Slow-rotation black hole perturbation theory

Nicola Franchini

Laboratoire Astroparticule & Cosmologie
Université Paris Cité

Septième Assemblée Générale du GdR Ondes Gravitationnelles

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BH perturbation in spherical symmetry

$$g_{ab} = g_{ab}^{(0)} + \varepsilon g_{ab}^{(1)}$$

$$\downarrow$$

$$R_{ab} = R_{ab}^{(0)} + \varepsilon R_{ab}^{(1)}$$

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Let's assume vacuum:

$R_{ab}^{(0)} = 0 \longrightarrow g_{ab}^{(0)} = g_{ab}^{\mathrm{Sch}}$ is the Schwarzschild metric

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G=c=2M=1

Let's assume vacuum:

$$R_{ab}^{(0)} = 0 \longrightarrow g_{ab}^{(0)} = g_{ab}^{\text{Sch}}$$

$$\begin{pmatrix} -1+\frac{1}{r} & 0 & 0 & 0 \\ 0 & \left(1-\frac{1}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

BH perturbation in spherical symmetry

For the perturbation we solve $R^{(1)}_{ab} = 0$

with ansatz $g^{(1)}_{ab} = g^{(1,+)}_{ab} + g^{(1,-)}_{ab}$

BH perturbation in spherical symmetry

For the perturbation we solve $R_{ab}^{(1)} = 0$

with ansatz $g_{ab}^{(1)} = g_{ab}^{(1,+)} + g_{ab}^{(1,-)}$

Polar component

$(-1)^{\ell}$

Parity transformation

$$\theta \to \pi - \theta$$
$$\varphi \to \varphi + \pi$$

Axial component

$(-1)^{\ell+1}$

BH perturbation in spherical symmetry

$$g_{ab}^{(1,+)} = \begin{pmatrix} H_0^\ell(r) & H_1^\ell(r) & 0 & 0 \\ H_1^\ell(r) & H_2^\ell(r) & 0 & 0 \\ 0 & 0 & r^2 K^\ell(r) & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta K^\ell(r) \end{pmatrix} Y^\ell(\theta,\varphi)\, \mathrm{e}^{-\mathrm{i}\omega t}$$

Regge-Wheeler gauge

$$g_{ab}^{(1,-)} = \begin{pmatrix} 0 & 0 & h_0^\ell(r) S_\theta^\ell(\theta,\varphi) & h_0^\ell(r) S_\varphi^\ell(\theta,\varphi) \\ 0 & 0 & h_1^\ell(r) S_\theta^\ell(\theta,\varphi) & h_1^\ell(r) S_\varphi^\ell(\theta,\varphi) \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t}$$

$$\left(S_\theta^{\ell m}, S_\varphi^{\ell m}\right) = \left(-\frac{Y_{,\varphi}^{\ell m}}{\sin\theta}, \sin\theta\, Y_{,\theta}^{\ell m}\right)$$

Regge, Wheeler 1957

BH perturbation in spherical symmetry

Black box - manipulate the equations

- 7x polar eqs. - 3x axial eqs.
- 2x 1st order polar eqs. - 2x 1st order axial eqs.
- Definition of master variables

BH perturbation in spherical symmetry

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$$\frac{\mathrm{d}^2\Psi_\pm^\ell}{\mathrm{d}r_*^2} + \left(\omega^2 - V_\pm^\ell\right)\Psi_\pm^\ell = 0$$

With tortoise coordinate defined as $\mathrm{d}r_* = \frac{\mathrm{d}r}{1-1/r}$

BH perturbation in spherical symmetry

Regge-Wheeler potential for axial tensor perturbation

$$V_-^\ell = \left(1 - \frac{1}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{3}{r^3}\right]$$

Zerilli potential for polar tensor perturbation, $\lambda = \ell(\ell+1) - 2$

$$V_+^\ell = \left(1 - \frac{1}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{3}{r^3}\frac{r^2\lambda(\lambda+4) + 6r - 3}{(3 + r\lambda)^2}\right]$$

