Gravitational-wave phasing of compact binary systems to the fourth-and-a-half post-Newtonian order

Quentin Henry

Albert Einstein Institute (Potsdam)

quentin.henry@aei.mpg.de

In collaboration with L. Blanchet, G. Faye, F. Larrouturou, T. Marchand, S. Marsat and D. Trestini

MAX-PLANCK-INSTITUT
FÜR GRAVITATIONSPHYSIK
(Albert-Einstein-Institut)

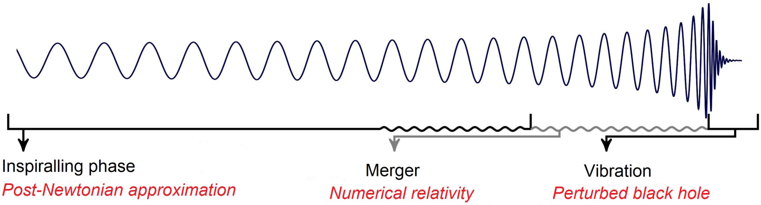
Groupement de recherche
Ondes gravitationnelles

October 16th 2023

Based on 10 papers

1. arXiv:1607.07601 [MBF16]
2. arXiv:2003.13672 [MHLMFB20]
3. arXiv:2105.10876 [HFB21]
4. arXiv:2110.02240 [LHBF21a]
5. arXiv:2110.02243 [LHBF21b]
6. arXiv:2204.11293 [BFL22]
7. arXiv:2209.02719 [TLB22]
8. arXiv:2301.09395 [TB23]
9. arXiv:2304.11185 [BFHLT23a]
10. arXiv:2304.11186 [BFHLT23b]

The post-Newtonian formalism



PN formalism:

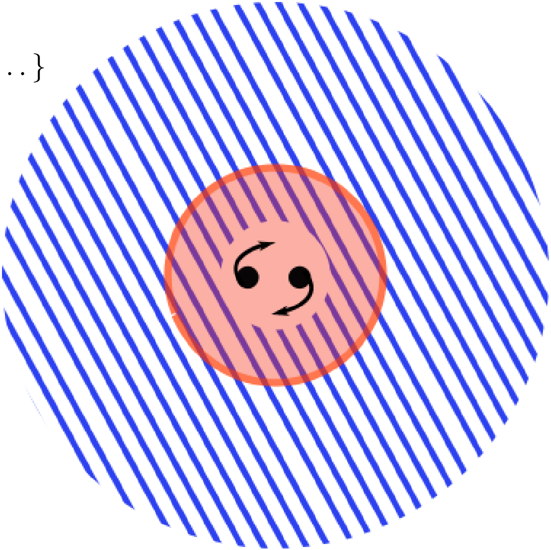
- Perturbative expansion of the equations of GR

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \quad \text{with} \quad \tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

- Weak field, small velocities : $(v/c)^2 \sim Gm/rc^2 \ll 1$
- 4.5PN order $\rightarrow O(1/c^9)$ beyond leading order
- Present work $\rightarrow$ non-spinning quasi-circular binaries

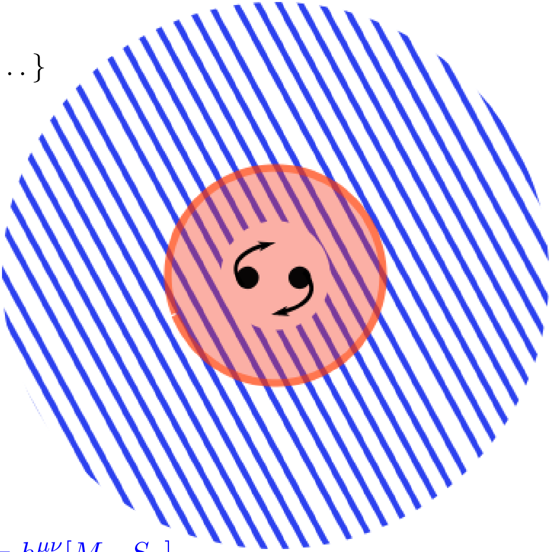
Different space zones in the PN-MPM formalism

Near zone, PN expansion
Exterior zone, PM, $\{I_L, J_L, \dots\}$
Buffer zone
Radiative zone



