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Tidal deformability and Love numbers of black holes

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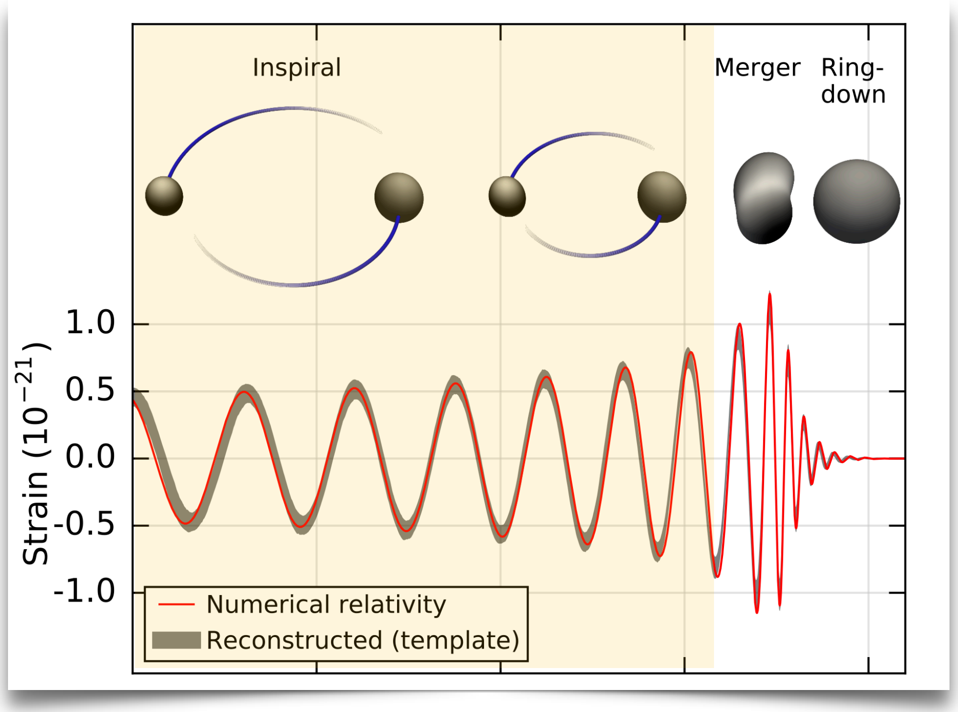
Tidal deformability of compact objects

- Unique opportunity to test General Relativity in the strong-field regime, and probe the fundamental nature of gravity and compact objects with gravitational waves.

- Toward the era of precision physics with gravitational waves: a key role is going to be played by the *tidal deformability*.

- The tidal deformability affects the dynamics during the inspiral: an alteration in the phase of the gravitational-wave signal can be used to constrain the tidal deformability of the objects.



Tidal deformability of compact objects

- The tidal deformability is characterized in terms of a set of *complex coefficients*:
  _ the real parts (a.k.a., the *Love numbers*) capture the conservative response of the body;
  _ the imaginary parts are associated with dissipative effects.

- They offer insights into the gravitational behavior and the body's internal structure:
  _ for a *neutron star*, the tidal deformability is tightly related to the type of particle species present inside the object, the density reached in the core, or the presence of phase transitions–i.e., the neutron star's *Equation of State* (EoS).
  _ for a *black hole*, the Love numbers depend on the physics at the horizon, and can be used to access and test the fundamental properties of gravity in the strong-field regime.

- An accurate measurement of the tidal effects would not only provide valuable insights into the characteristics of known objects, but could also potentially indicate the existence of new types of compact objects.