Regge, Wheeler 1957, Zerilli 1970

BH perturbation in slow rotation

BH perturbation in slow rotation

$$g_{ab} = g^{(0)}_{ab} + \varepsilon g^{(1)}_{ab}$$

$$\downarrow$$

$$R_{ab} = R^{(0)}_{ab} + \varepsilon R^{(1)}_{ab}$$

Let's assume vacuum with slow rotation:

$$R^{(0)}_{ab} = 0 \longrightarrow g^{(0)}_{ab} = g^{\mathrm{Sch}}_{ab} + a g^{\mathrm{SR},1}_{ab} + a^2 g^{\mathrm{SR},2}_{ab}$$

BH perturbation in slow rotation

$$g_{ab} = g_{ab}^{(0)} + \varepsilon g_{ab}^{(1)}$$

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Let's assume vacuum with slow rotation:

$$R_{ab}^{(0)} = 0 \longrightarrow g_{ab}^{(0)} = g_{ab}^{\mathrm{Sch}} + a g_{ab}^{\mathrm{SR},1}$$

$$\begin{pmatrix} -1+\frac{1}{r} & 0 & 0 & -\frac{a\sin^2\theta}{r} \\ 0 & \left(1-\frac{1}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ * & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

BH perturbation in slow rotation

$$g_{ab} = g_{ab}^{(0)} + \varepsilon g_{ab}^{(1)}$$

Same ansatz for perturbations!

$$\downarrow$$

$$R_{ab} = R_{ab}^{(0)} + \varepsilon R_{ab}^{(1)}$$

Let's assume vacuum with slow rotation:

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BH perturbation in slow rotation

Black box - manipulate the equations

- 10 mixed equations
- Formalism to decouple
  - Project onto proper angular basis
  - Introduce couplings with opposite parity $\mathcal{P}^{\ell} + a\overline{\mathcal{P}}^{\ell\pm1}$
- 7x polar-led eqs. - 3x axial-led eqs.
- 2x 1st order polar-led eqs. - 2x 1st order axial-led eqs.
- Definition of master variables that diagonalize the system

BH perturbation in slow rotation

$$\frac{d^2\Psi^\ell_\pm}{dr_*^2} + \left(\omega^2 - V^\ell_\pm\right)\Psi^\ell_\pm = 0$$

$$dr_* = \frac{dr}{(1 - 1/r)\,(1 + a f_1)}$$

$$V^\ell_\pm = V^\ell_{\pm,0} + a V^\ell_{\pm,1}$$

In the end, there is just a spin modification

Pani 2013, NF 2023

BH perturbation in slow rotation

$$\frac{\mathrm{d}^2\Psi_\pm^\ell}{\mathrm{d}r_*^2} + \left(\omega^2 - V_\pm^\ell\right)\Psi_\pm^\ell = 0$$

$$\mathrm{d}r_* = \frac{\mathrm{d}r}{(1-1/r)\left(1 + a f_1 + a^2 f_2\right)}$$

$$V_\pm^\ell = V_{\pm,0}^\ell + a V_{\pm,1}^\ell + a^2 V_{\pm,2}^\ell$$

In the end, there is just a spin modification also valid at second order in the spin

Pani 2013, NF 2023

Relation between potentials

Chandrasekhar found the *super-potential* that generates the RW/Z potential

$$V^{\ell}_{\pm,0} = \beta_0^2 W_0^{\ell 2} + \beta_0 \frac{\mathrm{d}W_0^{\ell}}{\mathrm{d}r_*} + \kappa_0 W_0^{\ell}$$

with

$$W_0^{\ell} = \frac{1-1/r}{r(3+\lambda r)}$$

$$\beta_0 = \pm 3, \quad \kappa_0 = \lambda(\lambda+2)$$

Chandrasekhar 1975

Relation between potentials

The superpotential still exists at least up to second order in the spin!

$$V^{\ell}_{(\pm)} = \beta_0^2 W^{\ell 2} + \beta_0 \frac{\mathrm{d}W^{\ell}}{\mathrm{d}r_*} + \kappa_0 W^{\ell} + \kappa_0 \kappa,$$

