Action-angle variables of post-Newtonian binary black holes

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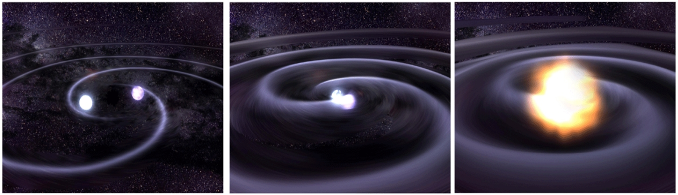
In collaboration with L. C. Stein, G. Cho, J. T. Gálvez Ghersi, and R. Samanta

Gravitational waves (GWs) from binary black holes



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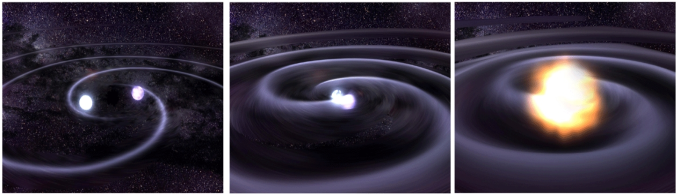

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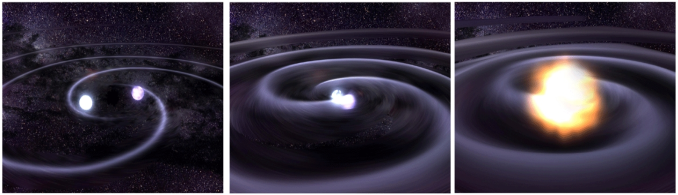

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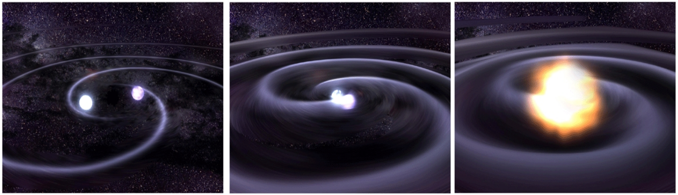

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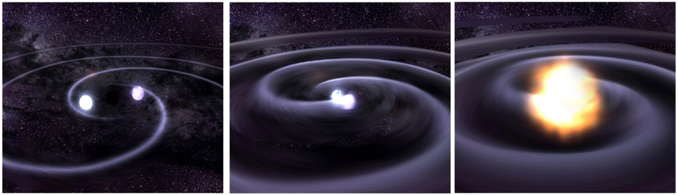

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- GWs are functions of black hole trajectories *(focus of the talk)*.

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$$0PN \quad 1PN \quad 1.5PN \quad 2PN$$

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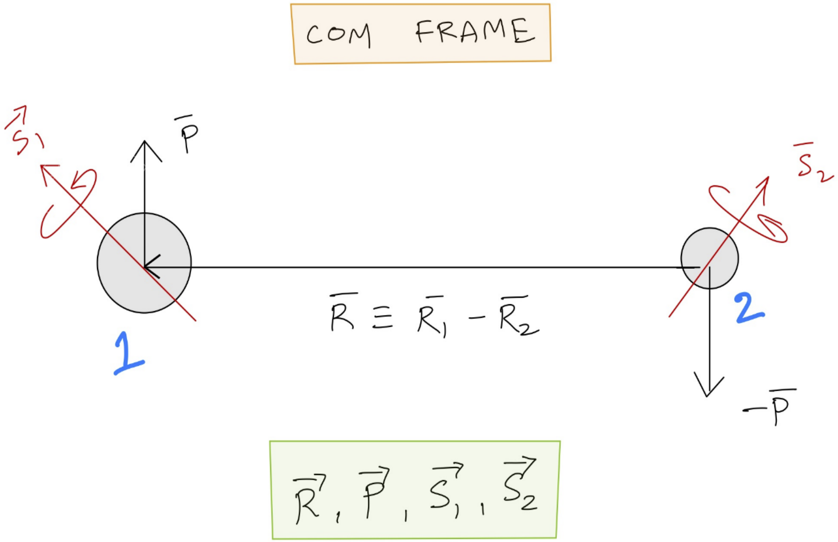
- Each factor of $1/c^2$ $\rightarrow$ one PN order.

Phase space of spinning BBHs



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- Solutions are crucial for fast GW template construction and data analysis.

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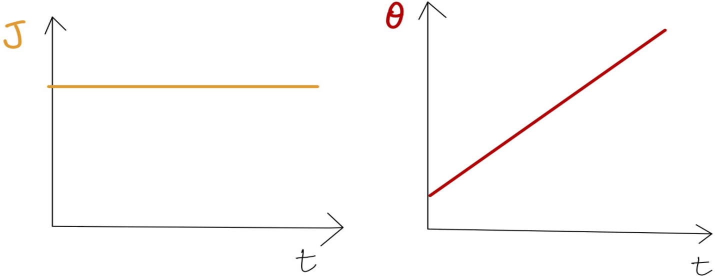

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- Extendable to higher PN via canonical pert. theory (Goldstein).

Expression of action variables

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- $\mathcal{J}_5$ is very, very lengthy.

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Refs:

- Papers: 2012.06586, 2110.15351, 2210.01605.
- Lecture notes: 2206.05799
- Mathematica package: github.com/sashwattanay/BBH-PN-Toolkit
- YouTube video on the package
- Contact: sashwat.tanay@obspm.fr

Thank you!
Questions?