Tidal deformability of compact objects

- The measurement of the tidal deformation is challenging with current detectors:
_ current measurement errors on the tidal Love numbers of neutron stars coming from GW170817 are $\lambda \lesssim 10^3$. [LIGO/Virgo C., 1805.11581], [Piovano, Maselli, Pani '22]
_ similar constraints on the Love numbers of black holes. [Chia et al '23]

- We can search for exotic compact objects (e.g., boson stars, DM stars…) with large λ.
_ Exotic compact objects can easily have Love numbers that are orders of magnitude larger than standard objects.
_ Results from the first matched-filtering search for binaries with compact objects having $10^2 \lesssim \lambda \lesssim 10^6$ has been recently reported in [Chia et al '23].

- Precise measurements of the tidal deformability coefficients will be possible in the future:
_ an EMRI detection by LISA could set constraints on the (dimensionless) Love numbers of highly-spinning central objects at $\sim 10^{-2}$–10^{-3} level; [Piovano, Maselli, Pani '22]
_ the Einstein Telescope will be able to observe the onset of tidal effects close to the merger and pin down very precisely the EoS of neutron stars. [Maggiore et al '19], [Iacovelli et al '23], […]

Tidal deformability of compact objects

- Given the promising observational prospects, the past few years have witnessed a surge of interest, at the theoretical level, in the tidal deformability, which has been accompanied by important breakthroughs in different areas of research:

  _ analytic calculation of Love numbers for different types of objects, in GR and beyond GR, in 4 and higher spacetime dimensions, static and time-dependent tides…;

  _ new results on connection formulas for Fuchsian equations (the Love numbers are the connection coefficients between two singular points);
  [Bonelli, Iossa, Lichtig, Tanzini '21, '22], [Consoli, Fucito, Morales, Poghossian '22], [Lisovyy, Naidiuk '22], […]

  _ Effective Field Theory (EFT) approach and symmetry properties of gravitational EFTs.

Outline

I will mainly focus on the theoretical aspects of the tidal Love numbers of black holes.

I will summarized some aspects of the recent progress in the field.

A simple analogy

- The Love numbers are analogous to the electric and magnetic susceptibilities in EM.

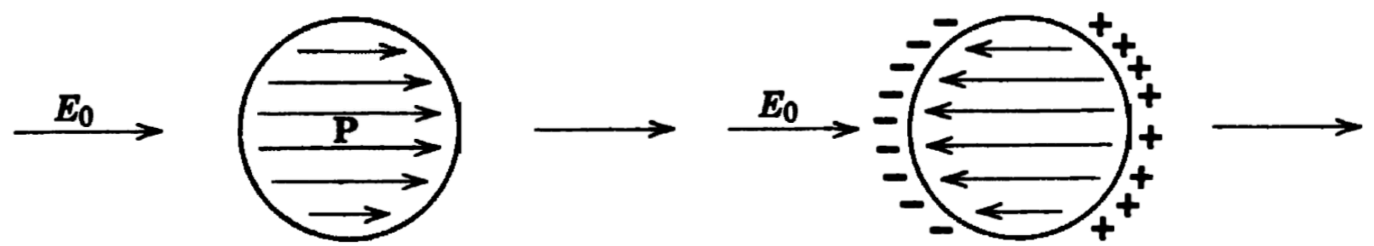


Figure 4.7 Dielectric sphere in a uniform field E_0, showing the polarization on the left and the polarization charge with its associated, opposing, electric field on the right.

[J. D. Jackson, "*Classical Electrodynamics*"]

- Solve $\vec{\nabla}^2\Phi = 0$:

$$\Phi_{\text{ext}} = A\left[r + \lambda\, r^{-2}\right]\cos\theta\,, \qquad\qquad \Phi_{\text{int}} = B\, r\cos\theta\,.$$

λ and B determined by regularity conditions across the surface (continuity of $\vec{E}_{\parallel}$ and $\vec{D}_{\perp}$):

$$\lambda = -\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} r_0^3\,,$$

where ϵ_0 and ϵ are the vacuum and dielectric permittivities. For more general $\vec{E}_0$:

$$\Phi_{\text{ext}} = \sum_{\ell} A_\ell\left[r^\ell + \lambda_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)\,.$$

λ_ℓ are the coefficients of the induced response.