Different space zones in the PN-MPM formalism

Near zone, PN expansion
Exterior zone, PM, $\{I_L, J_L, \dots\}$
Buffer zone
Radiative zone



$$h^{\mu\nu}[I_L, J_L, W_L, X_L, Y_L, Z_L] = h^{\mu\nu}[M_L, S_L]$$

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)} + \frac{1}{c^2}\left(\frac{1}{189}U_{ijk}^{(2)}U_{ijk}^{(2)} + \frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right) + \dots\right]$$

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left(\frac{1}{189}U^{(2)}_{ijk}U^{(2)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right) + \dots\right]$$

$$U_{ij} = M^{(2)}_{ij} + (\text{non-linear terms})$$

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)} + \frac{1}{c^2}\left(\frac{1}{189}U_{ijk}^{(2)}U_{ijk}^{(2)} + \frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right) + \dots\right]$$

$$U_{ij} = M_{ij}^{(2)} + (\text{non-linear terms})$$

$$M_{ij} = I_{ij} + (\text{gauge interactions})$$

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)} + \frac{1}{c^2}\left(\frac{1}{189}U_{ijk}^{(2)}U_{ijk}^{(2)} + \frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right) + \dots\right]$$

$$U_{ij} = M_{ij}^{(2)} + (\text{non-linear terms})$$

$$M_{ij} = I_{ij} + (\text{gauge interactions})$$

$$I_{ij} = \int \mathrm{d}^3x\, \hat{x}^{ij}\left(\bar{\Sigma} + \dots\right)$$

$\bar{\Sigma}$ related to the PN metric and the stress-energy tensor

The regularization problem: UV and IR

$\hookrightarrow$ Formalism written using the Hadamard *Partie Finie* regularization

$\hookrightarrow$ Difference between UV (self field) and IR (infinity) divergencies

- UV: bodies modelled as point particles
- IR: need for a regularization at infinity in the formalism itself

$\hookrightarrow$ But dimensional regularization is more suitable at high order

$$I^{(d)} = I^{(\mathrm{Had})} + \mathscr{D}I$$

$\hookrightarrow$ We compute using Hadamard regularization and difference with DR

$\hookrightarrow$ Every step has to be done with this procedure

Results

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)} + \frac{1}{c^2}\left(\frac{1}{189}U_{ijk}^{(2)}U_{ijk}^{(2)} + \frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right) + \dots\right]$$

$$U_{ij} = M_{ij}^{(2)} + (\text{non-linear terms})$$

$$M_{ij} = I_{ij} + (\text{gauge interactions})$$

$$\boxed{I_{ij} = \int \mathrm{d}^3x\, \hat{x}^{ij}\left(\bar{\Sigma} + \dots\right)}$$

The source mass quadrupole [MHLMFB20,LHFB21a,LHFB21b]

$$I_{ij}^{\text{renorm}} = m\nu \left(A\, x_{\langle i}x_{j\rangle} + B\, \frac{r^2}{c^2} v_{\langle i}v_{j\rangle} + \frac{G^2 m^2 \nu}{c^5 r}\, C\, x_{\langle i}v_{j\rangle} \right) + O\left(\frac{1}{c^9}\right)$$

$$\begin{aligned}
A = 1 &+ \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\
&+ \gamma^3\left(\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3\right) \\
&+ \gamma^4\left(-\frac{1067041075909}{12713500800} + \frac{31886}{2205}\ln\left(\frac{r}{r_0}\right) + \left[-\frac{85244498897}{470870400} - \frac{2783}{1792}\pi^2 - \frac{64}{7}\ln\left(16\gamma e^{2\gamma_E}\right)\right.\right. \\
&\qquad \left. - \frac{10886}{735}\ln\left(\frac{r}{r_0}\right) + \frac{8495}{63}\ln\left(\frac{r}{r_0'}\right)\right]\nu + \left[\frac{171906563}{4484480} + \frac{44909}{2688}\pi^2 - \frac{4897}{21}\ln\left(\frac{r}{r_0'}\right)\right]\nu^2 \\
&\qquad \left. - \frac{22063949}{5189184}\nu^3 + \frac{71131}{314496}\nu^4\right)
\end{aligned}$$