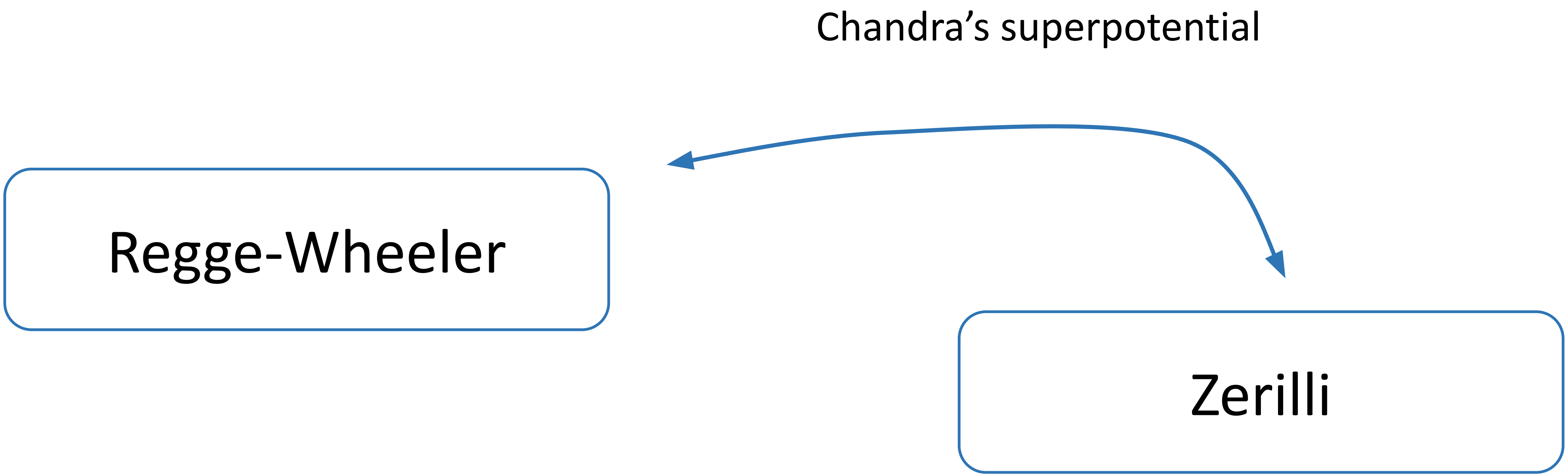
with

$$W^{\ell} = W^{\ell}_0 + a W^{\ell}_1 + a^2 W^{\ell}_2$$

$$\beta_0 = \pm 3, \qquad \kappa = a\kappa_1 + a^2 \kappa_2$$

Chandrasekhar 1975, NF 2023

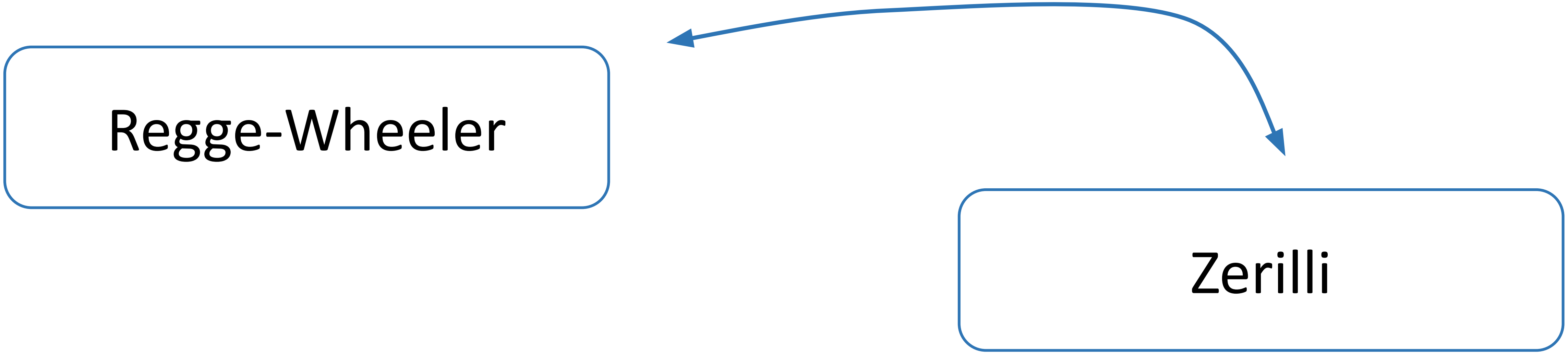