Tidal Love numbers of black holes

- An explicit calculation in general relativity shows that $\mathrm{Re}[\lambda_\ell] = 0$ for black holes, as opposed to other types of compact objects.

- For non-rotating Schwarzschild black holes this result was obtained years ago. [Fang and Lovelace '05], [Binnington and Poisson '09], [Damour and Nagar '09]

- For rotating Kerr black holes the vanishing of the conservative tidal response was unambiguously established much more recently. [Le Tiec, Casals '20], [Le Tiec, Casals, Franzin '20], [Chia '20], [Charalambous, Dubovsky, Ivanov '21]

Tidal Love numbers of black holes

- As opposed to the EM example, the calculation of the induced response in general relativity can be affected by ambiguity, in the choice of coordinate system and the source/response split.
  [Kol and Smolkin '11], [Hui, Joyce, Penco, LS, Solomon '20], [Le Tiec, Casals '20], [Charalambous, Dubovsky, Ivanov '21], […]

- A possible way to address the source/response split ambiguity is via an analytic continuation in ℓ or D. [Kol and Smolkin '11], [Le Tiec, Casals '20], [Rodriguez, LS, Solomon, Temoche '23]

- There is a completely unambiguous way of defining the tidal response coefficients based on an effective field theory.

Point-particle effective theory

- A conceptually clean way to define the induced response of a body is in terms of the worldline effective action. [Goldberger and Rothstein '04, '05, …], [Kol and Smolkin '11], [Porto '16], […]

- At distances large compared to the characteristic size of an object, there is an effective description where the object is modeled as a point particle. Corrections due to the object's finite size and its internal structure are encoded in higher-derivative operators in the effective theory.

- The lowest-order action describing the dynamics of a point particle is just the Nambu–Goto action on the worldline:

$$S_{\rm pp} = -M\int \mathrm{d}\tau\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\ .$$

To probe the properties of the object, we expose it to some external fields, whose (bulk) dynamics is simply

$$S_{\rm bulk} = \frac{M_{\rm Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\,R\ .$$

Point-particle effective theory

- Finite-size effects can be introduced perturbatively via effective operators consisting in derivatives and powers of the curvature tensor:

$$S_{\rm int} \supset \int \mathrm{d}\tau \left[Q_E^{\mu\nu} E_{\mu\nu} + Q_B^{\mu\nu} B_{\mu\nu}\right] + \text{higher multipoles}$$

where $E_{\mu\nu} \equiv C_{\mu\rho\nu\sigma} u^\rho u^\sigma$ and $B_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\gamma(\mu}{}^{\alpha\beta} C_{\alpha\beta\nu)\delta} u^\gamma u^\delta$. The EFT is $S = S_{\rm pp} + S_{\rm bulk} + S_{\rm int}$.

- According to linear-response theory, and in the adiabatic limit of quasi-static tides,

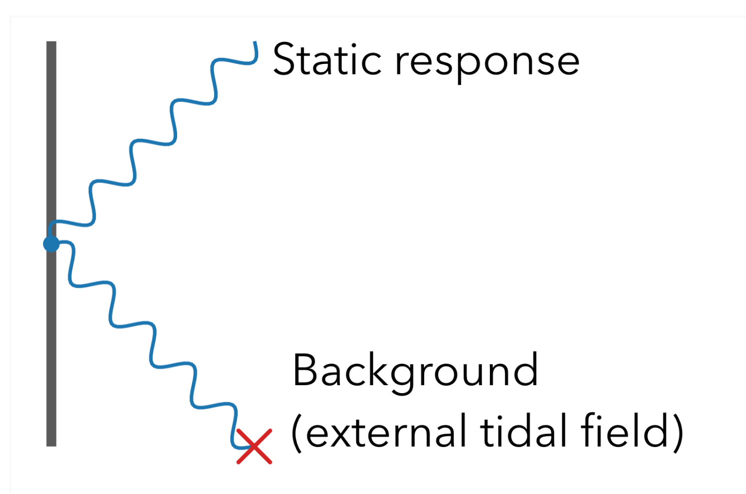
$$Q_E^{\mu\nu} = \sum_{n=0}^{\infty} \lambda_2^{(n)} \frac{\mathrm{d}^n E^{\mu\nu}}{\mathrm{d}\tau^n} \, .$$

- Responses come in two types: *conservative* (n even, i.e. time-reversal invariant) and *dissipative* (n odd, i.e. time-reversal breaking).

