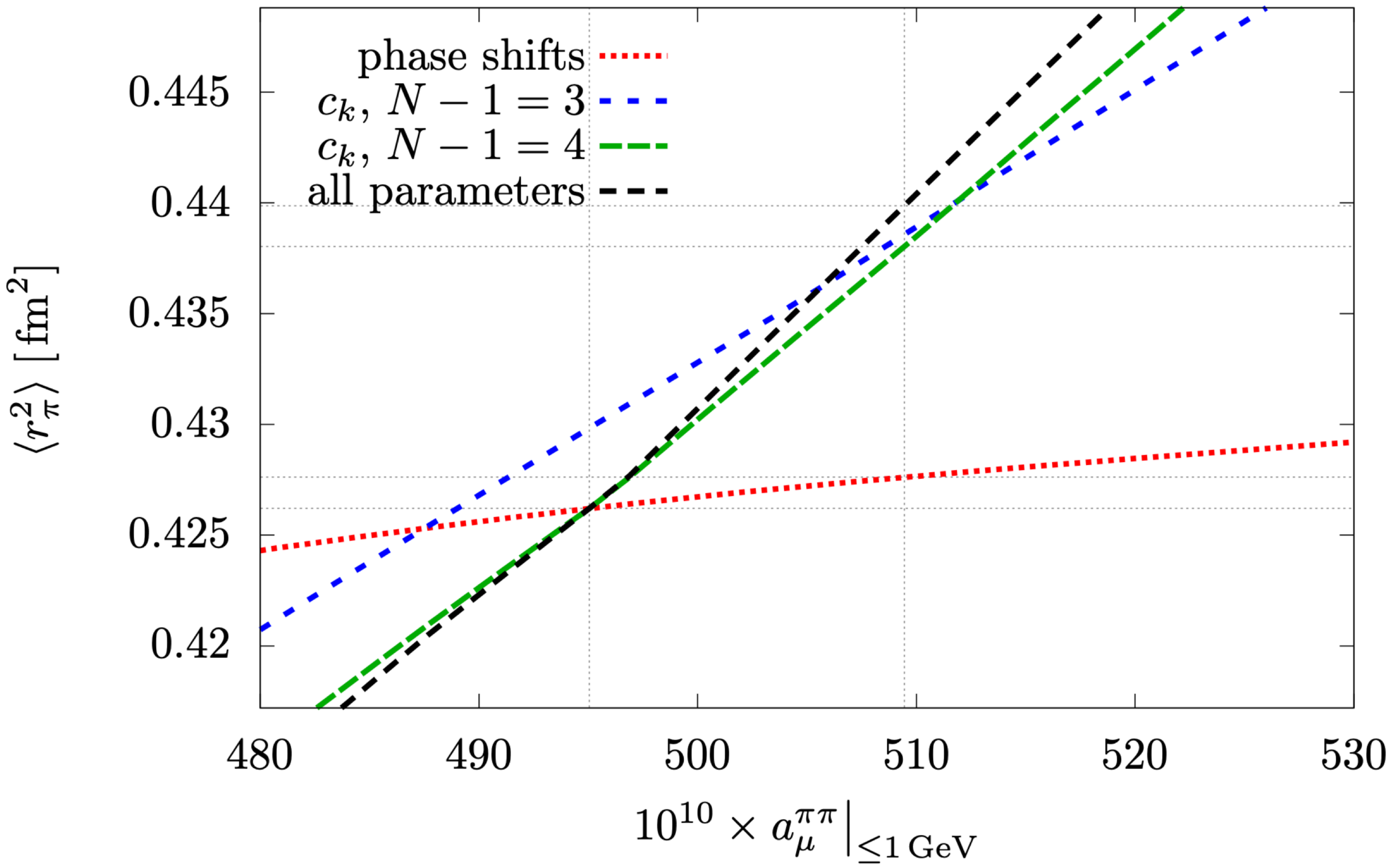

Hoferichter's talk @ LF(U)V Workshop '22

FUNDAMENTAL ISSUE:
there is a strong (positive) correlation among the value of aHVP and the pion charge radius!!

How can the pion charge radius help?

To give some numbers:

$$\langle r_\pi \rangle_{PDG} = 0.659 \pm 0.004 \text{ fm}$$

$$\left\{ \langle r_\pi \rangle \equiv \sqrt{\langle r_\pi^2 \rangle} \right\}$$

which is based on different determinations:

i) $\langle r_\pi \rangle = 0.656 \pm 0.005$ fm from an **average of dispersive analyses of timelike (e+e−) and spacelike data**

Colangelo et al, JHEP '19 [1810.00007]
Ananthanarayan et al, PRL '17 [1706.04020]

ii) $\langle r_\pi \rangle = 0.663 \pm 0.023$ fm from **spacelike data from the F2 experiment**

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SPOILER: $\langle r_\pi \rangle_{DM} = 0.703 \pm 0.027$ fm

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach to describe the FFs in the whole kinematical region!

- Pioneering works from S. Okubo [PRD, 3 (1971); PRD, 4 (1971)], C.'Bourrely et al [NPB, 189 (1981)] and L. Lellouch [NPB, 479 (1996)]
- New developments in PRD '21 (2105.02497)

The resulting description of the FFs

- **is entirely based on first principles (LQCD evaluation of 2- and 3-point Euclidean correlators)**
- **is independent of any assumption on the functional dependence of the FFs on the momentum transfer**
- **can be applied to theoretical calculations of the FFs, but also to experimental data**

→ No HQET, no series expansion, no perturbative bounds with respect to the well-known other parametrizations

How does it work?

The DM method

Let us focus on a generic FF *f*: **we will determine f(t) with $f(t_i)$ known at positions t_i (i=1, …, N)**

$$z(t) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}$$

$$t_+ = 4M_\pi^2 \qquad t_0 = 0$$

t: momentum transfer

How? We define

- inner product $$\langle h_1 | h_2 \rangle = \int_{|z|=1} \frac{dz}{2\pi i z} \bar{h}_1(z) h_2(z)$$

- auxialiary function $$g_t(z) \equiv \frac{1}{1 - \bar{z}(t) z}$$

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We build up the matrix M of the scalar products of ϕf, g_t, g_{t1}, …, g_{tN} :

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

A lot of pioneering works in the past:

L. Lellouch, NPB, 479 (1996), p. 353-391
C. Bourrely, B. Machet, and E. de Rafael, NPB, 189 (1981), pp. 157 – 181
E. de Rafael and J. Taron, PRD, 50 (1994), p. 373-380

The DM method

CENTRAL ISSUE: since **M** contains only inner products, its determinant is semipositive definite

$$\det \mathbf{M} \geq 0 \Rightarrow \text{Filtering of input data: } \chi_{DM} \leq \chi \Rightarrow f_{\rm lo}(z) \leq f(z) \leq f_{\rm up}(z)$$

DISPERSION RELATIONS:

$$0 \leq \langle \phi f | \phi f \rangle \leq \chi(q^2)$$

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

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The susceptibility χ_T

$$\chi_T(\overline{Q}_0^2) \equiv \frac{1}{4}\int_0^\infty d\tau \tau^4 \frac{j_1(\overline{Q}_0\tau)}{\overline{Q}_0\tau} V_{2\pi}(\tau)$$

Two ways of computing it numerically:

i) Direct lattice determination of the Euclidean correlator $V_{2\pi}$

ii) Data-driven determination of the Euclidean correlator $V_{2\pi}$ through e+e- data: in fact

$$V(\tau) = \frac{1}{12\pi^2}\int_{2M_\pi}^\infty d\omega \omega^2 R_{had}(\omega) e^{-\omega\tau}$$, where $$R_{had}(\omega) = \frac{3\omega^2}{4\pi\alpha_{em}^2}\sigma_{had}(\omega)$$

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We have used the pion FF results from **Colangelo et al, JHEP '19 [1810.00007]**

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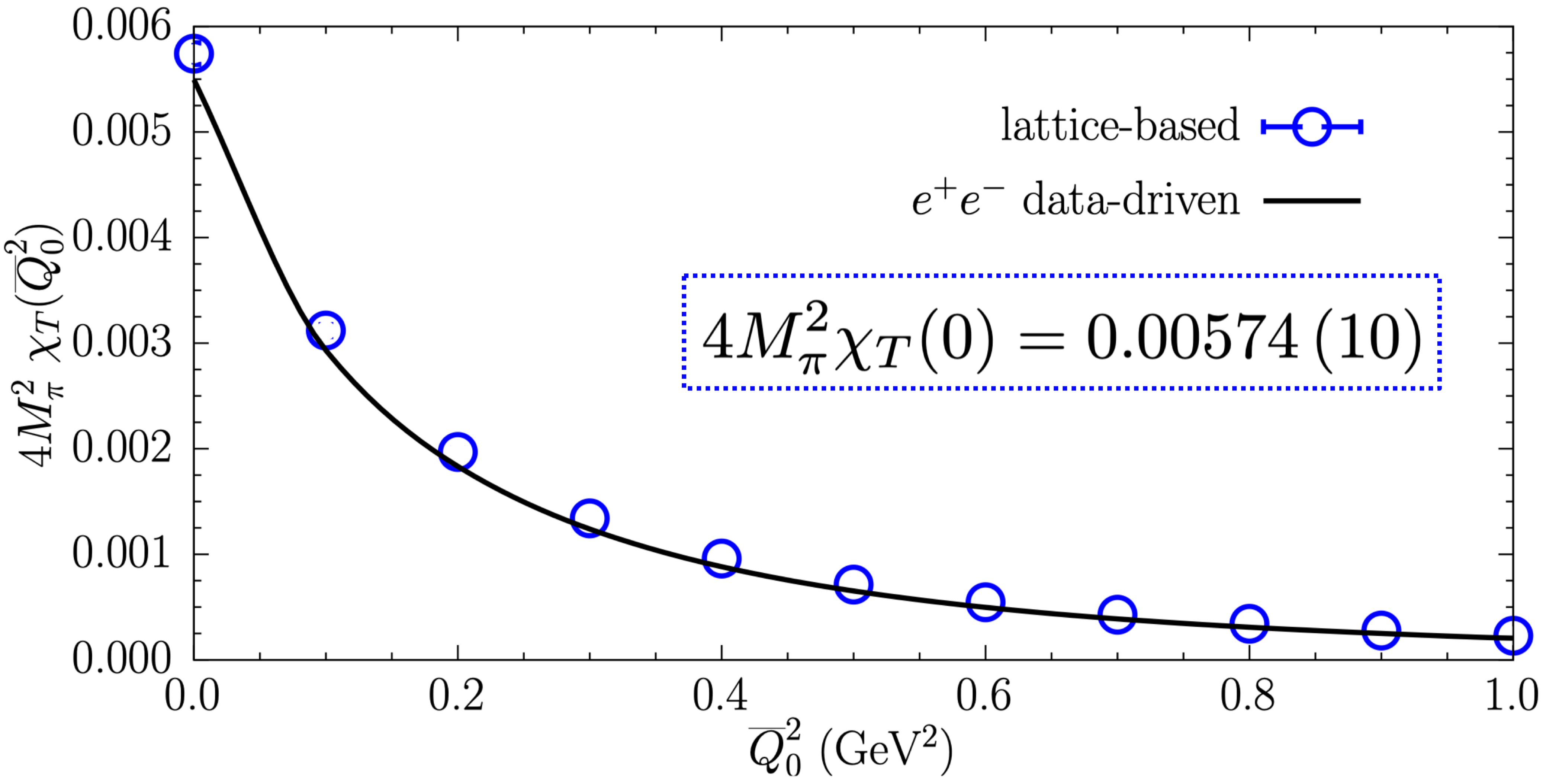


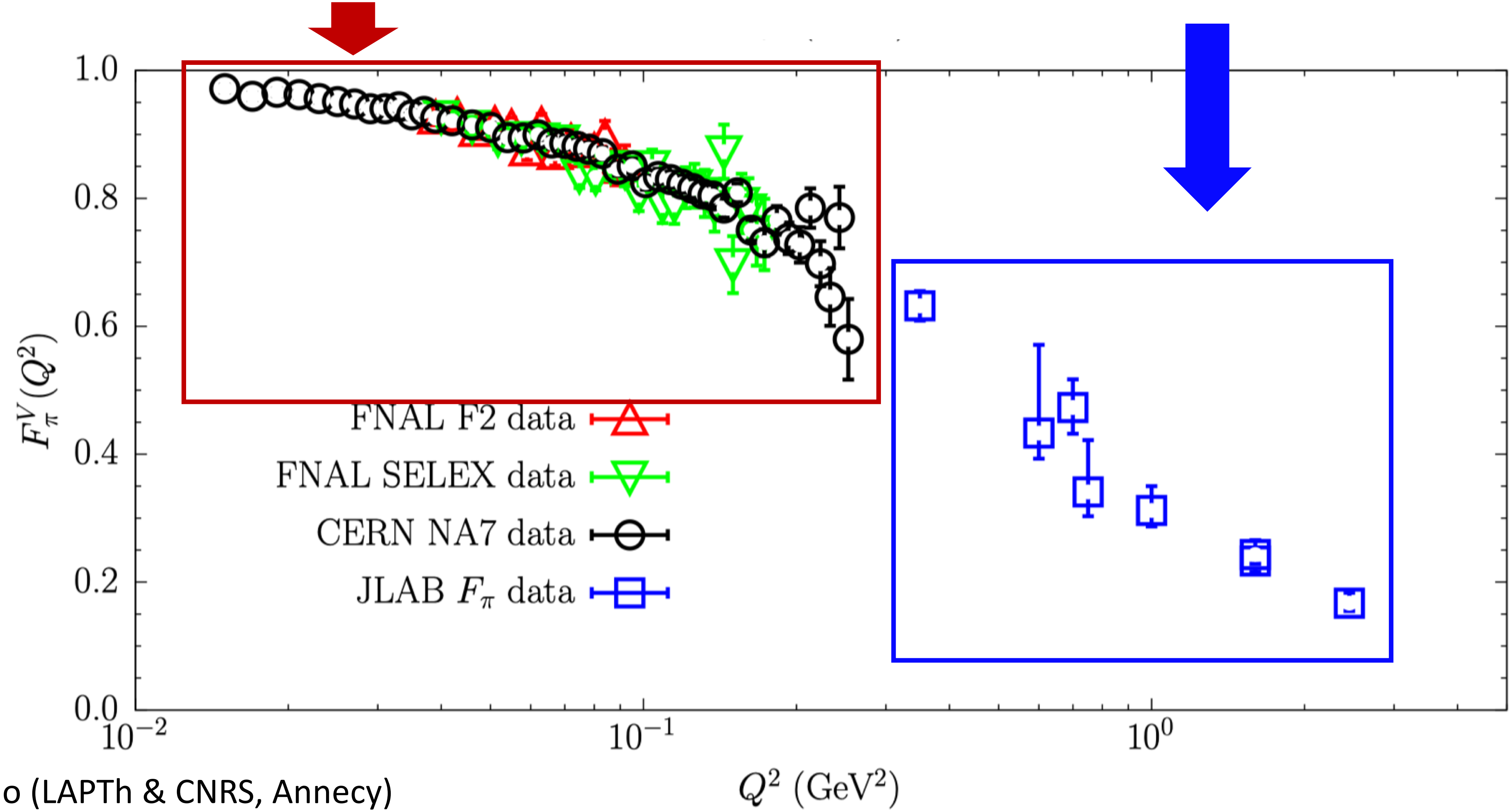
IMPORTANT: we trust our results in the region below 1 GeV !!