Relation between potentials



Chandrasekhar 1975, NF 2023

Relation between potentials

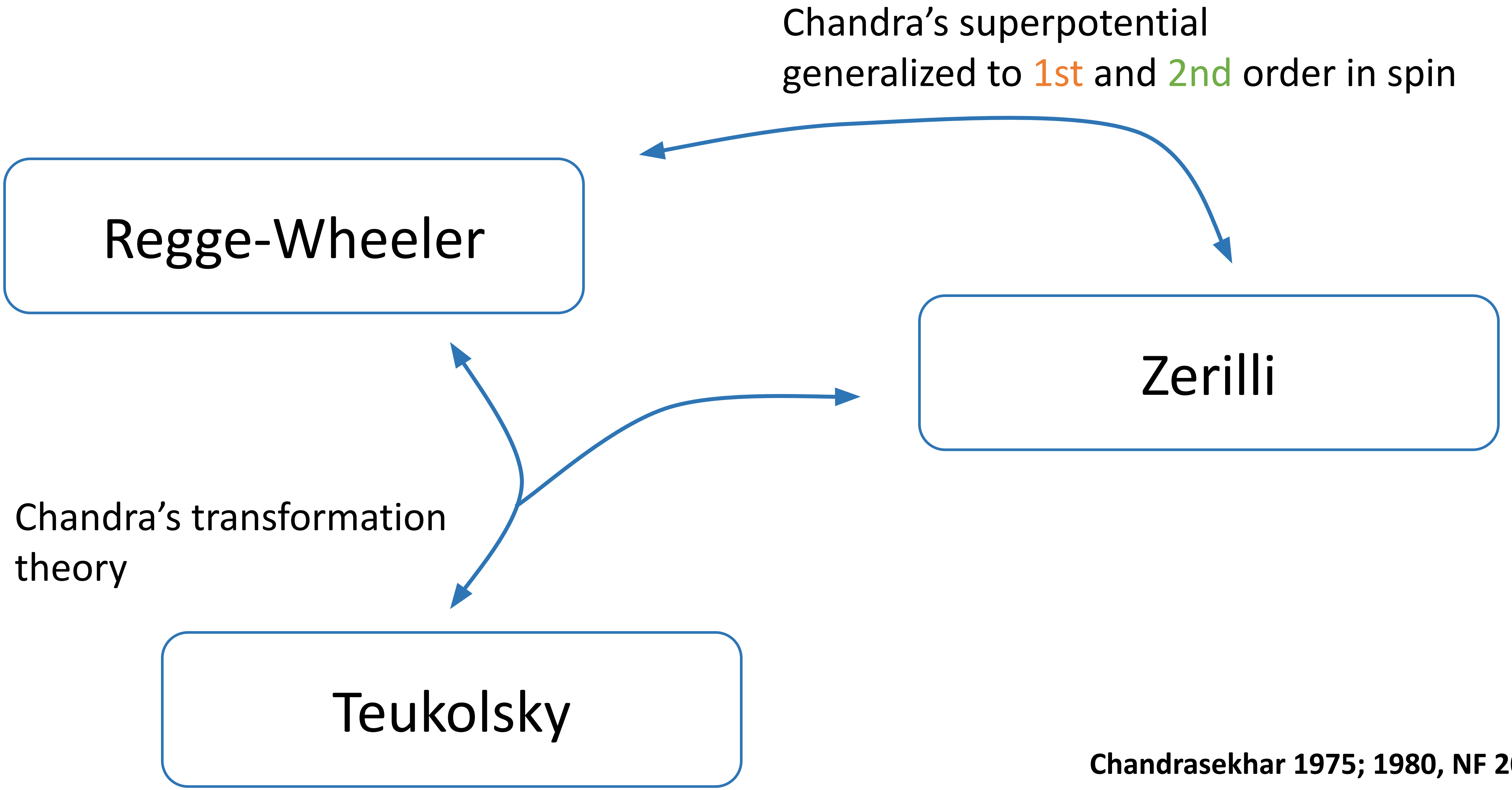
Chandra's superpotential
generalized to 1st and 2nd order in spin