$$\begin{aligned}
B = \frac{11}{21} - \frac{11}{7}\nu &+ \gamma\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) + \gamma^2\left(-\frac{357761}{19800} + \frac{428}{105}\ln\left(\frac{r}{r_0}\right) - \frac{92339}{5544}\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3\right) \\
&+ \gamma^3\left(\frac{23006898527}{1589187600} - \frac{4922}{2205}\ln\left(\frac{r}{r_0}\right) + \left[\frac{8431514969}{529729200} + \frac{143}{192}\pi^2 + \frac{32}{7}\ln\left(16\gamma e^{2\gamma_E}\right)\right.\right. \\
&\qquad \left. - \frac{1266}{49}\ln\left(\frac{r}{r_0}\right) - \frac{968}{63}\ln\left(\frac{r}{r_0'}\right)\right]\nu + \left[\frac{351838141}{5045040} - \frac{41}{24}\pi^2 + \frac{968}{21}\ln\left(\frac{r}{r_0'}\right)\right]\nu^2 \\
&\qquad \left. - \frac{1774615}{81081}\nu^3 - \frac{3053}{432432}\nu^4\right)
\end{aligned}$$

(r_0, r_0')

$$C = \frac{48}{7} + \gamma\left(-\frac{4096}{315} - \frac{24512}{945}\nu\right) - \frac{32}{7}\pi\,\gamma^{3/2}$$

$\hookrightarrow$ IR pole canceled by the ones of tails-of-tails and tails-of-memory contributions

The source current quadrupole [HFB21]

↪ In $3d$, J_{ij} defined with Levi-Civita tensor

↪ Needed to generalize the definition in d dimensions

$$J_{i|L} = \begin{array}{|c|c|c|c|}\hline i_\ell & \dots & i_2 & i_1 \\ \hline i & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad \underset{i|L}{\mathrm{Sym}} \equiv A_{ii_\ell} \underset{iL}{\mathrm{TF}} \; \underset{L}{\mathrm{STF}}$$

$$J^{[3]}_{ij} = \frac{1}{2}\, \varepsilon_{ab(i} \lim_{d\to 3} J_{\underline{a}|\underline{b}j)}$$

↪ Computed with full dimensional regularization to 3PN

$$J_{ij} = -\nu m \Delta \left[A\, L^{\langle i} x^{j\rangle} + B\, \frac{Gm}{c^3}\, L^{\langle i} v^{j\rangle} \right] + O\left(\frac{1}{c^7}\right)$$

$$\begin{aligned} A &= 1 + \gamma\left(\frac{67}{28} - \frac{2}{7}\nu\right) + \gamma^2\left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{1}{168}\nu^2\right) \\ &\quad + \gamma^3\left(\frac{2301023}{415800} - \frac{214}{105}\ln\left(\frac{r}{r_0}\right) + \left[-\frac{243853}{9240} + \frac{123}{128}\pi^2 - 22\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{44995}{5544}\nu^2 + \frac{599}{16632}\nu^3\right) \\ B &= \frac{188}{35}\nu\,\gamma \end{aligned}$$

(r_0, r'_0)

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left(\frac{1}{189}U^{(2)}_{ijk}U^{(2)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right) + \dots\right]$$

$$U_{ij} = M^{(2)}_{ij} + (\text{non-linear terms})$$

$$\boxed{M_{ij} = I_{ij} + (\text{gauge interactions})}$$

$$I_{ij} = \int \mathrm{d}^3x\, \hat{x}^{ij}\left(\bar{\Sigma} + \dots\right)$$

From source to canonical moments [BFL22,TLB22,BFHLT23a]

$\hookrightarrow$ In the exterior zone, we link the source to the canonical moments

$$h^{\mu\nu}[I_L, J_L, W_L, X_L, Y_L, Z_L] = h^{\mu\nu}[M_L, S_L]$$

$$\begin{aligned}
M_{ij} = I_{ij} &+ \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I^{(1)}_{ij}\right] \\
&+ \frac{4G}{c^7}\Bigg\{ \frac{4}{7} W^{(1)}_{a\langle i} I^{(3)}_{j\rangle a} + \frac{6}{7} W_{a\langle i} I^{(4)}_{j\rangle a} - \frac{1}{7} Y^{(3)}_{a\langle i} I_{j\rangle a} - Y_{a\langle i} I^{(3)}_{j\rangle a} - 2X\, I^{(3)}_{ij} \\
&\qquad - \frac{5}{21} W^{(4)}_a I_{ija} + \frac{1}{63} W^{(3)}_a I^{(1)}_{ija} - \frac{25}{21} Y^{(3)}_a I_{ija} - \frac{22}{63} Y^{(2)}_a I^{(1)}_{ija} + \frac{5}{63} Y^{(1)}_a I^{(2)}_{ija} \\
&\qquad + 2W^{(3)} W_{ij} + 2W^{(2)} W^{(1)}_{ij} - \frac{4}{3} W_{\langle i} W^{(3)}_{j\rangle} + 2W^{(2)} Y_{ij} - 4W_{\langle i} Y^{(2)}_{j\rangle} \\
&\qquad + \epsilon_{ab\langle i}\left[\frac{1}{3} I_{j\rangle a} Z^{(3)}_b - I^{(3)}_{j\rangle a} Z_b + \frac{4}{9} J_{j\rangle a} W^{(3)}_b - \frac{4}{9} J_{j\rangle a} Y^{(2)}_b + \frac{8}{9} J^{(1)}_{j\rangle a} Y^{(1)}_b\right]\Bigg\} \\
&+ O\left(\frac{1}{c^9}\right)
\end{aligned}$$