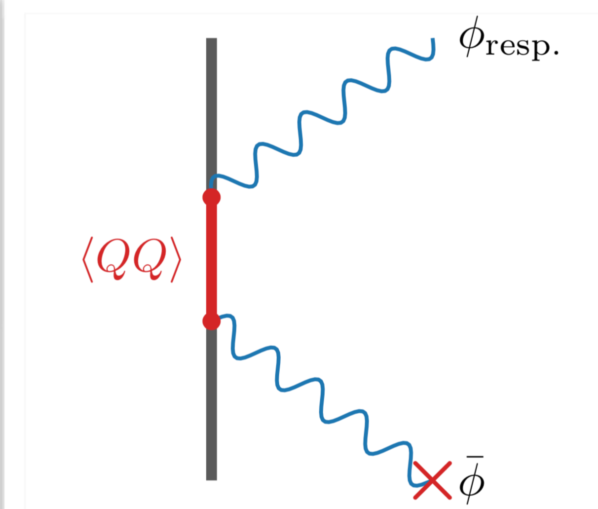
- The Wilson couplings $\lambda_\ell^{(2n)}$ are the Love numbers.

Local vs. nonlocal response

Conservative finite-size effects correspond to local operators on the worldline.



The dissipative response is modeled in terms of nonlocal operators on the worldline.



- Modeling dissipation requires to introduce additional degrees of freedom X that reside on the worldline. Conceptually, they absorb energy from the external fields: [Goldberger and Rothstein '20]

$$S_{\text{int}} \supset \int \mathrm{d}\tau \left[Q_E^{\mu\nu}(X) E_{\mu\nu} + Q_B^{\mu\nu}(X) B_{\mu\nu} \right] + \text{higher multipoles}$$

- The response can be obtained via computing $\langle \phi(x) \rangle$ in the in-in formalism, after the X variables are integrated out.

Vanishing of the Love numbers

- One generically expects: $\lambda_\ell \sim \mathcal{O}(1) r_s^{2\ell-1}$ and to find (classical) RG running.

- After matching with the UV result in $D=4$: $\lambda_\ell = 0$ and no running.

- Following 't Hooft's naturalness principle, the vanishing of the Love numbers is a naturalness puzzle from an EFT perspective. [Rothstein '14], [Porto '16]

- Looks like something that can very likely follow from a symmetry in the theory.

- Two proposals to solve this puzzle:
  - [Hui, Joyce, Penco, LS, Solomon '21]
  - [Charalambous, Dubovsky, Ivanov '21]

Hidden symmetries at finite frequency

[Hui, Joyce, Penco, LS and Solomon '22]

- Consider the action of a massless scalar field on a Schwarzschild black hole spacetime:

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}r\mathrm{d}\Omega_{S^2}\left[\frac{r^4}{\Delta}(\partial_t\phi)^2 - \Delta(\partial_r\phi)^2 + \phi\,\nabla^2_{\Omega_{S^2}}\phi\right].$$

- In the end we will take the static limit. However, it is convenient to start by keeping ω finite and define a *near-zone* approximation, valid for $r_s \leq r \ll 1/\omega$.
In practice, we will replace $(r^4/\Delta)\partial_t^2\phi$ with $(r_s^4/\Delta)\partial_t^2\phi$ in the action.

- This has the virtue of preserving the correct singularity as $r \to r_s$, while still accurately capturing the dynamics at larger r, as long as $\omega r \ll 1$.

