Exploring arithmetic geometry from France to Tokyo

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What is arithmetic geometry?

An arithmetic problem: the Pythagorean triples

Consider the equation

$$a^2 + b^2 = c^2$$

involved in Pythagora's theorem. Can you determine all the solutions with a, b, c three non zero integers?

Any such solution a, b, c is called a **Pythagorean triple**. For example,

$$3^2 + 4^2 = 5^2.$$

What is arithmetic geometry?

Dividing both sides by c^2, we get

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Write $x = \frac{a}{c}$ and $y = \frac{b}{c}$. The problem is equivalent to finding all solutions of the equation

$$x^2 + y^2 = 1$$

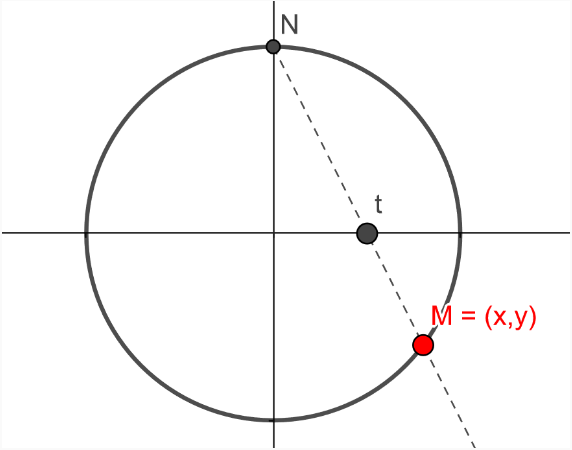
with x, y two rational numbers (ie. numbers that can be written as fractions of integers).

A geometric intuition towards the solution

The equation $x^2 + y^2 = 1$ represents the unit circle in the cartesian plane.

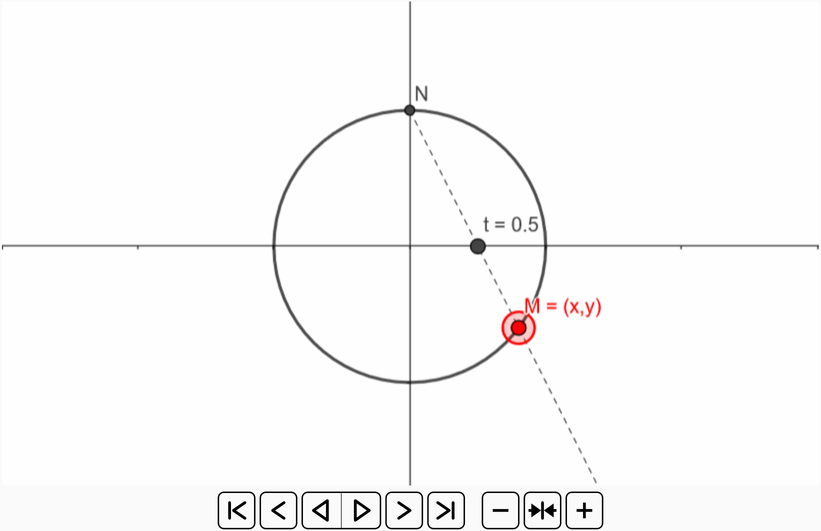
Let $M = (x, y)$ be a point of the circle, and let $N = (0, 1)$ be the point at the top of the circle. We are going to perform the **stereographic projection** to give a parametrization of the circle.

What is arithmetic geometry?



Draw the line joining N and M. It cuts the x-axis at a point of coordinate t. Any point of the circle is parametrized by a single value t (a real number, possibly equal to ∞).

What is arithmetic geometry?



What is arithmetic geometry?

By Thales's theorem, we have the relation

$$x = \frac{2t}{t^2+1}, \qquad\qquad y = \frac{t^2-1}{t^2+1}.$$

This is a **rational parametrization** of the unit circle.
We check that x and y are rational numbers if and only if t is a rational number itself.

Thus, all the rational solutions are given by the formula above with $t = \frac{p}{q}$ a fraction.

In conclusion

The equation

$$a^2 + b^2 = c^2$$

has non zero integer solutions, and even infinitely many.

In the 17th century, Fermat considered the similar equation

$$a^n + b^n = c^n$$

with n an integer ≥ 3. He stated, without a proof, that this equation admits **no solution** with a, b, c three positive integers.

This is the famous **Fermat's last theorem**. Building on previous works by various mathematicians (Taniyama, Shimura, Frey, Ribet, Serre, etc.), a valid proof was finally achieved in 1995 by Wiles and his student Taylor.

The proof required the development of many refined concepts, such as **modular curves**. These are special instances of **Shimura varieties**, which are expected to play a central role in the **Langlands program**, a far-reaching and very active research area in number theory.

My academic record

2016 - 2020: ENS de Lyon, Department of Mathematics

My academic record

Since 2009: **international cooperation program** between the ENS de Lyon and the University of Tokyo.
In particular, exchange program with the Graduate School of Mathematical Sciences in the University of Tokyo (Komaba campus).