$\hookrightarrow$ Hadamard treatment yields a 4PN term that is canceled with DR

Brief overview of the steps to compute the phase

$$\phi(\Omega)$$

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)} + \frac{1}{c^2}\left(\frac{1}{189}U_{ijk}^{(2)}U_{ijk}^{(2)} + \frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right) + \dots\right]$$

$$\boxed{U_{ij} = M_{ij}^{(2)} + (\text{non-linear terms})}$$

$$M_{ij} = I_{ij} + (\text{gauge interactions})$$

$$I_{ij} = \int \mathrm{d}^3x\, \hat{x}^{ij}\left(\bar{\Sigma} + \dots\right)$$

Tails-of-Memory [TB23]

↪ The 3.5PN corrections were already known

↪ Two types of terms: tail-like and double integral

$$
\begin{aligned}
U_{ij}^{4\text{PN}} = \frac{G^2 M}{c^8}\frac{8}{7}\Bigg\{ & \int_0^{+\infty}\mathrm{d}\rho\, M_{a\langle i}^{(4)}(u-\rho)\int_0^{+\infty}\mathrm{d}\tau\, M_{j\rangle a}^{(4)}(u-\rho-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)-\frac{1613}{270}\right]\\
&-\frac{5}{2}\int_0^{+\infty}\mathrm{d}\tau\,\left[M_{a\langle i}^{(3)}M_{j\rangle a}^{(4)}\right](u-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{3}{2}\ln\left(\frac{c\tau}{2b_0}\right)\right]\\
&-3\int_0^{+\infty}\mathrm{d}\tau\,\left[M_{a\langle i}^{(2)}M_{j\rangle a}^{(5)}\right](u-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\ln\left(\frac{c\tau}{2b_0}\right)\right]\\
&-\frac{5}{2}\int_0^{+\infty}\mathrm{d}\tau\,\left[M_{a\langle i}^{(1)}M_{j\rangle a}^{(6)}\right](u-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{3}{10}\ln\left(\frac{c\tau}{2b_0}\right)\right]\\
&-\int_0^{+\infty}\mathrm{d}\tau\,\left[M_{a\langle i}M_{j\rangle a}^{(7)}\right](u-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)-\frac{1}{4}\ln\left(\frac{c\tau}{2b_0}\right)\right]\\
&-2M_{a\langle i}^{(2)}\int_0^{+\infty}\mathrm{d}\tau\, M_{j\rangle a}^{(5)}(u-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{27521}{5040}\right]\\
&-\frac{5}{2}M_{a\langle i}^{(1)}\int_0^{+\infty}\mathrm{d}\tau\, M_{j\rangle a}^{(6)}(u-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{15511}{3150}\right]\\
&+\frac{1}{2}M_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\, M_{j\rangle a}^{(7)}(u-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)-\frac{6113}{756}\right]\\
&-\frac{7}{12}\mathrm{S}_a\epsilon_{ab\langle i}\int_0^{+\infty}\mathrm{d}\tau\, M_{j\rangle b}^{(6)}(u-\tau)\left[\ln\left(\frac{c\tau}{2b_0}\right)+2\ln\left(\frac{c\tau}{2r_0}\right)+\frac{1223}{1890}\right]\Bigg\}
\end{aligned}
$$

(r_0, b_0)

↪ This is the $3d$ result, the DR correction yields a pole which cancels the one of I_{ij}. It is incorporated in I_{ij}^{renorm}.