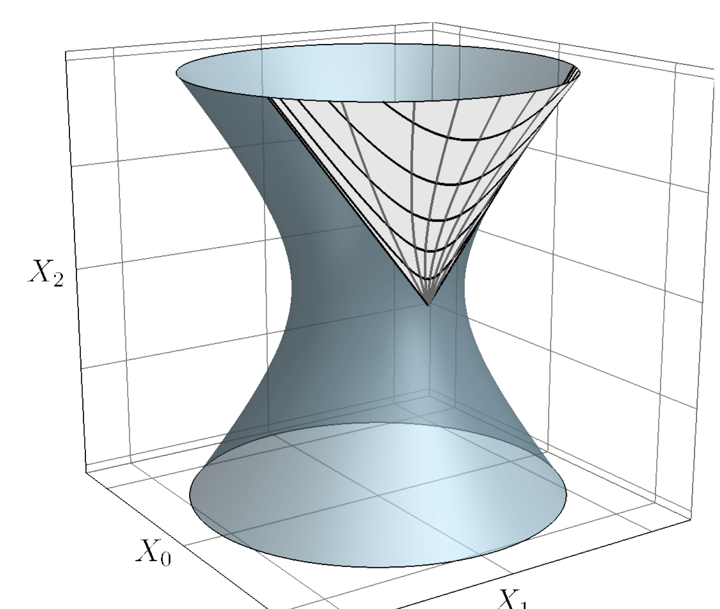
Hidden symmetries at finite frequency

[Hui, Joyce, Penco, LS and Solomon '22]

- In this limit, the scalar action is the same as that of a massless scalar minimally coupled to an *effective near-zone metric*:

$$\mathrm{d}s^2_{\text{near-zone}} = -\frac{\Delta}{r_s^2}\mathrm{d}t^2 + \frac{r_s^2}{\Delta}\mathrm{d}r^2 + r_s^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) .$$

- This metric has the following main property: it is a conformally-flat $\mathrm{AdS}_2 \times S^2$ spacetime ($\Rightarrow$ 6 KVs + 9 CKVs).



Hidden symmetries at finite frequency

[Hui, Joyce, Penco, LS and Solomon '22]

- The 6 KVs and 9 CKVs are:

$$L_0 = 2r_s\,\partial_t$$

$$L_\pm = e^{\pm t/2r_s}(2r_s\,\partial_r\sqrt{\Delta}\partial_t \mp \sqrt{\Delta}\partial_r)$$

$$J_{23} = \partial_\varphi$$

$$J_{12} = \cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi$$

$$J_{13} = \sin\varphi\,\partial_\theta + \cot\theta\cos\varphi\,\partial_\varphi$$

$$J_{01} = -\frac{2\Delta}{r_s}\cos\theta\,\partial_r - \frac{\partial_r\Delta}{r_s}\sin\theta\,\partial_\theta$$

$$J_{02} = -\cos\varphi\left[\frac{2\Delta}{r_s}\sin\theta\,\partial_r + \frac{\partial_r\Delta}{r_s}\left(\frac{\tan\varphi}{\sin\theta}\partial_\varphi - \cos\theta\partial_\theta\right)\right]$$

$$J_{03} = -\sin\varphi\left[\frac{2\Delta}{r_s}\sin\theta\,\partial_r - \frac{\partial_r\Delta}{r_s}\left(\frac{\cot\varphi}{\sin\theta}\partial_\varphi + \cos\theta\partial_\theta\right)\right]$$

$$K_\pm = e^{\pm t/2r_s}\frac{\sqrt{\Delta}}{r_s}\cos\theta\left(\frac{r_s^3}{\Delta}\partial_t \mp \partial_r\Delta\partial_r \mp 2\tan\theta\partial_\theta\right)$$