Study of spacelike data

Data measuring directly the scattering of high-energy pions off atomic electrons

+

Data extracted from pion electroproduction off the proton (off-shell VS on-shell pions)



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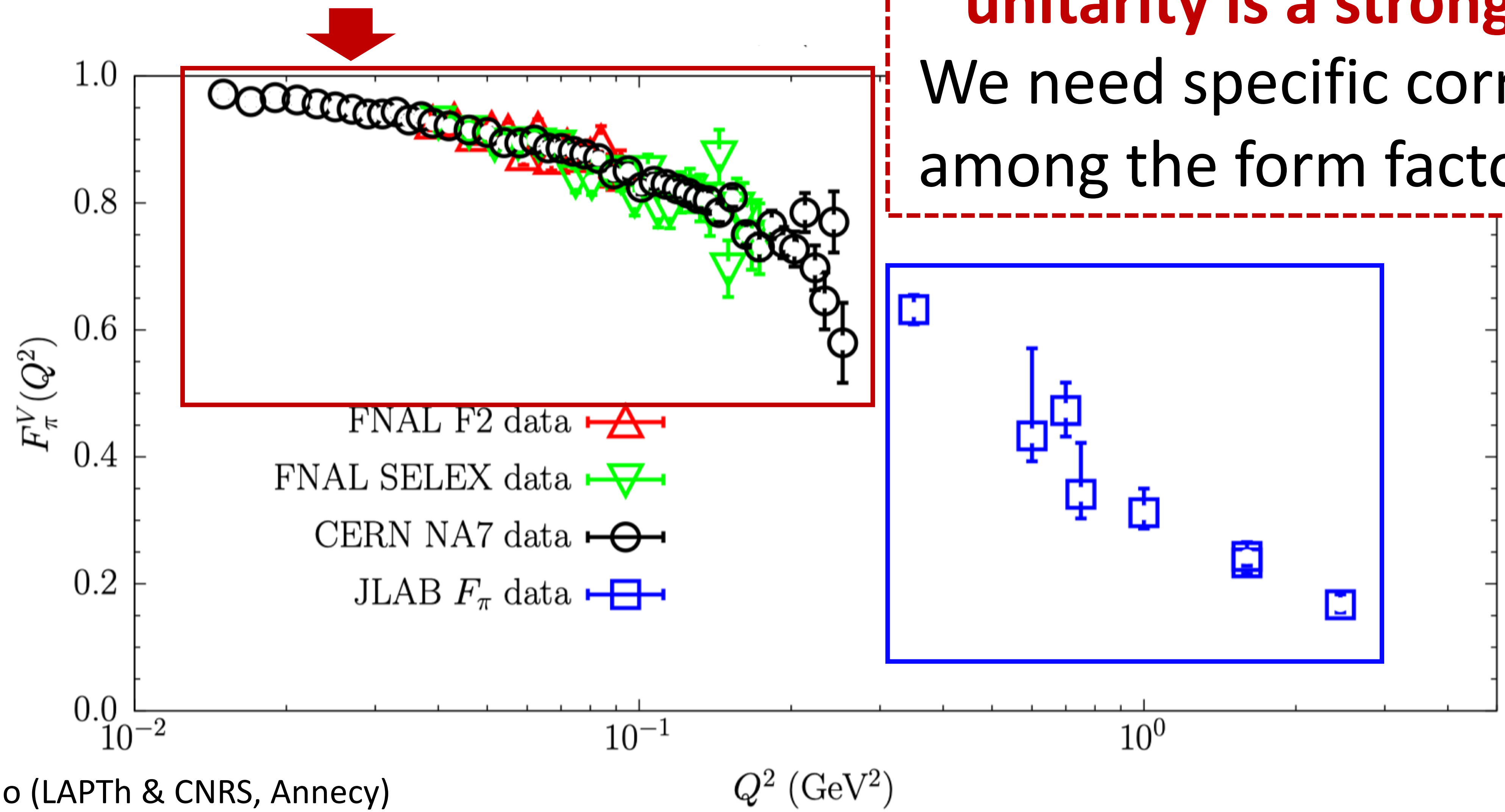
With a «large» input dataset, unitarity is a strong filter!

We need specific correlations among the form factor points



The Importance Sampling (IS) procedure

$$PDF(f_i) \propto e^{-\frac{1}{2}\sum_{i,j=0}^{N}(f_i - F_i)C_{ij}^{-1}(f_j - F_j)}$$

UNITARITY: $\chi_{DM} \leq \chi$

$$PDF_{DM}(f_i) \propto PDF(f_i) \cdot e^{-\frac{s}{4M_\pi^2 \chi_T(\overline{Q}_0^2)}\chi_{DM}(\overline{Q}_0^2)}$$

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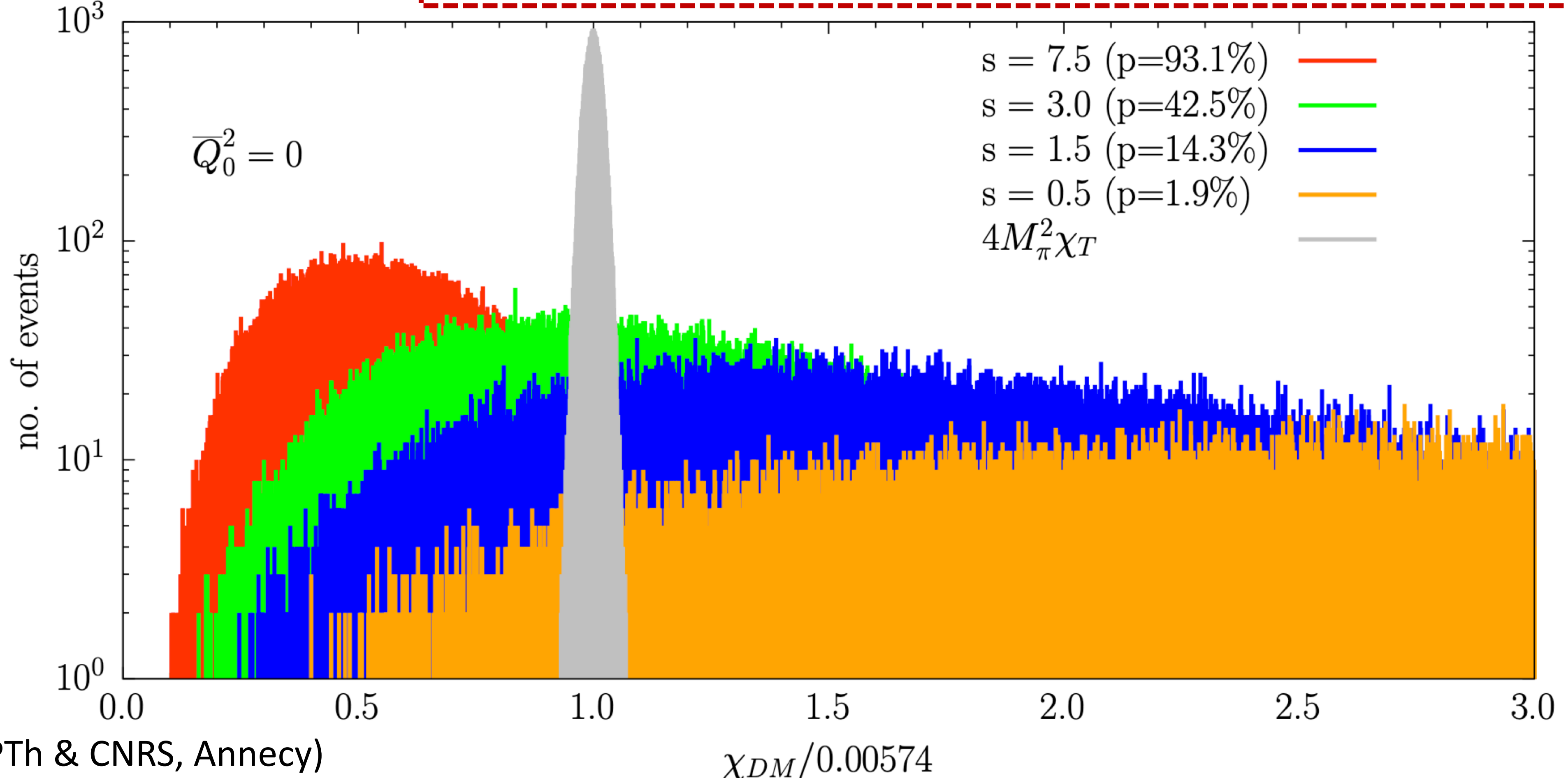
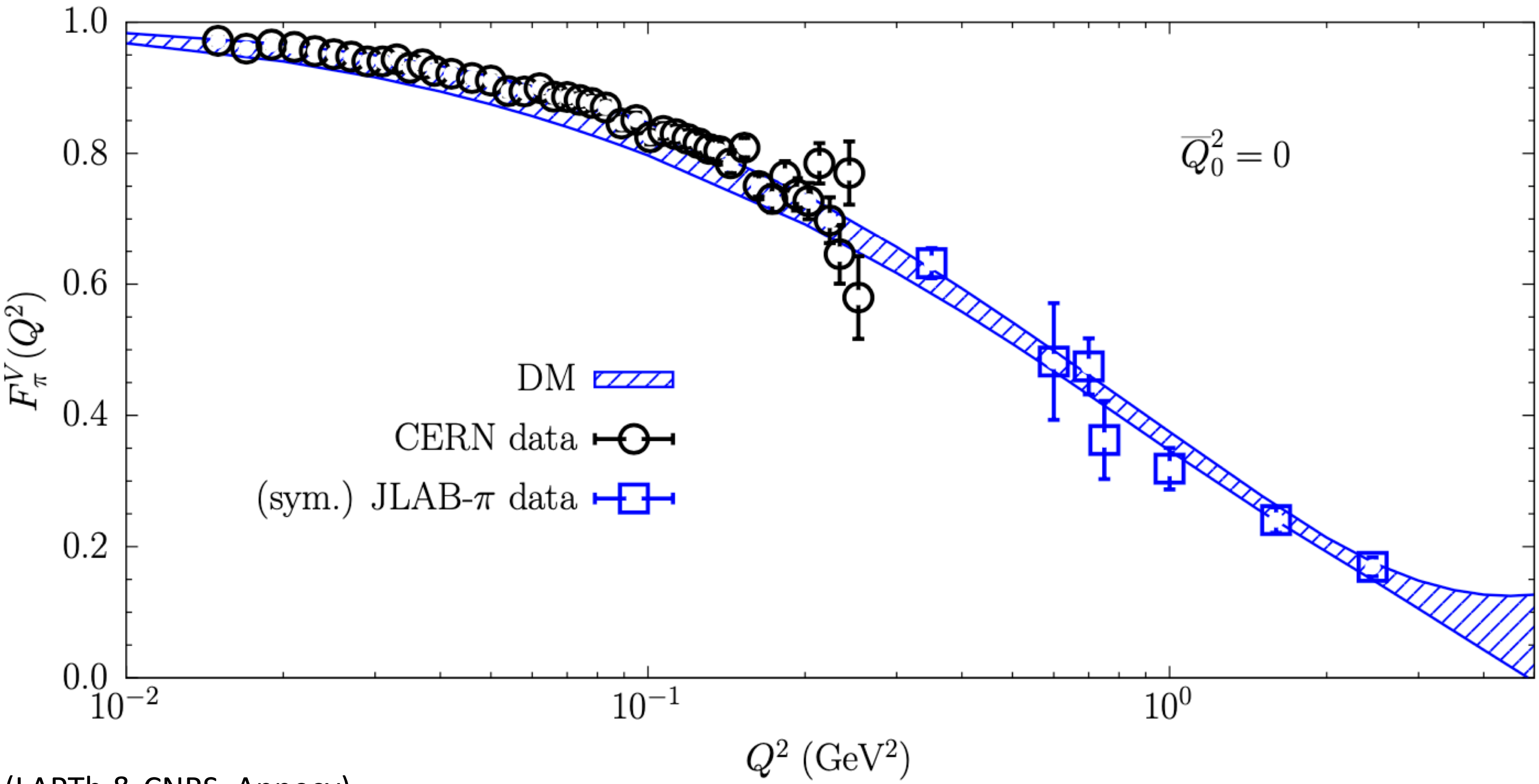