The transformation proves isospectrality between the two equations

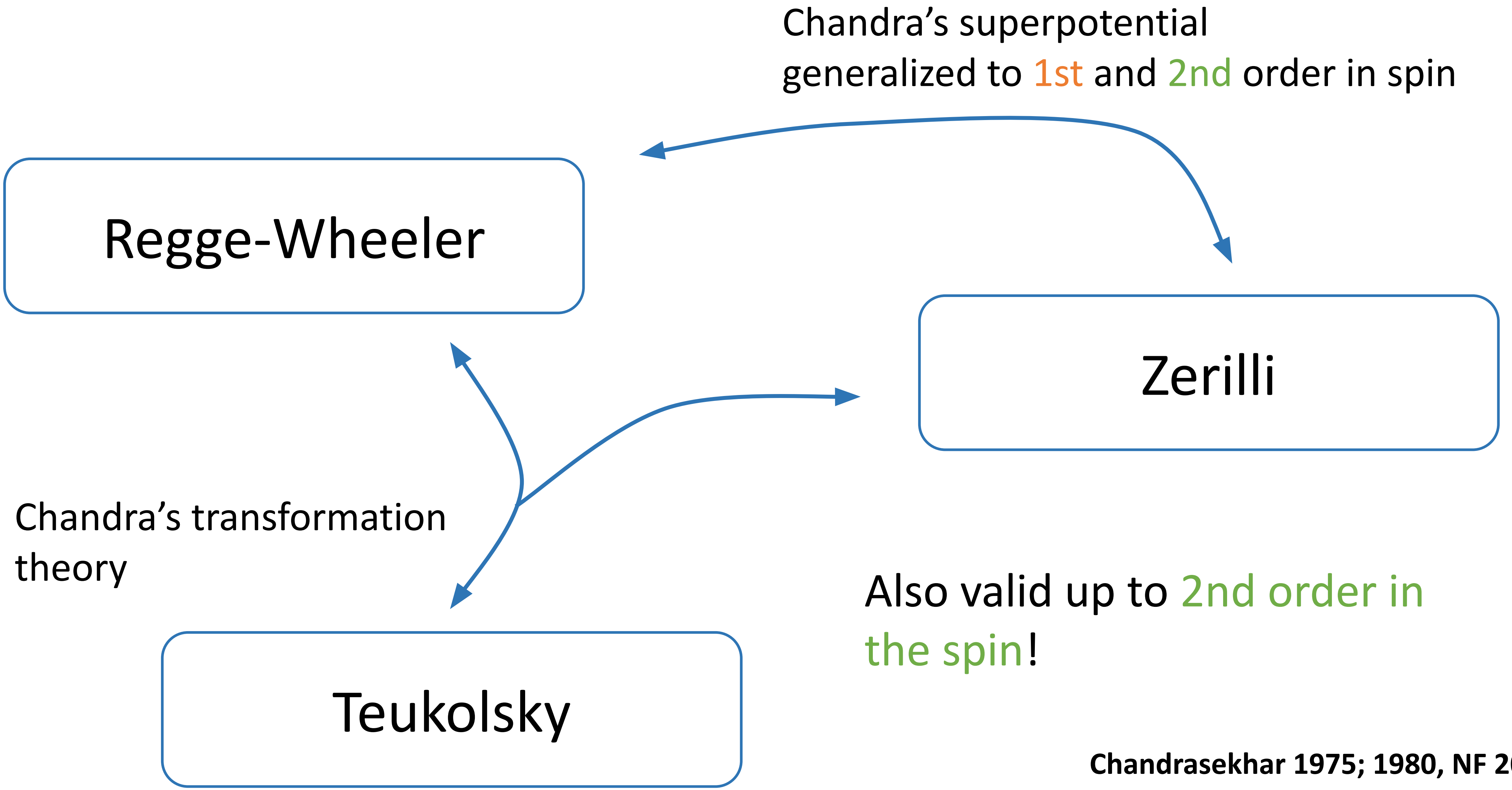
Chandrasekhar 1975, NF 2023

Relation between RW/Z and Teukolsky



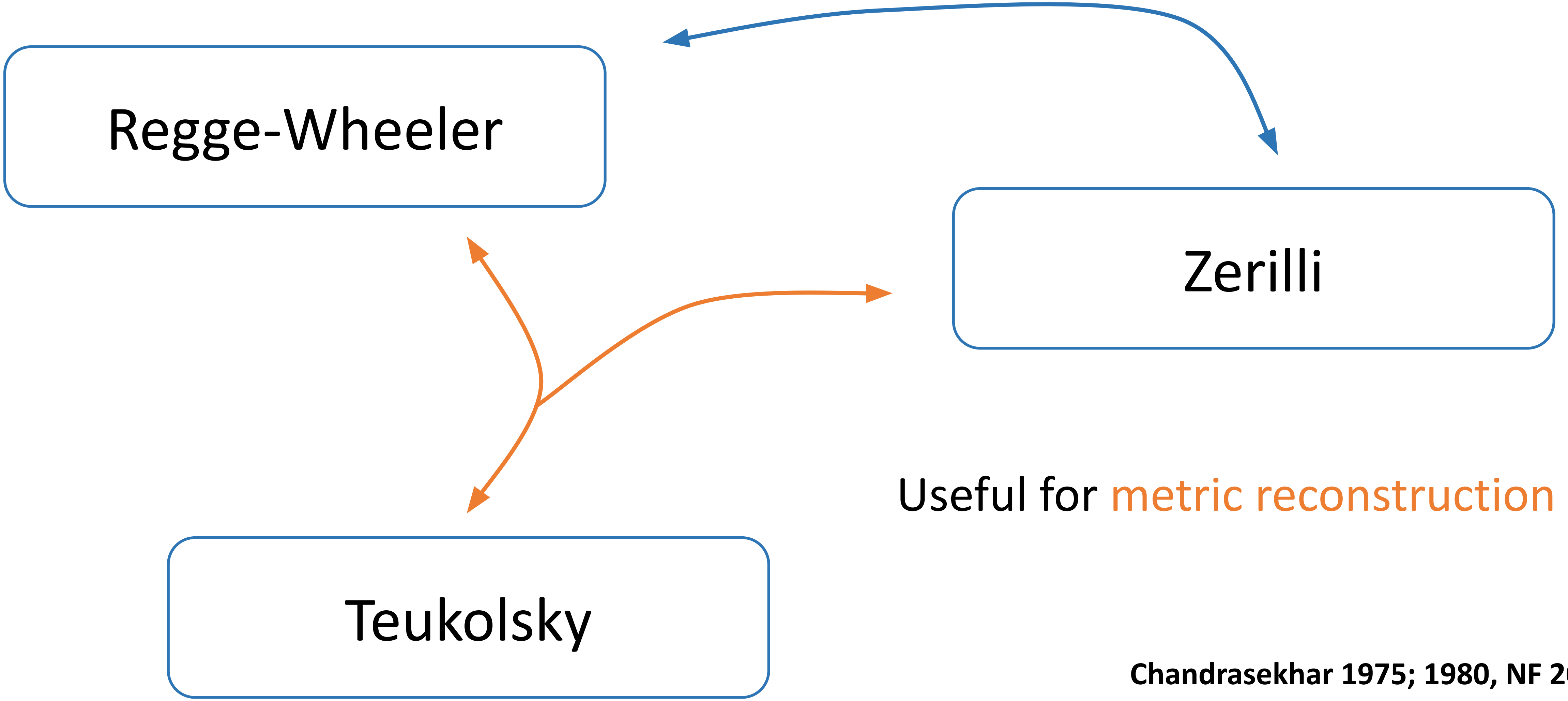
Chandrasekhar 1975; 1980, NF 2023

Relation between RW/Z and Teukolsky



Chandrasekhar 1975; 1980, NF 2023

Relation between RW/Z and Teukolsky



Useful for metric reconstruction

Chandrasekhar 1975; 1980, NF 2023

Conclusions

- We showed how to generalize the Regge-Wheeler and the Zerilli equation in a spin expansion

- Chandra's transformations between RWZ are still applicable, as well as those to slow-rotating Teukolsky

- Useful for "*perturbation of perturbation*" problem (second order perturbations, beyond-GR BHPT, etc…)

Open problem

Kerr metric perturbation conjecture

Is it possible to generalize the Regge-Wheeler and the Zerilli equations at arbitrary spin, without passing through the Teukolsky formalism?

Thanks for the attention =D