Using primitive variables to evolve isolated neutron stars with pseudospectral methods

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Content

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- Valencia formulation (Banyuls et al. 1997): conservative formulation of GR-hydro equations
- Development of high-resolution shock capturing (HSRC) methods

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- Valencia formulation (Banyuls et al. 1997): conservative formulation of GR-hydro equations
- Development of high-resolution shock capturing (HSRC) methods
- Conservative formulation comes with recovery procedures which use iterative algorithm to recover primitive variables from conservative ones: $(D, S_i, \tau) = f(e, p, T, U_i)$.
  - Be source of code failure (non-convergence of iterative algorithms, non-analytical EoS)
  - Computationally consuming (each grid point, each time step)

Goals

Building a set of hydrodynamical equations

- Using non-conservative, physical variables ("primitive" variables): energy, pressure, velocity field, ...
- Testing them in a simple numerical setup: oscillations of barotropic spherically symmetric isolated NS

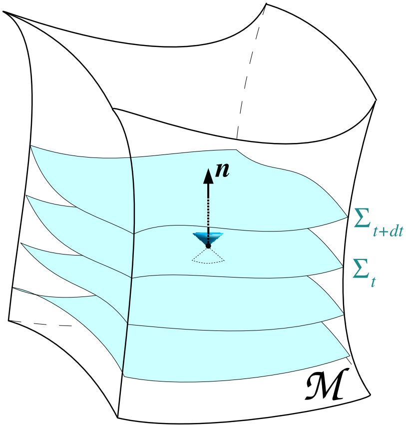


Figure: 3+1 foliation in spacelike hypersurfaces

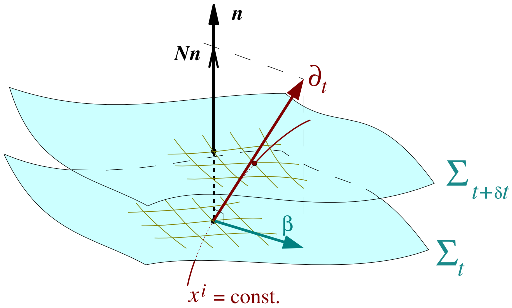


Figure: Illustration of the lapse and the shift

Figures credits: E. Gourgoulhon

Physical variables are called "primitive" variables:

- e the energy density, p the pressure
- n_B the baryon number density
- m_B a baryon mass
- $H = \ln\left(\frac{e+p}{m_B n_B}\right)$ the log-enthalpy
- U_i the Eulerian velocity field
- v_i the coordinate velocity field
- c_s the sound speed

Content

Conservative variables

The variables $D = m_B n_B \Gamma^2$, $S_j = (e+p)\Gamma^2 U_j$ and $\tau = (e+p)\Gamma^2 - p$, with $\Gamma = (1 - U_i U^i)^{-1/2}$ the Lorentz factor, are conserved in the sense that $\mathbf{u} = (D, S_j, \tau)$ obeys an equation that looks like

$$\partial_t \mathbf{u} + \mathrm{div}(F(\mathbf{u})) = \mathrm{source}$$

but the knowledge of e, p, U_i is compulsory to solve Einstein equations to compute the metric. The reference code CoCoNuT (Dimmelmeier et al. 2005) is based upon the conservative formulation of hydrodynamics.

Recovery procedures are needed to recover the primitive variables in multiple steps of CoCoNuT and account for a significant part of the computation:

- Solving for the metric
- Solving for the Riemann problems in each grid cell at each timestep

Full GR equations, using only primitive variables

From the principles of stress-energy and baryon number conservation:

$$\nabla_\mu (n_B u^\mu) = 0, \; \nabla_\mu T^{\mu\nu} = 0,$$

the following holds for a barotropic, non-reactive perfect fluid:

$$\partial_t U_i = -v^j D_j U_i - D_i N - \frac{N}{\Gamma^2}\left(D_i H - \frac{\Gamma^2(1-c_s^2)}{\Gamma^2 - c_s^2(\Gamma^2-1)} U_i U^j D_j H \right)$$
$$+ U_j D_i \beta^j + U_i U^j D_j N$$
$$+ \frac{N c_s^2}{\Gamma^2 - c_s^2(\Gamma^2-1)} U_i D_j U^j + \frac{N\Gamma^2(c_s^2-1)}{\Gamma^2 - c_s^2(\Gamma^2-1)} U_i U^j U^l K_{jl}$$

$$\partial_t H = -v^i D_i H - c_s^2 N \frac{\Gamma^2}{\Gamma^2 - c_s^2(\Gamma^2-1)} \left[U^i U^j K_{ij} - \frac{U^i}{\Gamma^2} D_i H + D_i U^i \right]$$

This new set of equations is **covariant** within the 3+1 formalism.
e, p, c_s are recovered in a single call to the EoS **(no iteration)**.

Content

Frequency extraction

The equations are implemented in a **spherically symmetric**, **semi-Lagrangian** C++ code based upon LORENE: **pseudospectral methods** (space) + explicit **finite-differences** (time).