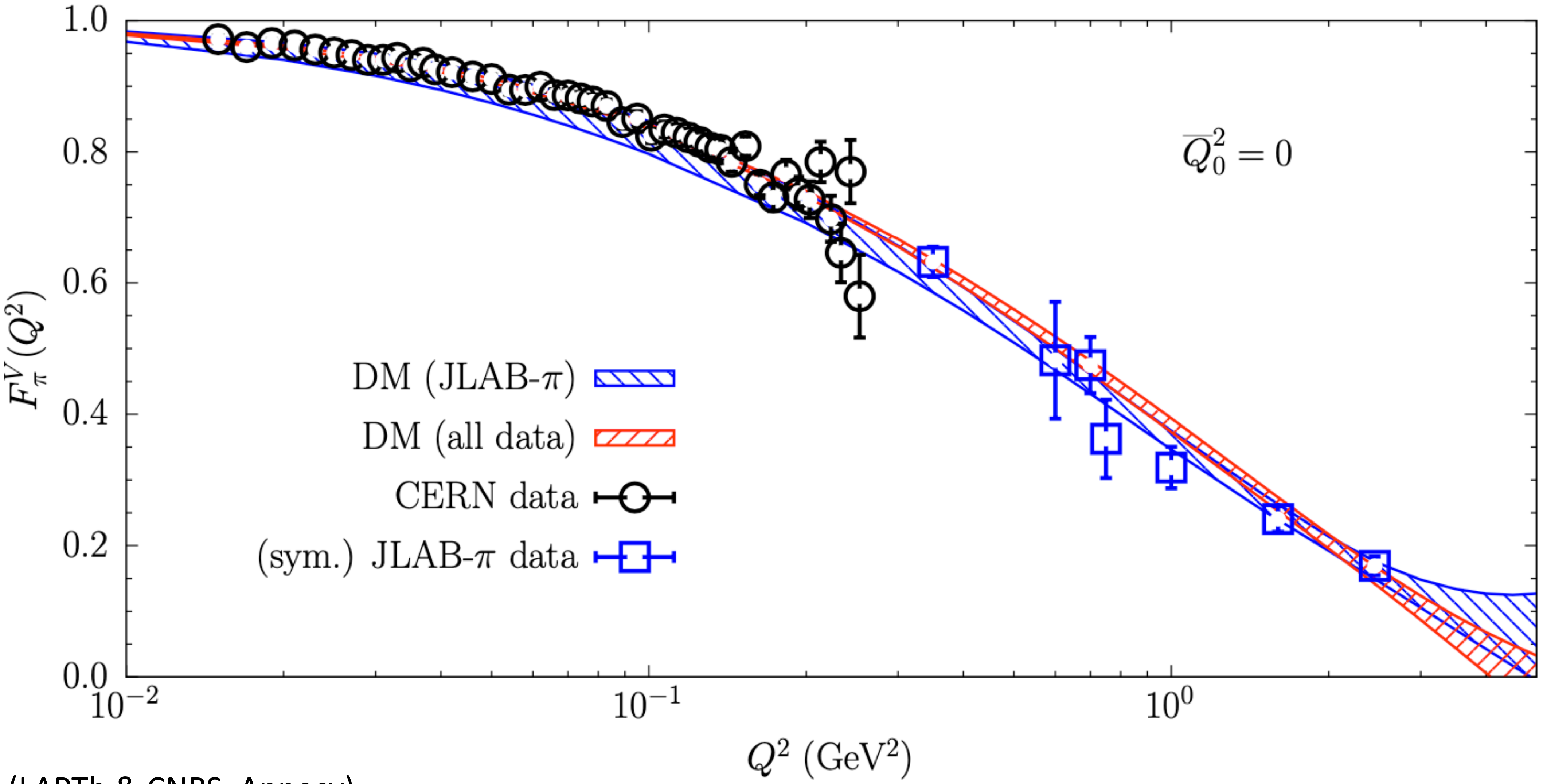
Relevant quantities for monitoring the results of IS DM

ONLY ELECTROPRODUCTION DATA AS INPUTS



Relevant quantities for monitoring the results of IS DM

ALL DATA AS INPUTS



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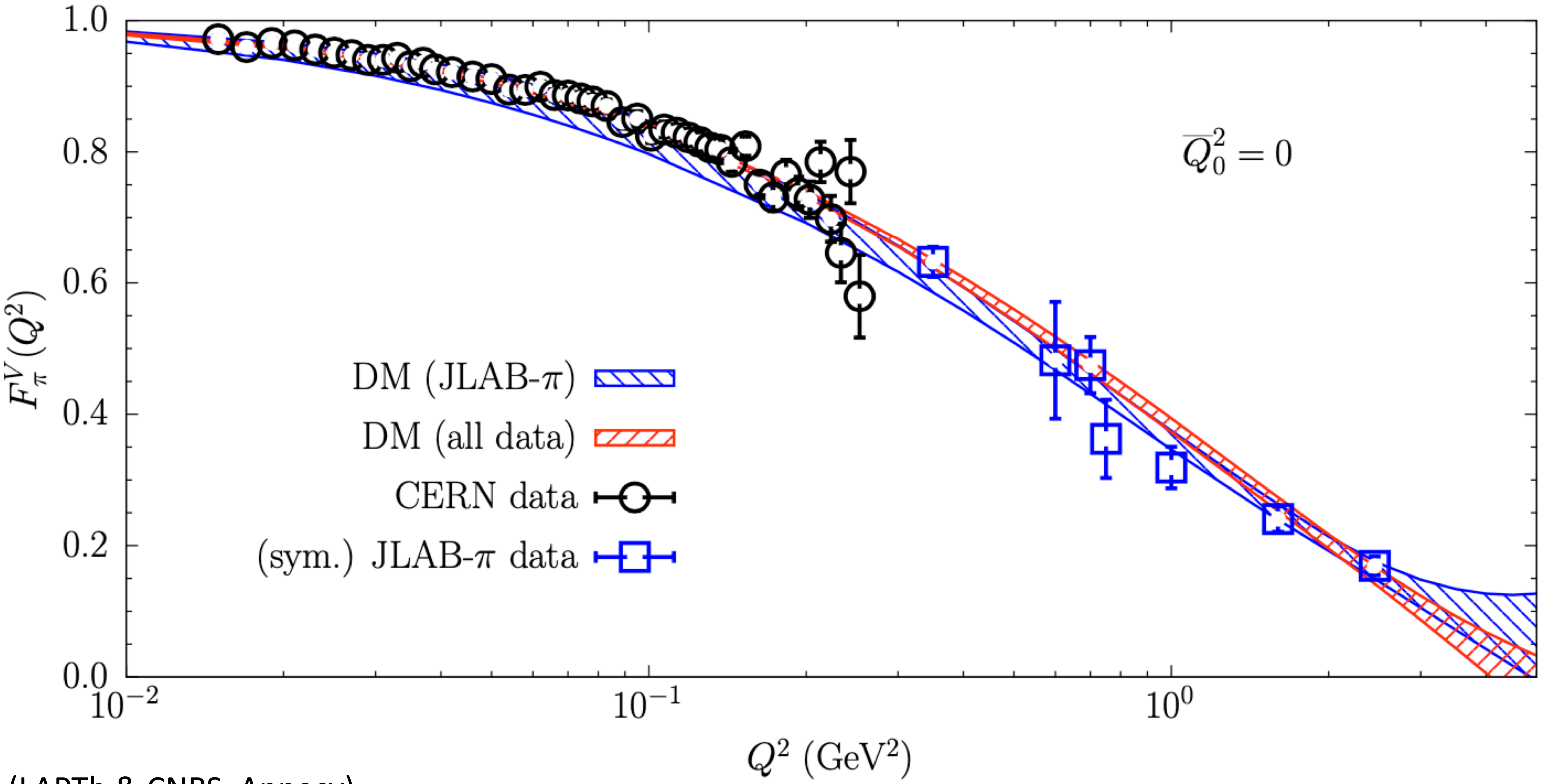
~~**SPOILER:**~~ $\langle r_\pi \rangle_{DM} = 0.703 \pm 0.027$ fm



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MONOPOLE:	***MONOPOLE+DIPOLE:***
$\langle r_\pi \rangle = 0.656 \pm 0.008 \text{ fm}$	$\langle r_\pi \rangle = 0.699 \pm 0.024 \text{ fm}$
$\chi^2/(d.o.f.) \simeq 1.0$	$\chi^2/(d.o.f.) \simeq 1.0$

Significative model dependence !!

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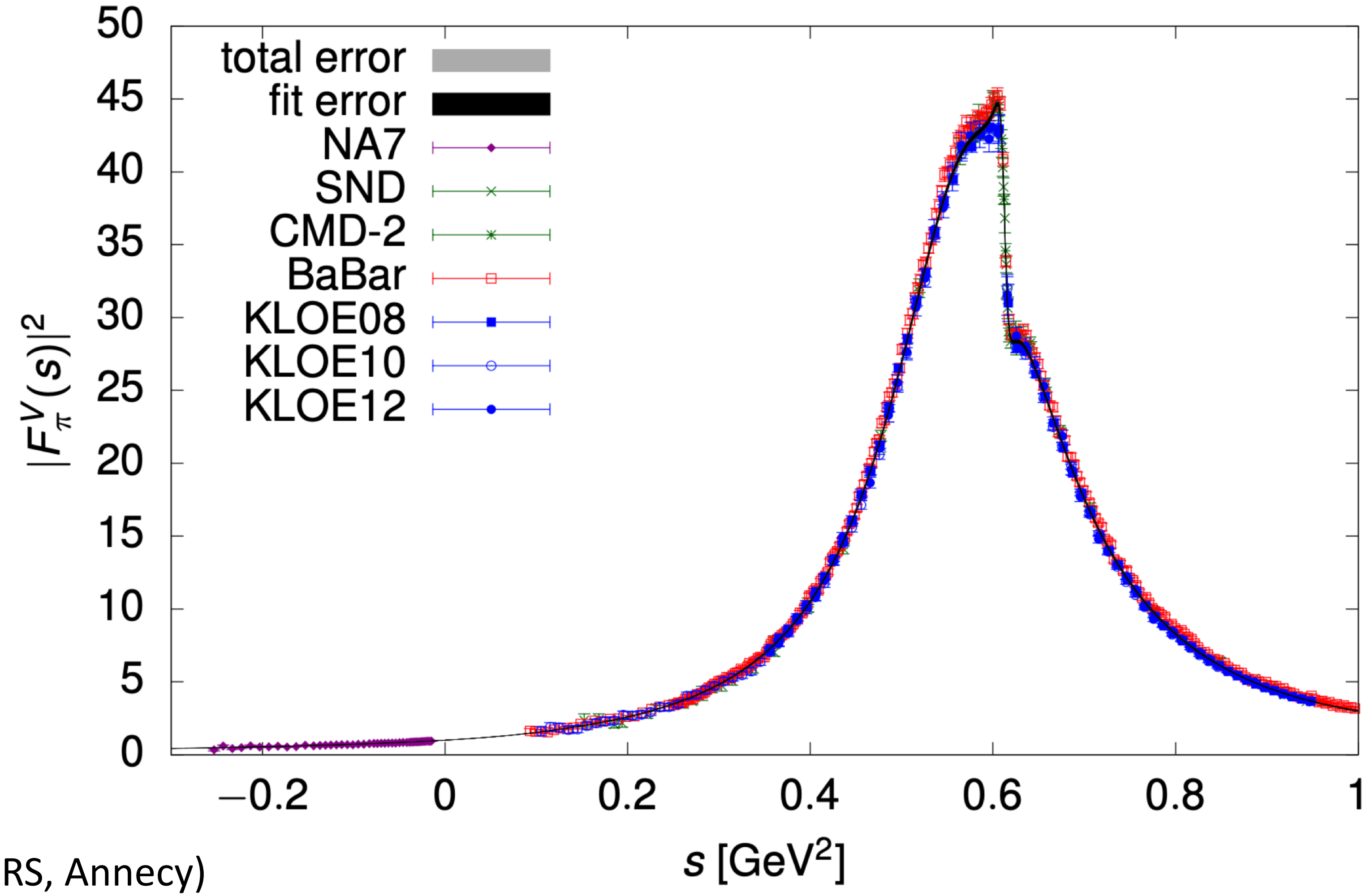