The 4.5PN flux for circular orbits [MBF16,BFHLT23a]

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left(\frac{1}{189}U^{(2)}_{ijk}U^{(2)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right) + \dots\right]$$

$\hookrightarrow$ Combine all radiative moments on circular orbits

The 4.5PN flux for circular orbits [MBF16,BFHLT23a]

$$\mathscr{F} = \frac{G}{c^5}\left[\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij} + \frac{1}{c^2}\left(\frac{1}{189}U^{(2)}_{ijk}U^{(2)}_{ijk} + \frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right) + \dots\right]$$

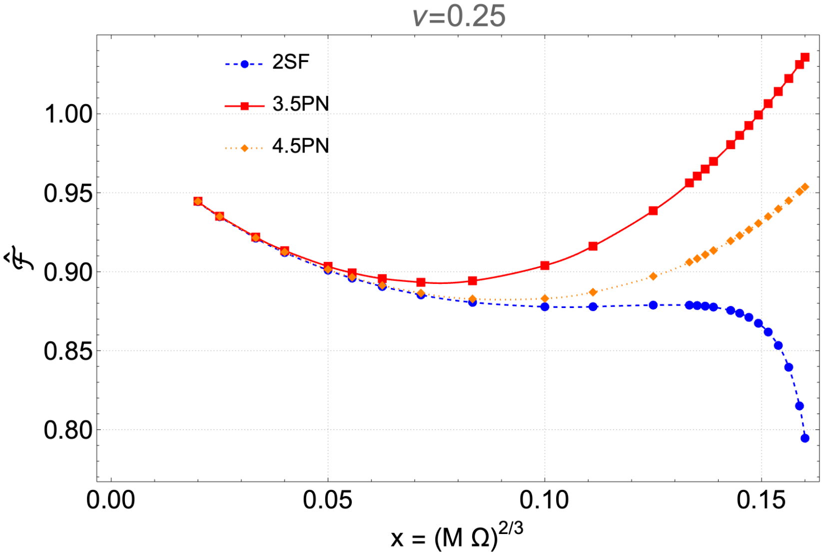
↪ Combine all radiative moments on circular orbits

$$\begin{aligned}
\mathscr{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathrm{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \\
& \qquad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathrm{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \qquad + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\Bigg]x^4 \\
& + \Bigg[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \\
& \qquad - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\Bigg]\pi x^{9/2} + O(x^5)\Bigg\}
\end{aligned}$$

$$x \equiv \left(\frac{Gm\Omega}{c^3}\right)^{2/3}$$

↪ The three arbitrary constants cancel out from observables

Comparison with second order self-force



Courtesy to H. Estellés

New feature: Orbital vs observable frequency [BFHLT23a]

$$h_{\ell m} = H_{\ell m} e^{-\mathrm{i}m\phi} = \hat{H}_{\ell m} e^{-\mathrm{i}m\psi}$$

↪ Absorb the logs (from tails) in $H_{\ell m}$ with a phase redifinition

$$\psi = \phi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right)$$

- Orbital frequency: $\omega \equiv \dot{\phi}$
- Apparent orbital frequency (at future null infinity): $\Omega \equiv \dot{\psi}$

↪ The observed orbital frequency differs by a 4PN correction

$$\Omega = \omega \left\{ 1 - \frac{192}{5} \nu \underbrace{\left(\frac{Gm\omega}{c^3}\right)^{8/3}}_{\text{4PN}} \left[\ln\left(\frac{\omega}{\omega_0}\right) + 1 \right] \right\} + O\left(\frac{1}{c^{10}}\right)$$

$\omega_0 \propto 1/b_0$

↪ Propagation of the GW on a Schwarzschild background

↪ **Crucial** to remove arbitrary constants from observables

The 4.5PN phase [BFHLT23b]

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t} \qquad \Rightarrow \qquad \psi = \int \mathrm{d}t\,\Omega = -\frac{c^3}{Gm}\int \mathrm{d}x\,\frac{x^{3/2}}{\mathscr{F}(x)}\,\frac{\mathrm{d}E}{\mathrm{d}x}$$