Through this program, I stayed twice in Tokyo:

- Summer 2018: 3 months long research internship for the completion of 1st year of Master degree.
- Academic year 2019-2020: exchange student in Tokyo University for the 2nd year of Master degree.

My academic record

2020 - 2023: Université Sorbonne Paris Nord, LAGA

My academic record

My advisor in Tokyo, Naoki Imai, introduced me to research on Shimura varieties. Meanwhile, I met with Pascal Boyer from Université Sorbonne Paris Nord.

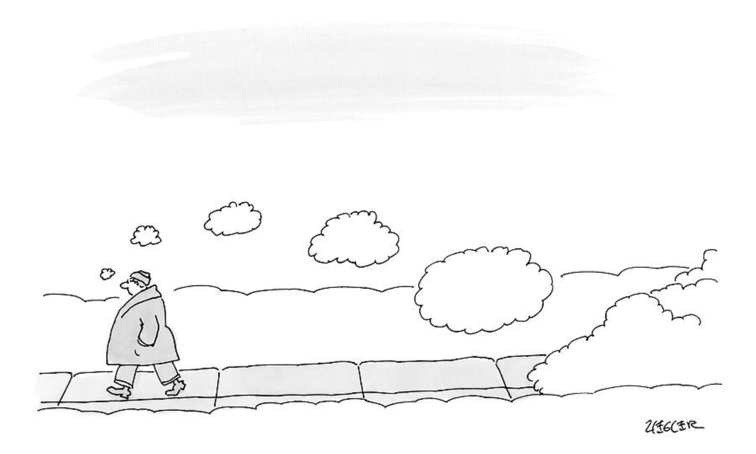
Together, we established a research program for my PhD under their supervision.

From September 2023, JSPS postdoctoral fellowship in the University of Tokyo.

A bit of Japanese culture

A bit of Japanese culture

When thinking about complicated problems, every mathematician has its own habits. In my case, I like have a walk.



A bit of Japanese culture

Usually, I end up sitting in a quiet place, like a church or a parc.

Saints Pierre et Paul church, 12th cent, Rosheim, my hometown.

A bit of Japanese culture

Parc de la tête d’or, Lyon.

A bit of Japanese culture

In Japan, I often end up visiting shrines and temples.

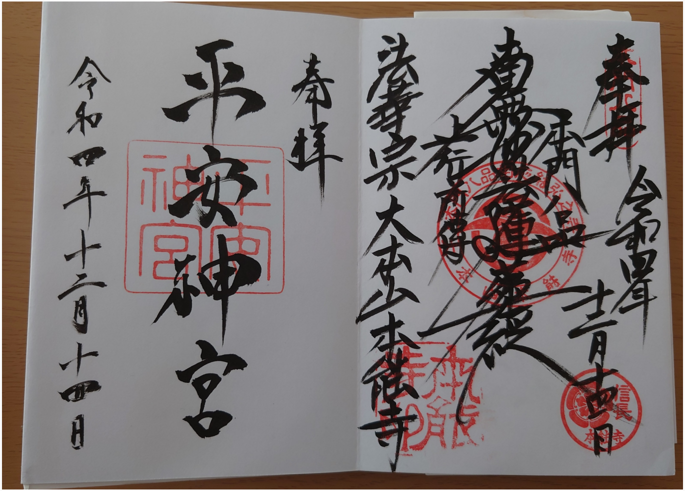
Kitazawa Hachiman jinja (北澤八幡神社) in Daizawa, Setagaya-ku.

A bit of Japanese culture

And so, I found myself collecting goshuin (御朱印) from various places around Japan! Here are my two goshuin cho (御朱印帳).

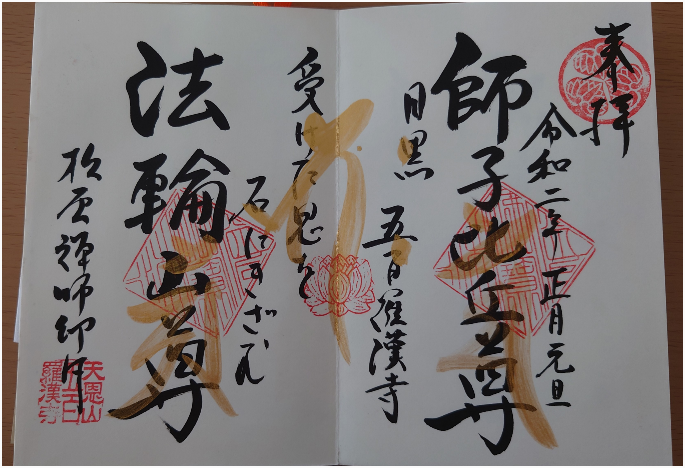
A bit of Japanese culture

Goshuin from Heian jingu (平安神宮) on the left, and Honno-ji (本能寺) on the right, in Kyoto.



A bit of Japanese culture

A special 2 pages goshuin from Gohyaku Rakan-ji (五百羅漢寺) in Meguro, delivered only on New Year's Eve.



Thank you for your attention!