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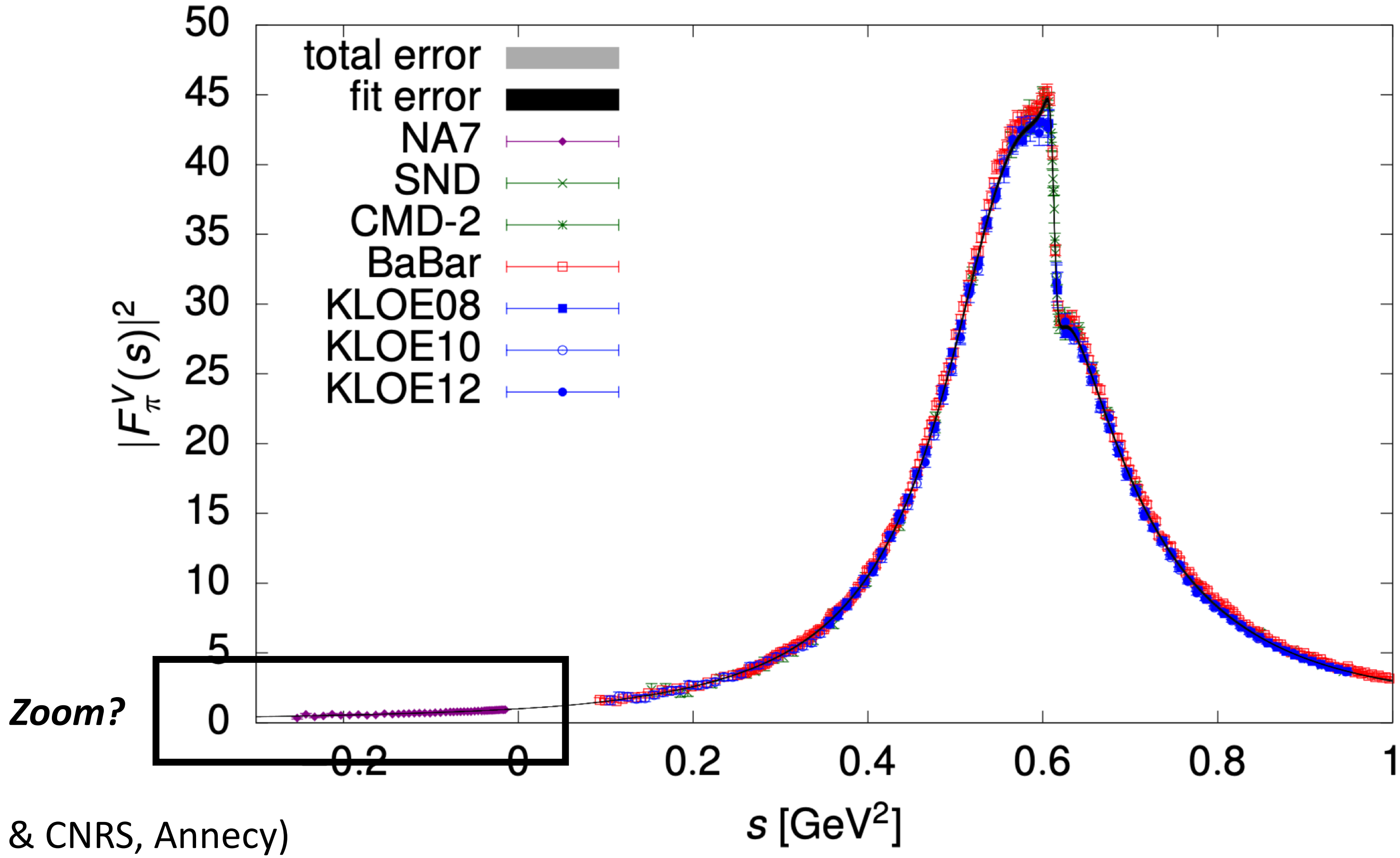

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Comparison with other results

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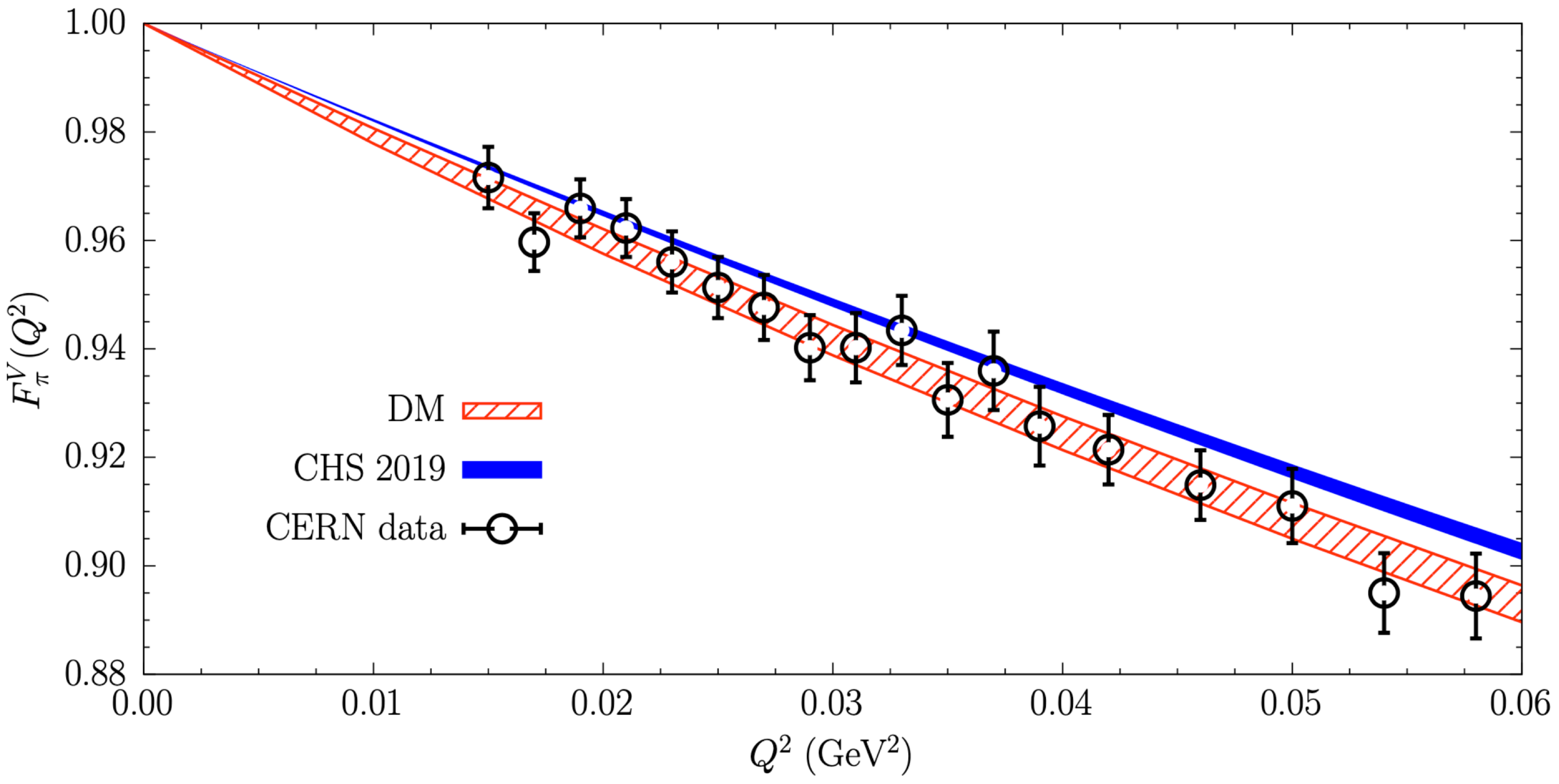
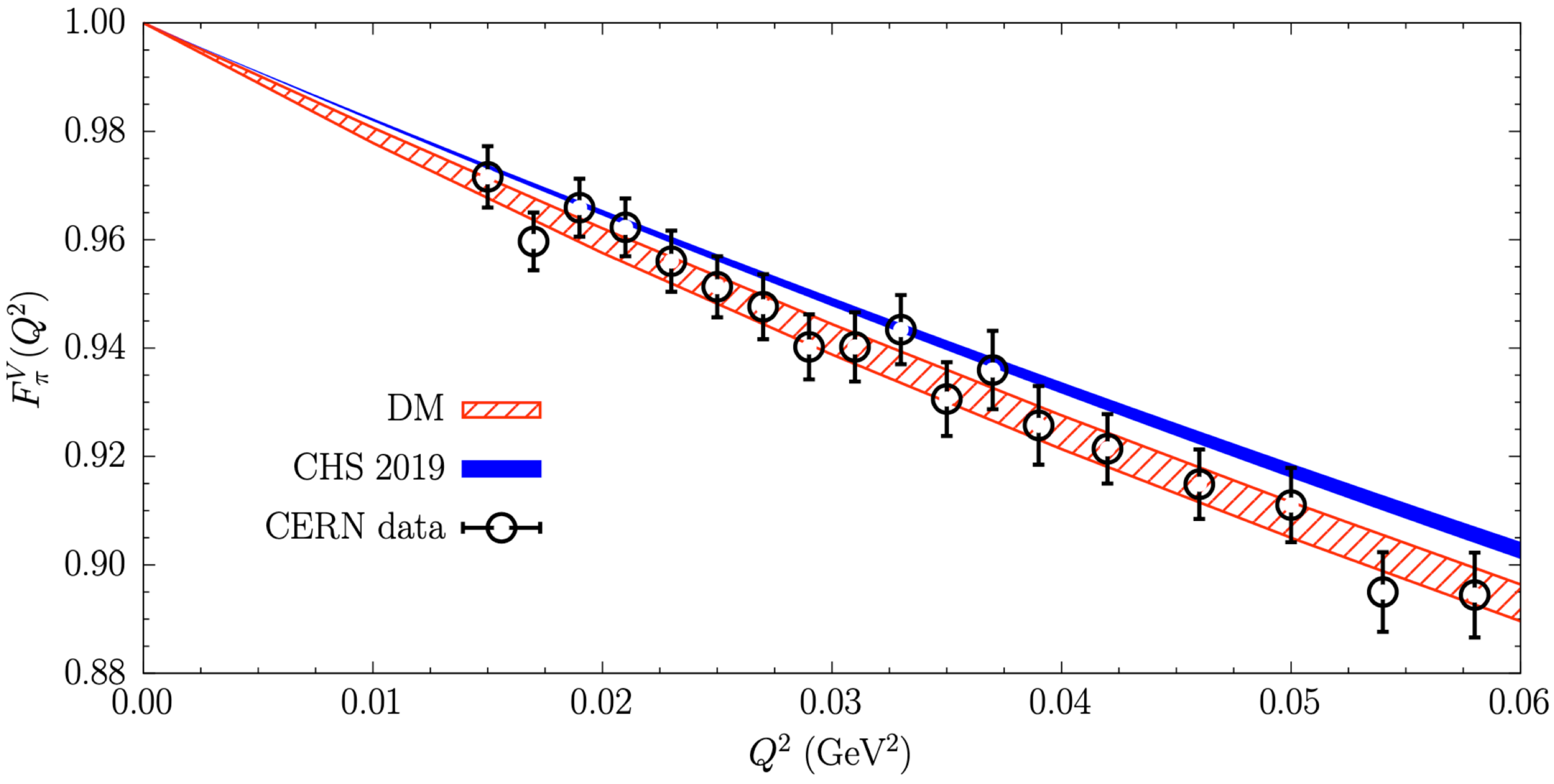
i) $\langle r_\pi \rangle = 0.656 \pm 0.005\ \text{fm}$ from an **average of dispersive analyses of timelike (e+e−) and spacelike data**



Comparison with other results



3 important comments:

i) the use of the **very precise and dense timelike e+e− data leads to the accurate result** for the pion charge radius

ii) the **DM band** is **in better agreement with the spacelike CERN data** w.r.t. to the blue one

iii) it is extremely interesting to see **what could be the possible impact of the recent CMD3 experimental data (arXiv:2302.08834 [hep-ex]) obtained in the timelike region on the dispersive estimate of the pion charge radius** …

Conclusions

The **experimental data on the em form factor of charged pions available at spacelike momenta** have been analyzed using **the DM approach, which describes the momentum dependence of hadronic form factors without introducing any explicit parameterization and includes properly the constraint coming from unitarity and analyticity**.

Main take-home messages :

i) Our value of the pion charge radius is higher than the PDG

ii) Unitarity and model-independence matter!!

iii) If we analyze separately spacelike and timelike data, we obtain different values of the pion charge radius… What about CMD-3 (arXiv:2302.08834) ?

iv) The Importance Sampling (IS) procedure allows to include an arbitrarily high number of input data

Conclusions

For those who are interested: in the paper two other technical issues have been analyzed in detail (no time to tell you this in detail here):

i) Comparison among the DM method and BGL/BCL fitting procedures

ii) Impact of non-zero values of $\overline{Q}_0$

iii) Issue of the onset of pQCD at large spacelike momenta (sensitivity study)

iv) (related to ii)) Insight on the pre-asymptotic effects related to the scale dependence of the pion distribution amplitude

THANKS FOR
YOUR ATTENTION!

BACK-UP SLIDES

aHVP from experimental data

Some basic definitions:

$$a^{HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 K(Q^2)[\Pi(Q^2) - \Pi(0)]$$

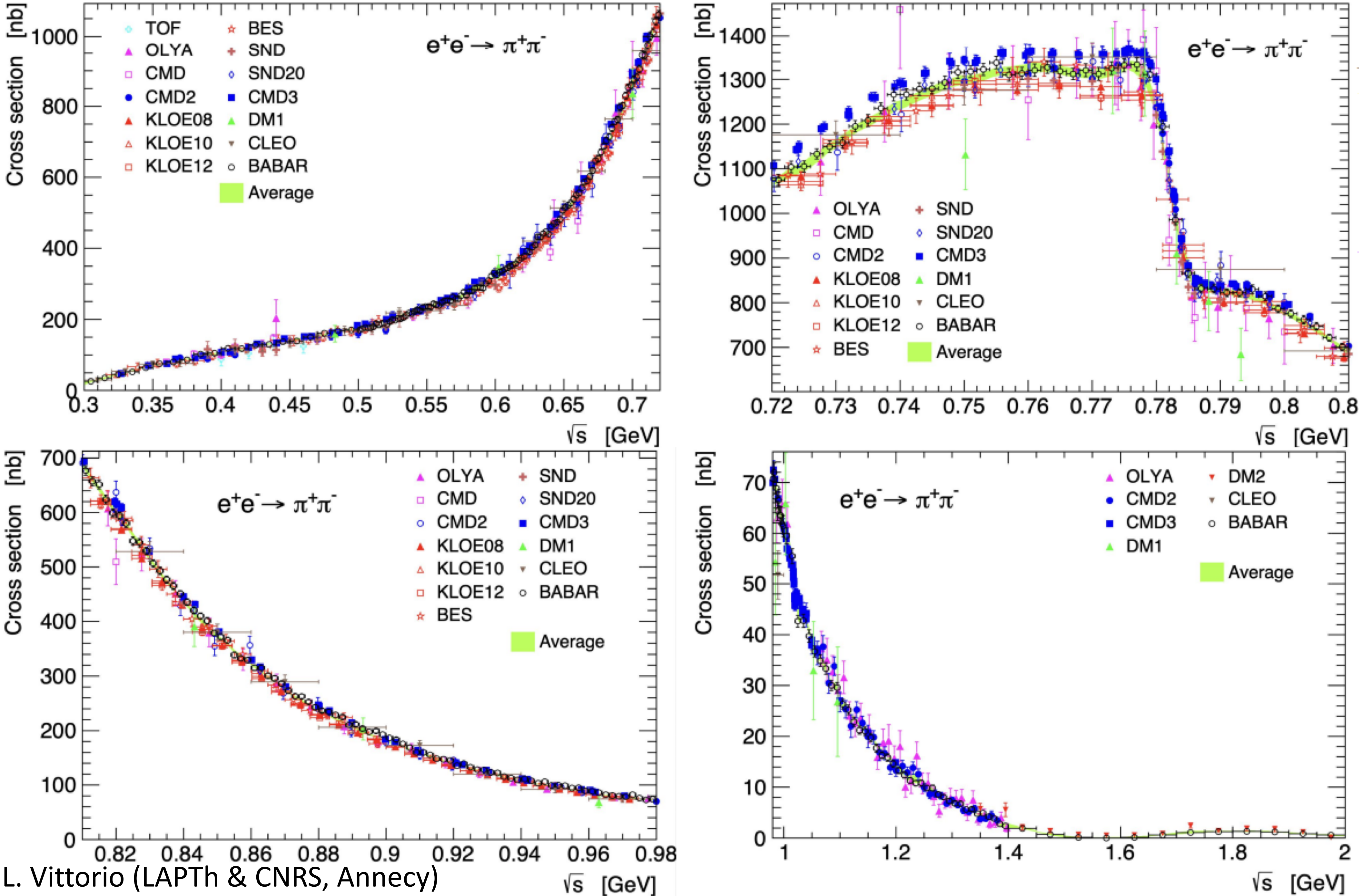
The Hadronic Vacuum Polarization (HVP) tensor is defined as