$$M_\pm = e^{\pm t/2r_s}\cos\varphi\left[\frac{r_s^2}{\sqrt{\Delta}}\sin\theta\partial_t \mp \frac{\sqrt{\Delta}\partial_r\Delta\sin\theta}{r_s}\partial_r \pm \frac{2\sqrt{\Delta}}{r_s}\cos\theta\partial_\theta \mp \frac{2\sqrt{\Delta}}{r_s}\frac{\tan\varphi}{\sin\theta}\partial_\varphi\right]$$

$$N_\pm = e^{\pm t/2r_s}\sin\varphi\left[\frac{r_s^2}{\sqrt{\Delta}}\sin\theta\partial_t \mp \frac{\sqrt{\Delta}\partial_r\Delta\sin\theta}{r_s}\partial_r \pm \frac{2\sqrt{\Delta}}{r_s}\cos\theta\partial_\theta \pm \frac{2\sqrt{\Delta}}{r_s}\frac{\cot\varphi}{\sin\theta}\partial_\varphi\right]$$

- $L_{0,\pm}$ form a $\mathrm{SL}(2,\mathbb{R})$ algebra: $[L_m, L_n] = (n-m)L_{n+m}$, $n, m = -1, 0, +1$.
  Static solutions belong to a finite-dimensional representation of $\mathrm{SL}(2,\mathbb{R})$ and are finite polynomial in r. [Charalambous, Dubovsky, Ivanov '21]
- Only L_0, J_{ij} and J_{0i} remain good symmetries at $\omega = 0$.
- J_{01} is the generator of ladder symmetries. [Hui, Joyce, Penco, LS, Solomon '21]

Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- Let's decompose $\phi = \sum_\ell \int \mathrm{d}\omega\, \mathrm{e}^{-i\omega t} \phi_\ell(r) Y_{\ell,m}(\theta, \varphi)$.
- In the static limit and for $\ell = 0$, the scalar equation is simply:

$$\partial_r \left(\Delta \partial_r \phi_0 \right) = 0\,, \qquad\qquad \Delta = r(r - r_s)\,.$$

- $P_0 \equiv \Delta \partial_r \phi_0$ is the conserved charge associated with a symmetry of the (static) scalar action.

- It is useful because it allows to connect asymptotics:

$$\phi_0 \sim r^0 \quad \text{as} \quad r \to +\infty \quad \to \quad P_0 = 0 \quad \to \quad \phi_0 \sim \text{const.} \quad \text{as} \quad r \to r_s$$
$$\phi_0 \sim r^{-1} \quad \text{as} \quad r \to +\infty \quad \to \quad P_0 \neq 0 \quad \to \quad \phi_0 \sim \log(r - r_s) \quad \text{as} \quad r \to r_s$$

Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- In spherical harmonics, the J_{01} generator acts on the scalar ϕ as:

$$\delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1},$$

$$D^+_\ell \equiv -\Delta\partial_r + \frac{\ell+1}{2}(r_s - 2r) \quad \text{and} \quad D^-_\ell \equiv \Delta\partial_r + \frac{\ell}{2}(r_s - 2r)\,.$$

- $D^\pm_\ell$ are *ladder symmetries*, i.e. $\phi_{\ell\pm1} \propto D^\pm_\ell \phi_\ell$. They allow to extend P_0 to higher ℓ's:

$$\begin{array}{lllll}\phi_\ell \sim r^\ell \ \text{as} \ r \to +\infty & \to & P_\ell = 0 & \to & \phi_\ell \sim \text{const.} \ \text{as} \ r \to r_s \\ \phi_\ell \sim r^{-(\ell+1)} \ \text{as} \ r \to +\infty & \to & P_\ell \neq 0 & \to & \phi_\ell \sim \log(r - r_s) \ \text{as} \ r \to r_s\end{array}$$

- The vanishing of the Love numbers follows from two facts: (1) the purely decaying solution ($\sim 1/r^{\ell+1}$ at large r) is divergent at the horizon, and (2) the solution that is regular at the horizon is a finite polynomial going as $\sim 1 + r + \ldots + r^\ell$.