$$\begin{aligned}
\psi = -\frac{x^{-5/2}}{32\nu}\Bigg\{ & 1+\left(\frac{3715}{1008}+\frac{55}{12}\nu\right)x-10\pi x^{3/2}+\left(\frac{15293365}{1016064}+\frac{27145}{1008}\nu+\frac{3085}{144}\nu^2\right)x^2+\left(\frac{38645}{1344}-\frac{65}{16}\nu\right)\pi x^{5/2}\ln x \\
&+\left[\frac{12348611926451}{18776862720}-\frac{160}{3}\pi^2-\frac{1712}{21}\gamma_\mathrm{E}-\frac{856}{21}\ln 16x-\left(\frac{15737765635}{12192768}-\frac{2255}{48}\pi^2\right)\nu+\frac{76055}{6912}\nu^2-\frac{127825}{5184}\nu^3\right]x^3 \\
&+\left(\frac{77096675}{2032128}+\frac{378515}{12096}\nu-\frac{74045}{6048}\nu^2\right)\pi x^{7/2} \\
&+\Bigg[\frac{2550713843998885153}{2214468081745920}-\frac{9203}{126}\gamma_\mathrm{E}-\frac{45245}{756}\pi^2-\frac{252755}{2646}\ln 2-\frac{78975}{1568}\ln 3-\frac{9203}{252}\ln x \\
&\quad+\left(-\frac{680712846248317}{337983528960}-\frac{488986}{1323}\gamma_\mathrm{E}+\frac{109295}{1792}\pi^2-\frac{1245514}{1323}\ln 2+\frac{78975}{392}\ln 3-\frac{244493}{1323}\ln x\right)\nu \\
&\quad+\left(\frac{7510073635}{24385536}-\frac{11275}{1152}\pi^2\right)\nu^2+\frac{1292395}{96768}\nu^3-\frac{5975}{768}\nu^4\Bigg]x^4 \\
&+\Bigg[-\frac{93098188434443}{150214901760}+\frac{1712}{21}\gamma_\mathrm{E}+\frac{80}{3}\pi^2+\frac{856}{21}\ln(16x) \\
&\quad+\left(\frac{1492917260735}{1072963584}-\frac{2255}{48}\pi^2\right)\nu-\frac{45293335}{1016064}\nu^2-\frac{10323755}{1596672}\nu^3\Bigg]\pi x^{9/2}+O(x^5)\Bigg\}
\end{aligned}$$

The 4.5PN phase [BFHLT23b]

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t} \qquad \Rightarrow \qquad \psi = \int \mathrm{d}t\,\Omega = -\frac{c^3}{Gm}\int \mathrm{d}x\,\frac{x^{3/2}}{\mathscr{F}(x)}\,\frac{\mathrm{d}E}{\mathrm{d}x}$$

$$\begin{aligned}
\psi = -\frac{x^{-5/2}}{32\nu}\Bigg\{ & 1+\left(\frac{3715}{1008}+\frac{55}{12}\nu\right)x - 10\pi x^{3/2} + \left(\frac{15293365}{1016064}+\frac{27145}{1008}\nu+\frac{3085}{144}\nu^2\right)x^2 + \left(\frac{38645}{1344}-\frac{65}{16}\nu\right)\pi x^{5/2}\ln x \\
& + \left[\frac{12348611926451}{18776862720}-\frac{160}{3}\pi^2-\frac{1712}{21}\gamma_\mathrm{E}-\frac{856}{21}\ln 16x - \left(\frac{15737765635}{12192768}-\frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\right]x^3 \\
& + \left(\frac{77096675}{2032128}+\frac{378515}{12096}\nu-\frac{74045}{6048}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[\frac{2550713843998885153}{2214468081745920}-\frac{9203}{126}\gamma_\mathrm{E}-\frac{45245}{756}\pi^2-\frac{252755}{2646}\ln 2-\frac{78975}{1568}\ln 3-\frac{9203}{252}\ln x \\
& \quad + \left(-\frac{680712846248317}{337983528960}-\frac{488986}{1323}\gamma_\mathrm{E}+\frac{109295}{1792}\pi^2-\frac{1245514}{1323}\ln 2+\frac{78975}{392}\ln 3-\frac{244493}{1323}\ln x\right)\nu \\
& \quad + \left(\frac{7510073635}{24385536}-\frac{11275}{1152}\pi^2\right)\nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4\Bigg]x^4 \\
& + \Bigg[-\frac{93098188434443}{150214901760}+\frac{1712}{21}\gamma_\mathrm{E}+\frac{80}{3}\pi^2+\frac{856}{21}\ln(16x) \\
& \quad + \left(\frac{1492917260735}{1072963584}-\frac{2255}{48}\pi^2\right)\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\Bigg]\pi x^{9/2} + O(x^5)\Bigg\}
\end{aligned}$$

Thank you for your attention!