$$\Pi_{\mu\nu}(Q^2) = \int d^4x e^{iQ\cdot x} \langle j_\mu(x) j_\nu(0) \rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

$$j_\mu(x) \equiv \sum_f q_f \bar{\psi}_f(x)\gamma_\mu \psi_f(x)$$

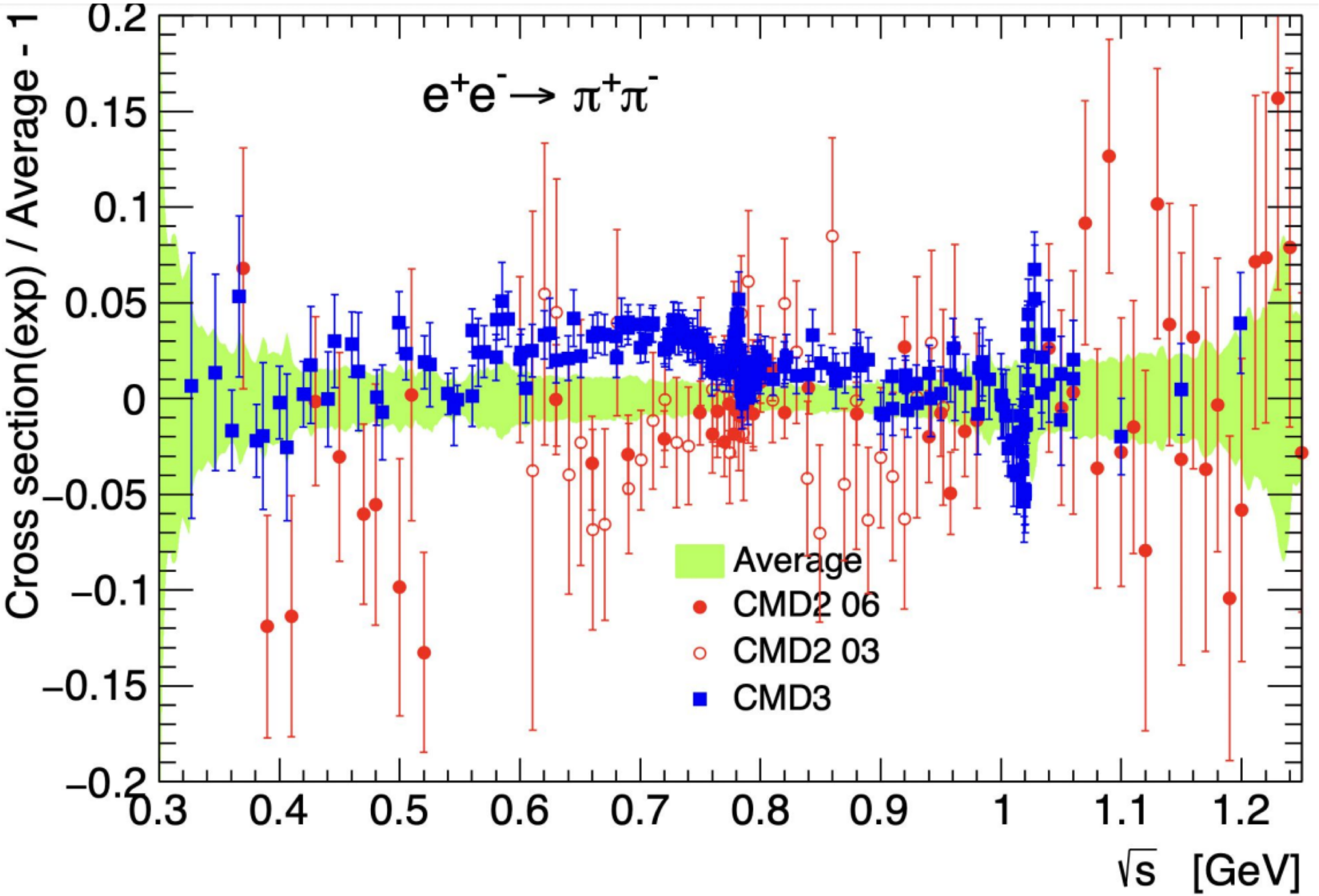
Dispersion relations:

$$\Pi_{ren} = \Pi(k^2) - \Pi(0) = \frac{k^2}{\pi} \int_{s_{thr}}^{\infty} ds \frac{\operatorname{Im}\Pi(s)}{s(s-k^2)}$$



Febr. '23: CMD3 results

Malaescu's talk @ Muon g-2 Theory Initiative '23



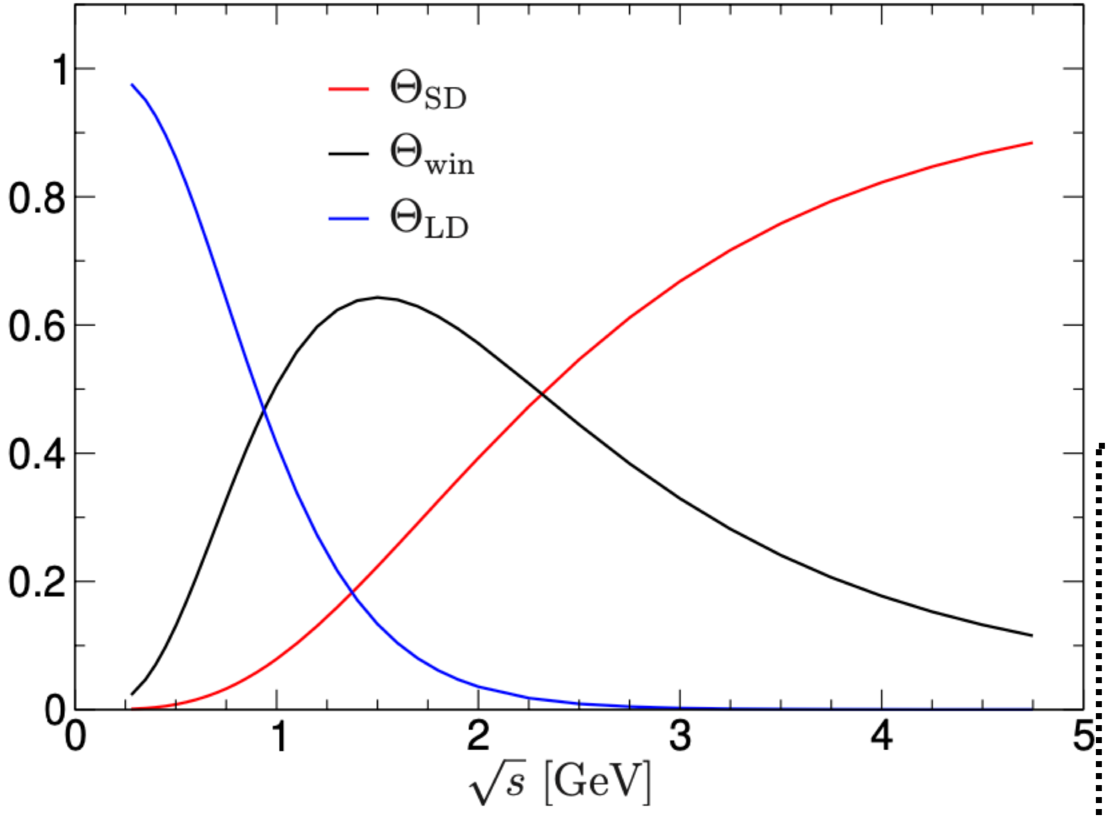
Tensions among CMD3 and CMD2 results

Important Exp. Issue!

Malaescu's talk @ Muon g-2 Theory Initiative '23

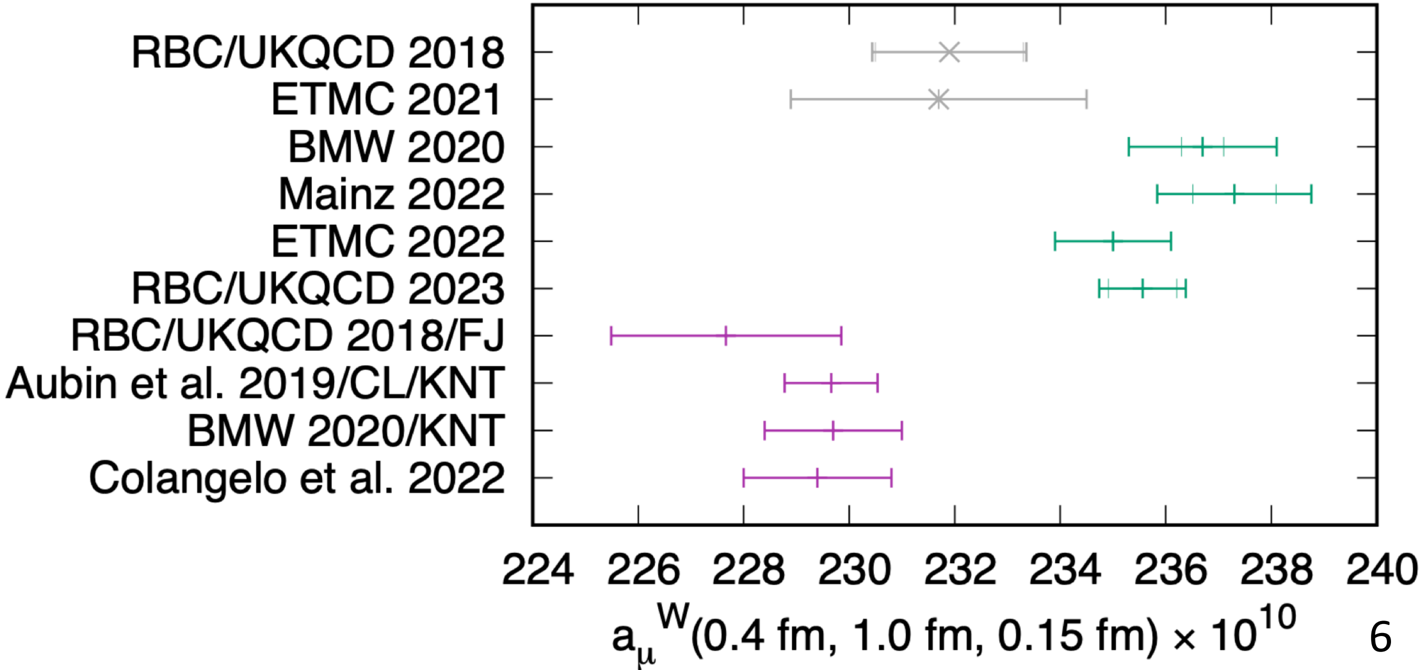
The *new* g-2 puzzle

Many lattice computations are now compatible with BMW Collaboration:



Weigth functions (*windows*)
[proposal by RBC/UKQCD Coll. (2018)]

The intermediate region is the one where the tension among dispersive analyses and lattice is stronger!!



The DM method

The positivity of the original inner products guarantee that $\det \mathbf{M} \geq 0$: the solution of this inequality can be computed analitically, bringing to

LOWER bound $\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$ *UPPER bound*

$$\beta(z) \equiv \frac{1}{\phi(z) d(z)} \sum_{i=1}^{N} \phi_i f_i d_i \frac{1 - z_i^2}{z - z_i} \qquad \gamma = \frac{1}{d^2(z) \phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

UNITARITY FILTER: unitarity is satisfied if γ is semipositive definite, namely if

$$\chi \geq \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$$

Unitarity is built-in!

Statistical and systematic uncertainties

How can we finally combine all the N_U lower and upper bounds of both the FFs??