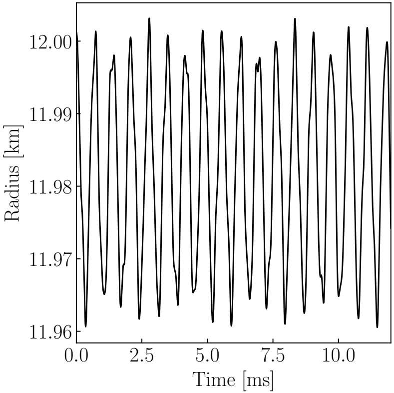


Figure: Radius vs time

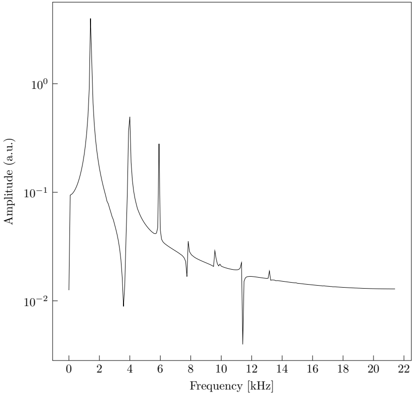


Figure: $\hat{R}(f)$ spectrum

Validation of the code: frequency extraction

- Reproduction of polytropic frequencies given in Font et al. 2002 and Hartle & Friedman 1975 with less than 1% precision in full GR:

	κ	γ	$H_c\,[c^2]$	$M\,[M_\odot]$	R [km]	Fund. [kHz]	1st ov. [kHz]	2nd ov. [kHz]
Font et al. [25]	100	2	0.2279	1.4	14.15	1.450	3.958	5.935
This work	100	2	0.2279	1.401	14.16	1.442	3.954	5.915
Relative difference						0.6%	0.1%	0.3%
Hartle & Friedman [26]	7.308	5/3	6.720×10^{-2}	$\times$	$\times$	0.824	1.94	2.86
This work	7.308	5/3	6.720×10^{-2}	0.4866	16.49	0.823	1.95	2.86
Relative difference						0.1%	0.5%	0.0%

- In Cowling approximation:

	κ	γ	$H_c\,[c^2]$	$M\,[M_\odot]$	R [km]	Fund. [kHz]	1st ov. [kHz]	2nd ov. [kHz]
Font et al. [25]	100	2	0.2279	1.4	14.15	2.696	4.534	6.346
This work	100	2	0.2279	1.401	14.16	2.685	4.548	6.339
Relative difference						0.4%	0.3%	0.1%

Validation of the code: Migration test

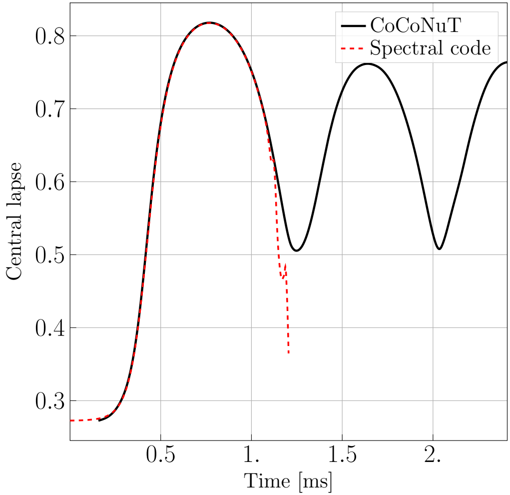


Figure: Central lapse

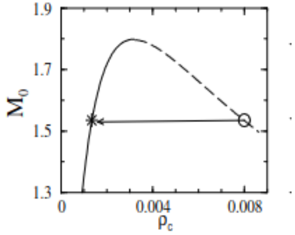


Figure: Migration test principle

- Shock formation during the oscillations
- Gibbs phenomenon: the spectral code crashes
- 10s (spectral code) vs 6min (CoCoNuT)

Content

Recap

- New set of **covariant** 3+1 equation for any configuration of the perfect fluid with primitive variables
- Primitive variables allow to simulate subsonic, smooth flows
- No shock treatment + spectral methods = shocks are prohibitive (cf migration test)
- 3D code is **under development**.

Thank you !

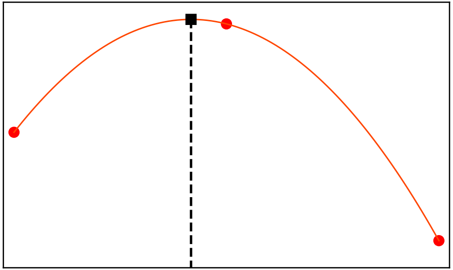
arxiv:2212.10853

Newtonian equations

The set of Newtonian equations is:

$$\partial_t \rho + \nabla_i(\rho v^i) = 0$$
$$\partial_t v^i + v^j \nabla_j v^i = -\nabla^i (h + \Phi)$$
$$\Delta\Phi = 4\pi\rho$$
$$h = \frac{\kappa\gamma}{\gamma - 1}\rho^{\gamma-1}$$

Frequency extraction : principle



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