- The growing branch respects the symmetry, while the decaying branch spontaneously breaks the symmetry.

(See also [Achour, Livine, Mukohyama, Uzan '22])

Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- At large *r*, $\delta\phi$ reduces to a SCT, $\delta\phi = c_i\left(x^i - \vec{x}^{\,2}\partial^i + 2x^i\vec{x}\cdot\vec{\partial}\,\right)\phi.$
- We claim that this is the sought-after infrared symmetry that forbids Love number couplings in the point-particle effective action.
- Straightforward the generalization to Kerr and higher spins (via spin-ladder operators). Similar conclusions for Reissner–Nordström, although more involved. [Rai and LS, *to appear*]

More on tidal deformability of black holes

- The vanishing of the Love numbers is a fragile property of general relativity in $D = 4$.

- In higher dimensions Love numbers are generically non-zero, depending on $\hat{\ell} \equiv \frac{\ell}{D-3}$.
[Kol, Smolkin '11], [Hui, Joyce, Penco, LS, Solomon '20], [Charalambous, Ivanov '21], [Pereñiguez, Cardoso '21], [Rodriguez, LS, Solomon, Temoche '23], [...]

- For a scalar field:

$$k_\ell = \frac{\Gamma(-2\hat{\ell}-1)}{\Gamma(-\hat{\ell})^2}\frac{\Gamma(\hat{\ell}+1)^2}{\Gamma(2\hat{\ell}+1)} = \frac{2\hat{\ell}+1}{2\pi}\frac{\Gamma(\hat{\ell}+1)^4}{\Gamma(2\hat{\ell}+2)^2}\tan(\pi\hat{\ell}),$$

[Kol, Smolkin '11], [Hui, Joyce, Penco, LS, Solomon '20]

- Richer phenomenology beyond spherical symmetry: Myers–Perry black holes, multiple axes of rotation, non-trivial topologies (black ring, strings…).
[Charalambous and Ivanov '23], [Rodriguez, LS, Solomon, Temoche '23], [Glazer, Joyce, Rodriguez, LS, Solomon, Temoche, *in prep.*]

More on tidal deformability of black holes

- Black hole Love numbers are generically non-zero in theories beyond GR.
  [Cardoso et al '17], [Cardoso, Kimura, Maselli, Senatore '18], [...]

- In the EFT of GR:

$$S \sim \int \mathrm{d}^4x \sqrt{-g} \left[\frac{M_{\rm Pl}^2}{2} R + g_n \frac{R^n}{\Lambda^{2n-4}} \right] .$$

- The quadrupolar electric and magnetic tidal Love numbers are:

$$k_2^E = -\frac{1008}{25}\epsilon_1 , \quad k_2^B = \frac{432}{25}\epsilon_1 ,$$

[Cardoso, Kimura, Maselli, Senatore '18]

where ϵ_1 is here the coupling of $(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2$.

- The breaking of the equality of the Love numbers can be understood from the breaking of the even-odd duality of black holes in GR.

More on tidal deformability of black holes

- Dynamical Love numbers of black holes are non-zero and display logarithmic running.
[Mano, Suzuki, Takasugi '96], [Charalambous, Dubovsky, Ivanov '21], [Saketh, Zhou, Ivanov '23], [Mandal et al '23], [Perry, Rodriguez '23], [...]

$$k_\ell^{Schw}(\omega) = \frac{\Gamma(1+\ell-s-2iM\omega)\Gamma(1+\ell-2iM\omega)}{(2\ell+1)!\,\Gamma(2\ell+1)\,\Gamma(-\ell-s-2iM\omega)\,\Gamma(-\ell-2iM\omega)}\log\left(\frac{2M}{r}\right)$$

[Perry, Rodriguez '23]

Conclusions

Conclusions and open directions

- Hidden symmetries strongly constrain the linear response of black holes in GR.
- Love numbers represent an important window on the physics at the horizon.
- Observing non-zero Love numbers in black-hole binary systems would be a clear smoking gun of new physics.
- In the future, we will benefit from a better understanding of the symmetry structure of the EFT, dynamical Love numbers, nonlinearities, the rich phenomenology of higher dimensional black holes, etc.