One bootstrap event case:
after a single extraction, we have one value of the lower bound f_L and one value of the upper one f_U for each FF. Assuming that the true value of each FF can be **everywhere inside the range (f_U - f_L) with equal probability**, we associate to the FFs a *flat* distribution

$$P(f_{0(+)}) = \frac{1}{f_{U,0(+)} - f_{L,0(+)}}\Theta(f_{0(+)} - f_{L,0(+)})\Theta(f_{U,0(+)} - f_{0(+)})$$

Many bootstrap events case:
how to mediate over the whole set of bootstrap events? Since the lower and the upper bounds of a generic FF are deeply correlated, we will assume a multivariate Gaussian distribution:

$$P(f_L, f_U) = \frac{\sqrt{\det \rho}}{2\pi} \exp\left[-\frac{\rho_{up,up}(f_U - \langle f_U\rangle)^2 + \rho_{lo,lo}(f_L - \langle f_L\rangle)^2 + 2\rho_{lo,up}(f_U - \langle f_U\rangle)(f_L - \langle f_L\rangle)}{2}\right]$$

In conclusion, we can combine the bounds of each FF in a final mean value and a final standard deviation, defined as

$$\langle f\rangle = \frac{\langle f_L\rangle + \langle f_U\rangle}{2},$$

$$\sigma_f = \frac{1}{12}(\langle f_U\rangle - \langle f_L\rangle)^2 + \frac{1}{3}(\sigma^2_{f_{lo}} + \sigma^2_{f_{up}} + \rho_{lo,up}\sigma_{f_{lo}}\sigma_{f_{up}})$$

NO PARAMETRIZATION ADOPTED!!!

Parametrization of pion FF in CHS '19

$$F_\pi^V(s) = \Omega_1^1(s) G_\omega(s) G_{\text{in}}^N(s)$$

$$\Omega_1^1(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

Omnes
(δ_1^1 is the phase shift of elastic $\pi\pi$ scattering)

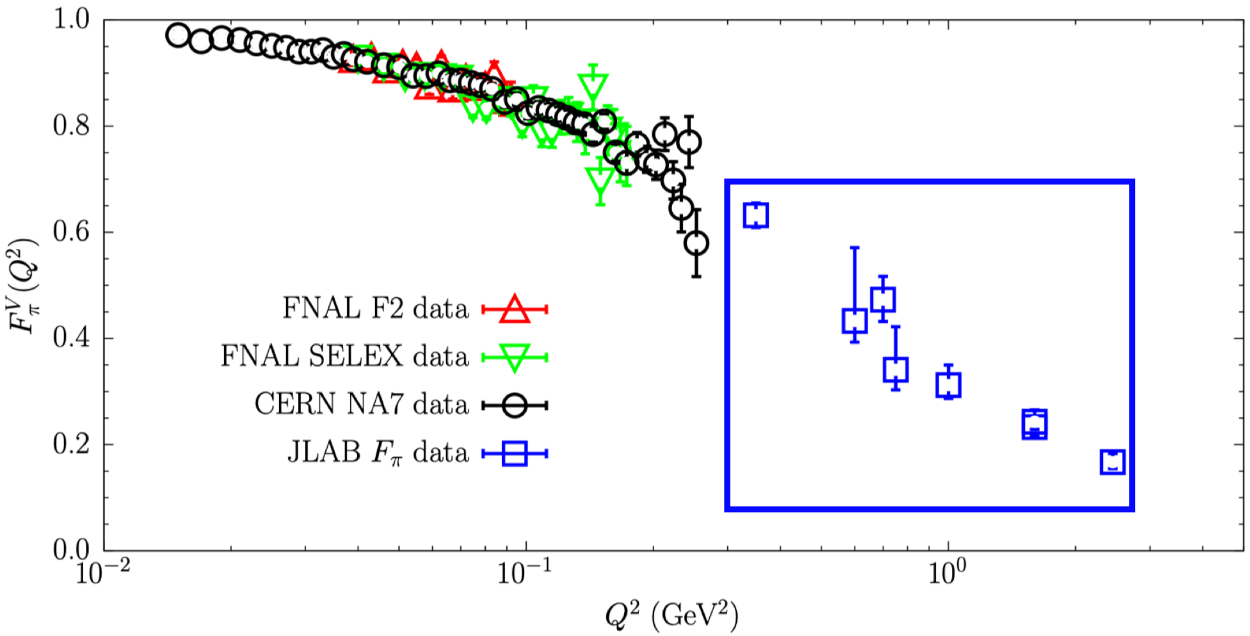
ρ-ω mixing

$$G_\omega(s) = 1 + \frac{s}{\pi} \int_{9M_\pi^2}^{\infty} ds' \frac{\text{Im}\, g_\omega(s')}{s'(s'-s)} \left(\frac{1 - \frac{9M_\pi^2}{s'}}{1 - \frac{9M_\pi^2}{M_\omega^2}} \right)^4$$

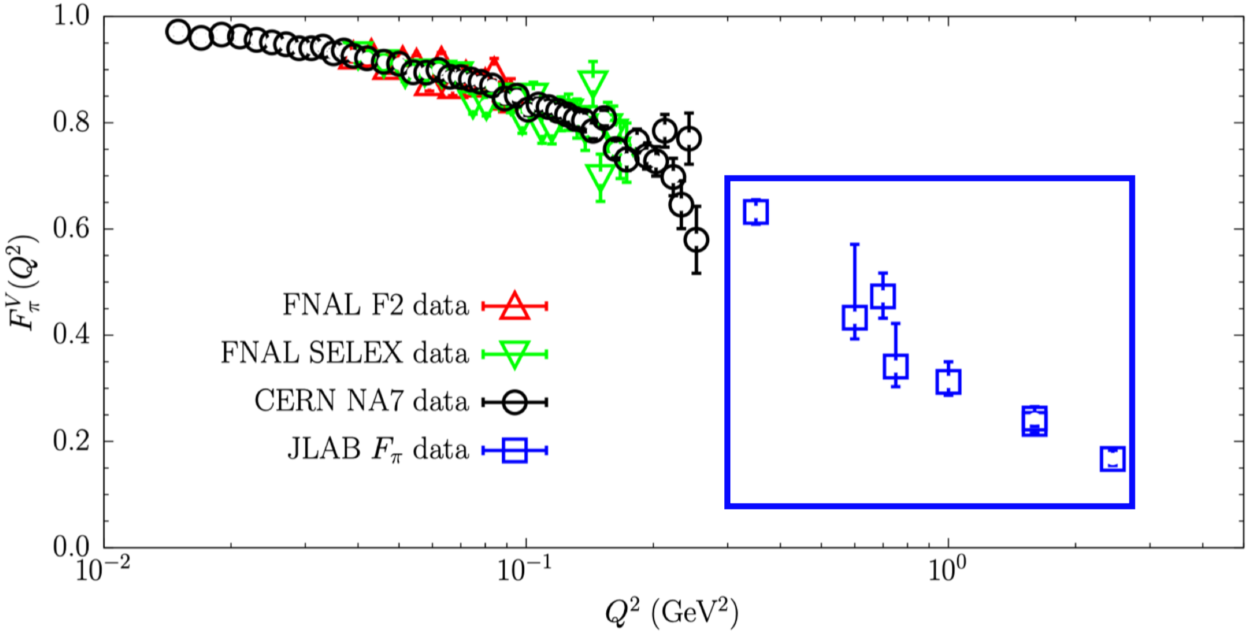
$$g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}$$

$$G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k (z^k(s) - z^k(0))$$

Further inelastic scattering
(starting from $16M_\pi$^2)



i	Q_i^2 (GeV^2)	z_i	F_i	F_i (sym.)	d_i	$\phi_i(\overline{Q}_0^2 = 0)$
0	0.0	0.0	1.0	1.0	$-7.02 \cdot 10^1$	0.0144
1	0.35	0.402	0.632^{+23}_{-23}	0.632 (23)	$+2.88 \cdot 10^4$	0.0219
2	0.60	0.494	0.433^{+138}_{-40}	0.482 (89)	$-1.26 \cdot 10^6$	0.0229
3	0.70	0.519	0.473^{+44}_{-41}	0.475 (43)	$+5.48 \cdot 10^6$	0.0230
4	0.75	0.530	0.341^{+81}_{-38}	0.363 (60)	$-4.73 \cdot 10^6$	0.0231
5	1.00	0.576	0.312^{+38}_{-25}	0.319 (32)	$+5.34 \cdot 10^5$	0.0233
6	1.60	0.645	0.238^{+21}_{-17}	0.240 (19)	$-5.66 \cdot 10^4$	0.0232
7	2.45	0.701	0.167^{+16}_{-12}	0.169 (14)	$+6.45 \cdot 10^3$	0.0228



DM master formulae:

$$\beta - \sqrt{\gamma} \leq f(z) \leq \beta + \sqrt{\gamma}$$

$$\beta(z) \equiv \frac{1}{\phi(z) d(z)} \sum_{i=1}^{N} \phi_i f_i d_i \frac{1-z_i^2}{z-z_i}$$

$$\gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1-z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1-z_i^2)(1-z_j^2)}{1-z_i z_j} \right]$$

$$d(z) \equiv \prod_{m=1}^{N} \frac{1-z z_m}{z-z_m} \qquad d_i \equiv \prod_{m \neq i=1}^{N} \frac{1-z_i z_m}{z_i-z_m}$$

i	Q_i^2 (GeV^2)	z_i	F_i	F_i (sym.)	d_i	$\phi_i(\overline{Q}_0^2 = 0)$
0	0.0	0.0	1.0	1.0	$-7.02 \cdot 10^1$	0.0144
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7	2.45	0.701	0.167^{+16}_{-12}	0.169 (14)	$+6.45 \cdot 10^3$	0.0228

With a «large» input dataset, unitarity is a strong filter!

A very delicate compensation in γ is required and this naturally implies specific correlations among the form factor points

The Importance Sampling (IS) procedure

The basic idea is a substitution of the usual probability density function (PDF) adopted in our analyses:

$$PDF(f_i) \propto e^{-\frac{1}{2}\sum_{i,j=0}^{N}(f_i - F_i)C_{ij}^{-1}(f_j - F_j)}$$

$$\Downarrow$$

$$PDF_{DM}(f_i) \propto PDF(f_i) \cdot e^{-\frac{s}{\chi_T(\overline{Q}_0^2)}\chi_{DM}(\overline{Q}_0^2)}$$

$$\beta - \sqrt{\gamma} \leq f(z) \leq \beta + \sqrt{\gamma}$$

χDM

$$\beta = \frac{1}{d(z)\phi(z)}\sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)}\frac{1}{1 - z^2}\left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}\right]$$

The Importance Sampling (IS) procedure

The basic idea is a substitution of the usual probability density function (PDF) adopted in our analyses:

$$PDF_{DM}(f_i) \propto e^{-\frac{1}{2}\sum_{i,j=0}^{N}(f_i-\widetilde{F}_i)\widetilde{C}_{ij}^{-1}(f_j-\widetilde{F}_j)}$$

In short: a new set of input data $\{\widetilde{F}_i, \widetilde{C}_{ij}\}$ is introduced in order to increase the likelihood of small values of χDM !

$$\beta - \sqrt{\gamma} \leq f(z) \leq \beta + \sqrt{\gamma}$$

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \underbrace{\sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}}_{\chi\text{DM}} \right]$$

Relevant quantities for monitoring the results of IS DM

Recall that the **DM** remains a **fitting procedure with a vanishing value of the χ2-variable in a frequentist language**!
Then, we have to monitorate the deviation of the new input data from the initial ones thorugh the quantities

$$\Delta \equiv \left\{ \frac{1}{N+1} \sum_{i,j=0}^{N} (\widetilde{F}_i - F_i) C_{ij}^{-1} (\widetilde{F}_j - F_j) \right\}^{1/2}$$

Δ < 1 means that on average the new data deviate from the original ones by less than one standard deviation

$$\eta \equiv \left\{ \frac{1}{N+1} \sum_{i=0}^{N} \frac{\widetilde{F}_i^2}{F_i^2} \right\}^{1/2}$$

The value of η can be less or larger than unity depending on whether the new data are (on average) less or larger than original ones

$$\epsilon \equiv \left\{ \frac{1}{N+1} \sum_{i=0}^{N} \frac{\widetilde{C}_{ii}}{C_{ii}} \right\}^{1/2} = \left\{ \frac{1}{N+1} \sum_{i=0}^{N} \frac{\widetilde{\sigma}_i^2}{\sigma_i^2} \right\}^{1/2}$$

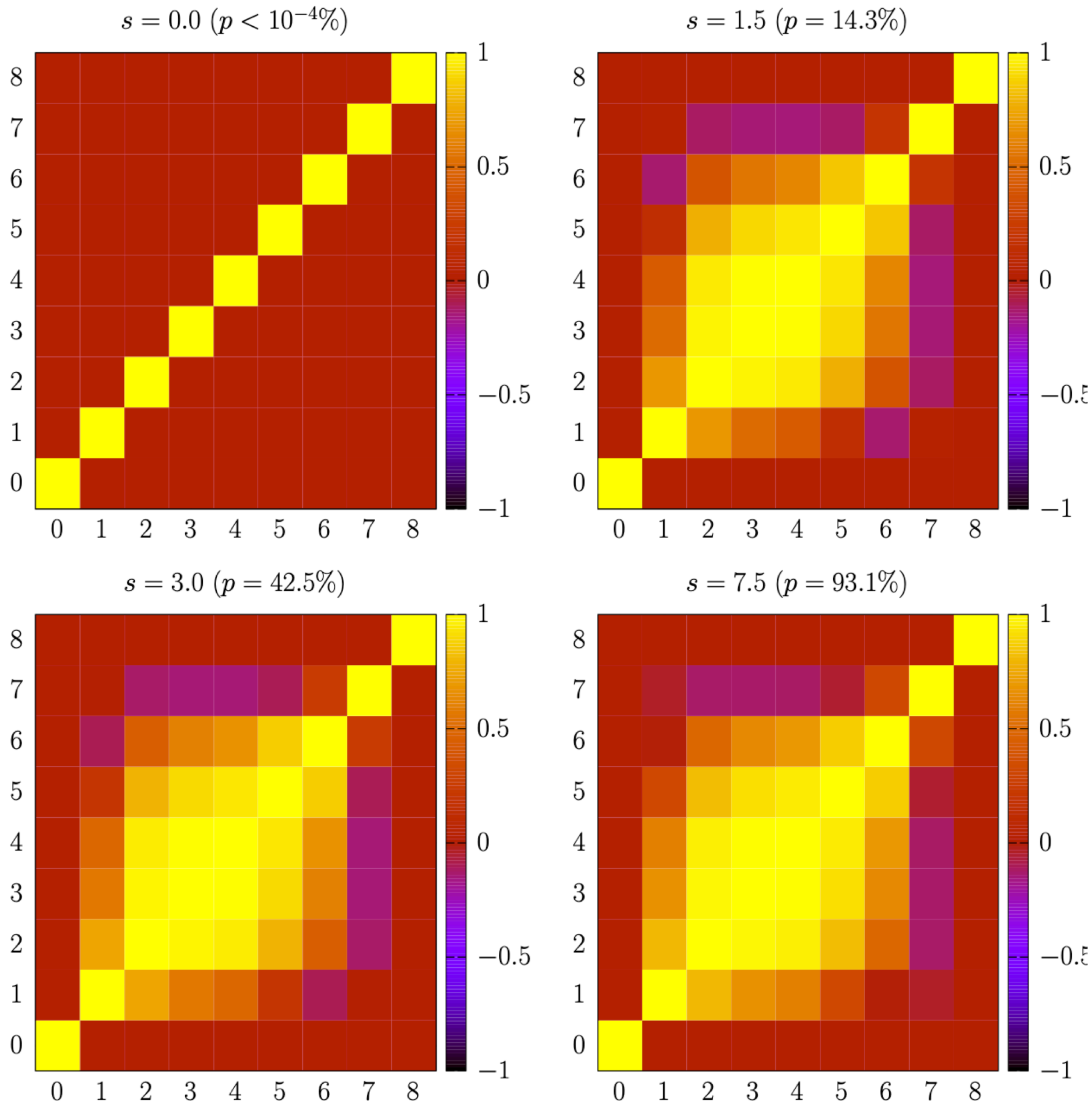
Same physical meaning of η, but now referred to the uncertaintities of the new data in comparison to the original ones

Relevant quantities for monitoring the results of IS DM

Q^2 (GeV^2)	F (sym.)	$\widetilde{F}$ ($s=0.5$)	$\widetilde{F}$ ($s=1.5$)	$\widetilde{F}$ ($s=3.0$)	$\widetilde{F}$ ($s=7.5$)
0.35	0.632 (23)	0.629 (22)	0.624 (21)	0.619 (21)	0.608 (20)
0.60	0.482 (89)	0.481 (18)	0.482 (16)	0.483 (16)	0.483 (16)
0.70	0.475 (43)	0.438 (18)	0.440 (16)	0.442 (16)	0.445 (15)
0.75	0.363 (60)	0.419 (18)	0.422 (17)	0.424 (16)	0.429 (15)
1.00	0.319 (32)	0.342 (18)	0.346 (17)	0.350 (16)	0.358 (15)
1.60	0.240 (19)	0.234 (15)	0.237 (15)	0.241 (14)	0.249 (13)
2.45	0.169 (14)	0.170 (14)	0.168 (14)	0.167 (14)	0.164 (14)
Δ	0.0	0.54	0.56	0.61	0.76
η	1.0	1.02	1.02	1.02	1.03
ϵ	1.0	0.72	0.70	0.69	0.69
p (%)	$< 10^{-4}$	1.9	14.3	42.5	93.1

Using the events surviving to the unitarity filter, I compute new data $\{\overline{F}_i, \overline{\sigma}_i\}$, with which we are finally able to get rid off the problem of unitarity and to compute the **final DM band for the em pion form factor**!

The Importance Sampling (IS) procedure

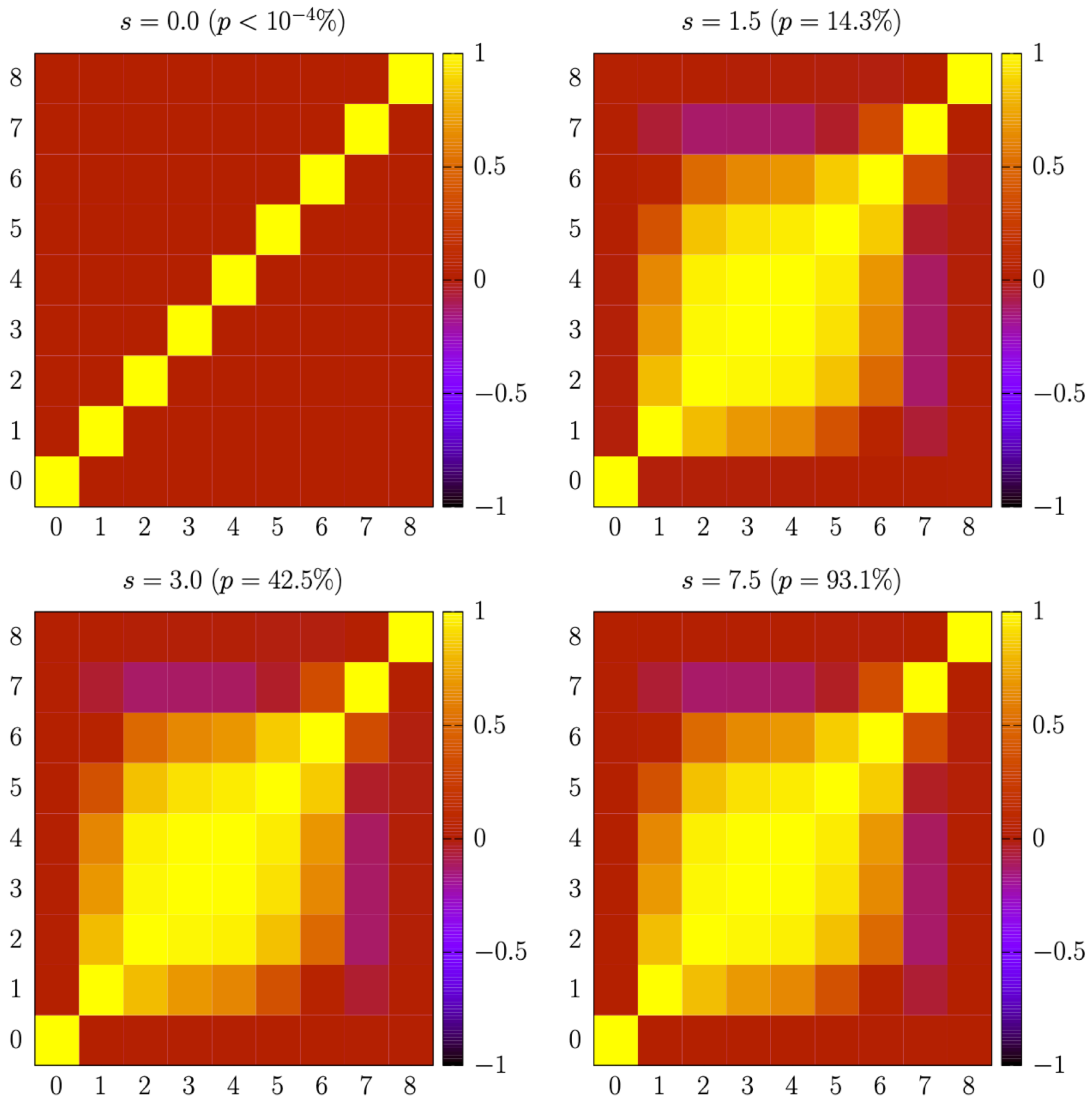


$$\widetilde{\rho}_{ij} \equiv \widetilde{C}_{ij}/(\widetilde{\sigma}_i\widetilde{\sigma}_j)$$

The Importance Sampling (IS) procedure

Q^2 (GeV^2)	F (sym.)	$\overline{F}$ ($s = 0.5$)	$\overline{F}$ ($s = 1.5$)	$\overline{F}$ ($s = 3.0$)	$\overline{F}$ ($s = 7.5$)
0.35	0.632 (23)	0.615 (19)	0.614 (19)	0.612 (19)	0.607 (20)
0.60	0.482 (89)	0.483 (16)	0.483 (16)	0.483 (16)	0.483 (16)
0.70	0.475 (43)	0.443 (16)	0.444 (16)	0.444 (15)	0.446 (15)
0.75	0.363 (60)	0.426 (16)	0.426 (16)	0.427 (15)	0.429 (15)
1.00	0.319 (32)	0.353 (15)	0.354 (15)	0.355 (15)	0.359 (15)
1.60	0.240 (19)	0.244 (13)	0.244 (13)	0.246 (13)	0.249 (13)
2.45	0.169 (14)	0.166 (14)	0.165 (14)	0.165 (14)	0.163 (14)
Δ	0.0	0.68	0.68	0.70	0.78
η	1.0	1.03	1.03	1.03	1.03
ϵ	1.0	0.66	0.66	0.66	0.66
N_{sample}	–	1900	14300	42500	93100

The Importance Sampling (IS) procedure



$$\overline{\rho}_{ij} \equiv \overline{C}_{ij}/(\overline{\sigma}_i \overline{\sigma}_j)$$

Original covariances of data

$$C = \begin{pmatrix} C^{\mathrm{CERN}} & 0 \\ 0 & C^{\mathrm{JLAB}-\pi} \end{pmatrix}$$

$$C_{ij}^{\text{J-}\pi} = \sigma_i^2 \delta_{ij}$$

$$C_{ij}^{\mathrm{CERN}} = \sigma_i^2 \delta_{ij} + F_i F_j \delta r^2$$

Comparison with other results

~~***SPOILER:***~~ $\langle r_\pi \rangle_{DM} = 0.703 \pm 0.027 \text{ fm}$

$$M(Q^2) = \frac{1}{1+ < r_\pi^2 > Q^2/6}$$

$$M\&D(Q^2) = \frac{a_1}{1+ < r_\pi^2 > Q^2/6} + \frac{1-a_1}{(1+KQ^2)^2}$$

MONOPOLE:	***MONOPOLE+DIPOLE:***
$\langle r_\pi \rangle = 0.656 \pm 0.008 \text{ fm}$	$\langle r_\pi \rangle = 0.699 \pm 0.024 \text{ fm}$
$\chi^2/(d.o.f.) \simeq 1.0$	$\chi^2/(d.o.f.) \simeq 1.0$

Significative model dependence !!

Impact of non-zero values of $\overline{Q}_0$

Non-zero values of $\overline{Q}_0$ affect both the susceptibility χ and the kinematical functions ϕ:

$$4M_\pi^2\overline{\chi}_T(\overline{Q}_0^2) \equiv \frac{4M_\pi^2\chi_T(\overline{Q}_0^2)}{(1-\overline{z}_0)^6}\,,$$

$$\left\{ (1-\overline{z}_0) = 4M_\pi/\overline{Q}_0 + \mathcal{O}(1/\overline{Q}_0^2) \right\}$$

$$\overline{\phi}(z,\overline{Q}_0^2) \equiv \frac{\phi(z,\overline{Q}_0^2)}{(1-\overline{z}_0)^3} = \frac{1}{\sqrt{1536\pi}}(1+z)^2\frac{\sqrt{1-z}}{(1-\overline{z}_0 z)^3}\,.$$

$$\overline{\beta}(z) - \sqrt{\overline{\gamma}(z)} \leq F_\pi^V(z) \leq \overline{\beta}(z) + \sqrt{\overline{\gamma}(z)}\,,$$

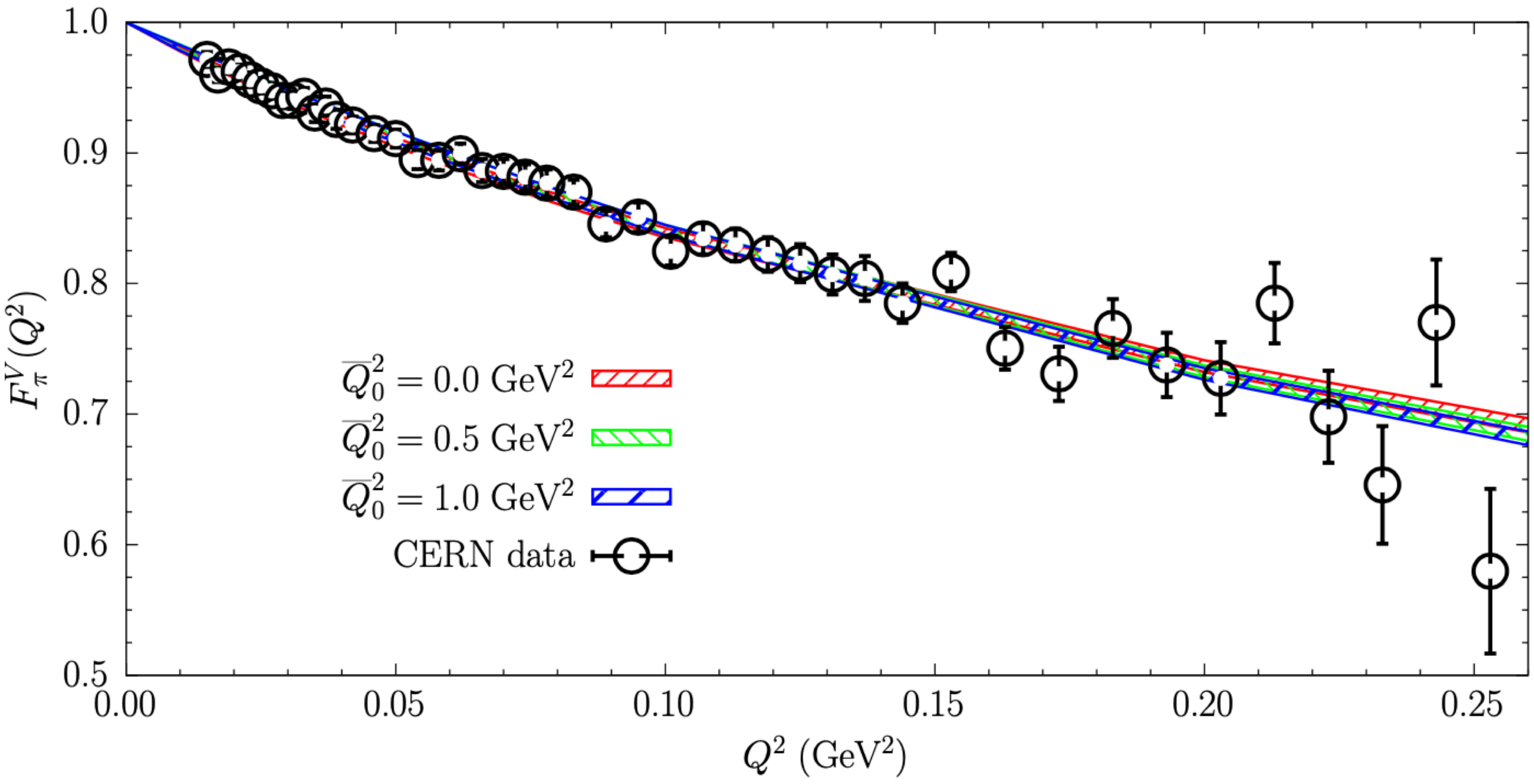
$$\overline{\beta}(z) = \frac{1}{\overline{\phi}(z,\overline{Q}_0^2)d(z)}\sum_{i=0}^{N}\overline{\phi}_i F_i d_i \frac{1-z_i^2}{z-z_i}\,,$$

$$\overline{\gamma}(z) = \frac{1}{(1-z^2)\overline{\phi}^2(z,\overline{Q}_0^2)d^2(z)}\left[4M_\pi^2\overline{\chi}_T(\overline{Q}_0^2) - \overline{\chi}_{\rm DM}(\overline{Q}_0^2)\right]$$