Backup slides

Ladder in Kerr: static limit

[Hui, Joyce, Penco, LS and Solomon '21]

- The previous algebraic ladder structure has a direct analog in a Kerr background:

$$\mathrm{d}s^2 = -\frac{\rho^2 - r_s r}{\rho^2}\mathrm{d}t^2 - \frac{2ar_s r\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\mathrm{d}\varphi^2$$

with $\rho^2 \equiv r^2 + a^2\cos^2\theta$ and $\Delta \equiv r^2 - rr_s + a^2$.

- The static Klein-Gordon equation, $\partial_r(\Delta\partial_r\phi_\ell)+\frac{a^2m^2}{\Delta}\phi_\ell - \ell(\ell+1)\phi_\ell = 0$, has both *ladder* and *horizontal* symmetries.
- The ladder symmetries $D_\ell^\pm$ descend from a CKV of the 3D-static metric:

$$\mathrm{d}s_K^2 = \frac{\rho^2 - rr_s}{\Delta}\left(\mathrm{d}r^2 + \Delta\mathrm{d}\theta^2 + \frac{\Delta^2\sin^2\theta}{\rho^2 - rr_s}\mathrm{d}\varphi^2\right).$$

- $\xi^\mu = (0, \Delta\cos\theta, \frac{1}{2}(2r - r_s)\sin\theta, 0)$ is the CKV that induces
$\delta\phi = \xi^\mu\partial_\mu\phi+\frac{1}{2}(2r-r_s)\cos\theta\,\phi \qquad \Rightarrow \qquad \delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1}$,
- The conserved charges P_ℓ associated with the horizontal symmetries, evaluated for the "growing branch", are non-zero (and imaginary), unlike in the Schwarzschild case:

$$P_\ell \propto iq\prod_{k=1}^{\ell}\left(k^2 + 4q^2\right), \qquad q \equiv \frac{am}{r_+ - r_-},$$

which reproduces the dissipative response [Le Tiec and Casals '20].

Ladder in Spin: From Scalar to Vector and Tensor

[Hui, Joyce, Penco, LS and Solomon '21, '22]

- Ladder operators in the spin, $E_s^{\pm}$, raise and lower s in the Teukolsky equation ($E_s^{\pm}\phi_\ell^{(s)} = \phi_\ell^{(s\pm1)}$),

$$\partial_r\Big(\Delta\partial_r\phi_\ell^{(s)}\Big) + s(2r-r_s)\partial_r\phi_\ell^{(s)} + \Big(\frac{a^2m^2 + is(2r-r_s)am}{\Delta} - (\ell-s)(\ell+s+1)\Big)\phi_\ell^{(s)} = 0\,,$$

- Allow to extend the previous results from scalar to vector and tensor fields.

- $E_s^{\pm}$ are related to what are known as Teukolsky–Starobinsky identities.
In Chandrasekhar's notation,

$$\phi^{(-1)} = \Delta\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\Delta\phi^{(1)}, \quad \phi^{(1)} = \mathscr{D}_0\mathscr{D}_0\phi^{(-1)}, \quad \phi^{(-2)} = \Delta^2\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\Delta^2\phi^{(2)}, \quad \phi^{(2)} = \mathscr{D}_0\mathscr{D}_0\mathscr{D}_0\mathscr{D}_0\phi^{(-2)},$$

where $\mathscr{D}_0 \equiv \partial_r + i[am - \omega(r^2+a^2)]/\Delta$.

The new twist we are adding is that, in the static limit, we can truncate these operations, enabling us to increment s by unity, $E_s^{\pm}\phi_\ell^{(s)} = \phi_\ell^{(s\pm1)}$.