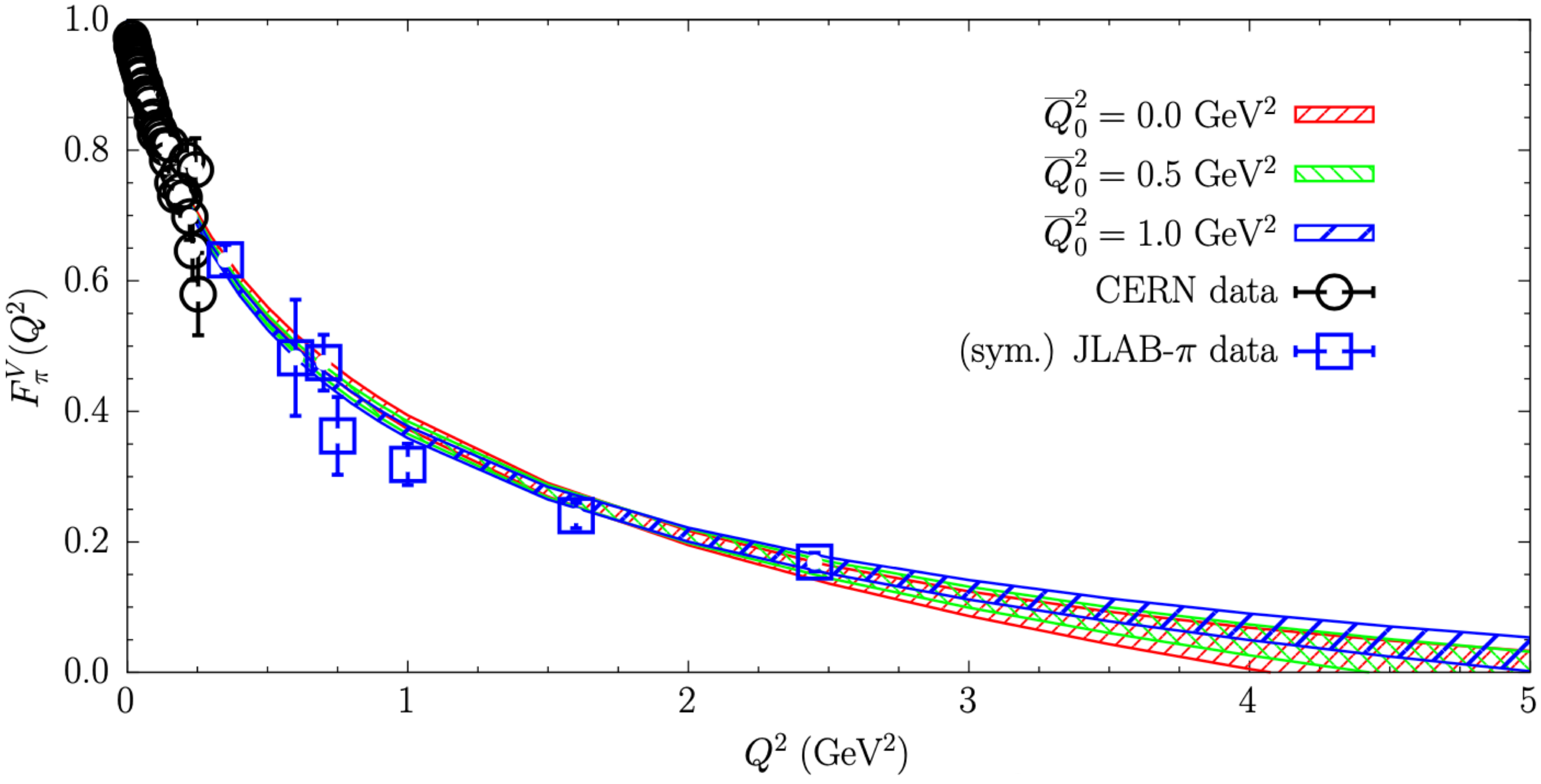
ACHTUNG:

$$4M_\pi^2\overline{\chi}_T(\infty) = \frac{1}{1536\pi^2}\int_{2M_\pi}^{\infty}\frac{d\omega}{2M_\pi}\left(\frac{\omega}{2M_\pi}\right)^3\left(1-\frac{4M_\pi^2}{\omega^2}\right)^{3/2}|F_\pi^V(\omega)|^2$$

Impact of non-zero values of $\overline{Q}_0$

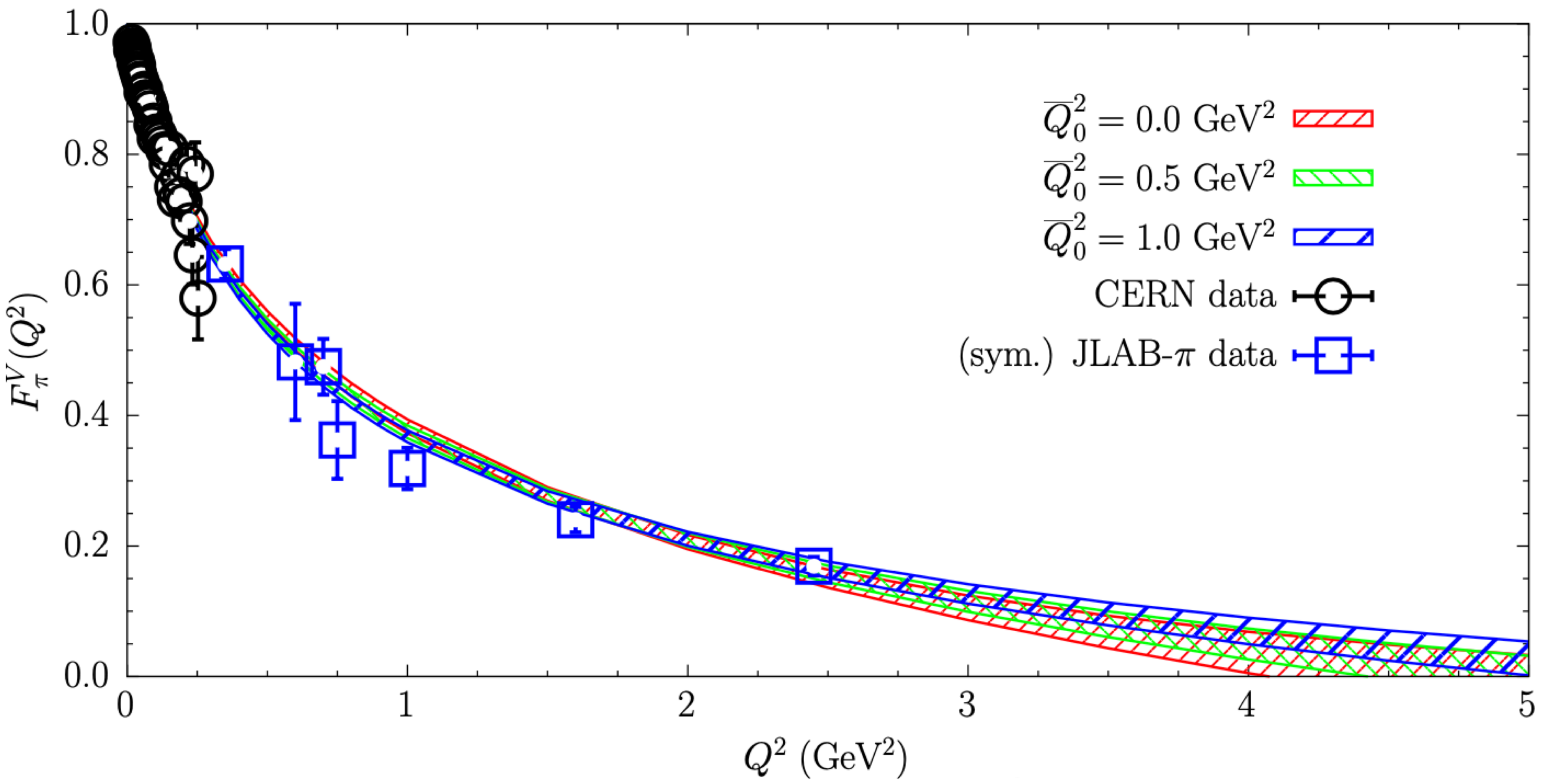


Impact of non-zero values of $\overline{Q}_0$



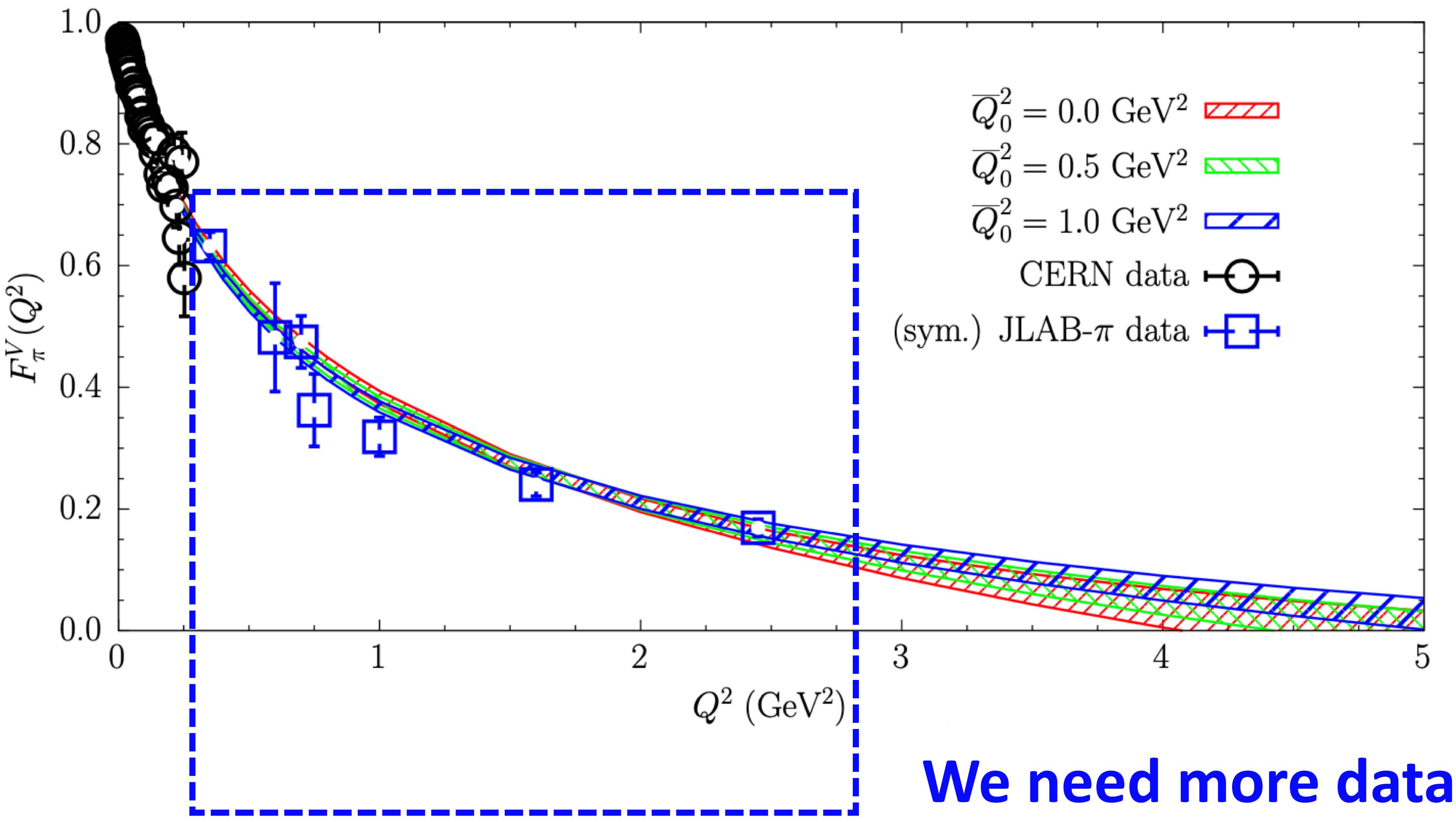
For increasing $\overline{Q}_0$, the impact of the electroproduction data increases!

Impact of non-zero values of $\overline{Q}_0$



ACHTUNG 2: pion form factor is the Fourier transform of a charge distribution proportional to the square of the pion wave function, thus zeros of the FFs have to be excluded!

Impact of non-zero values of $\overline{Q}_0$



We need more data here to go at large Q^2 !