Gravitational Wave Turbulence

- Multiple Time Scale Analysis -

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Introduction

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As non-linear interactions lead to turbulent cascades, we aim to study how it occurs for GW. For that purpose, we base our approach on wave turbulence [Galtier & Nazarenko, PRL, 2017].

Model assumptions

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- There is a reference frame (t, x, y, z) where the metric is diagonal and ∂_z is a Killing field [Hadad and Zakharov, 2014]:

$$g_{\mu\nu} = \begin{pmatrix} -e^{-2\varphi}(1+\gamma)^2 & 0 & 0 & 0 \\ 0 & e^{-2\varphi}(1+\beta)^2 & 0 & 0 \\ 0 & 0 & e^{-2\varphi}(1+\alpha)^2 & 0 \\ 0 & 0 & 0 & e^{2\varphi} \end{pmatrix}.$$

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- The field φ has small amplitude oscillation:

$$|\varphi| \propto \varepsilon \ll 1.$$

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- At first non-linear order, we have dynamical equation:

$$\Box\varphi = -\partial_t\left[\left(\alpha+\beta-\gamma\right)\partial_t\varphi\right] + \partial_x\left[\left(\alpha-\beta+\gamma\right)\partial_x\varphi\right] + \partial_y\left[\left(-\alpha+\beta+\gamma\right)\partial_y\varphi\right],$$

and the constraint equations:

$$\partial_x\partial_t\alpha = -2\partial_t\varphi\partial_x\varphi, \quad \partial_y\partial_t\beta = -2\partial_t\varphi\partial_x\varphi, \quad \partial_x\partial_y\gamma = -2\partial_x\varphi\partial_y\varphi.$$

Wave Turbulence approach: a plunge into Fourier space

We introduce the normal variables:

$$a^s(\mathbf{k},t) = \frac{1}{\varepsilon}\left(\sqrt{\frac{k}{2}}\hat{\varphi}(\mathbf{k},t) + \frac{is}{\sqrt{2k}}\partial_t\hat{\varphi}(\mathbf{k},t)\right)e^{is\omega_{\mathbf{k}}t} \text{ with } s=\pm 1 \text{ and } \omega_{\mathbf{k}} = k.$$

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We combine the expanded Einstein's equations to get their evolution. It can be written as the standard form:

$$\partial_t a^s(\mathbf{k}) = \varepsilon^2 \int_{\mathbb{R}^6} \sum_{s_1,s_2,s_3} \mathbf{L}^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\; a^{s_1}(\mathbf{k}_1)a^{s_2}(\mathbf{k}_2)a^{s_3}(\mathbf{k}_3)\; e^{i\Omega^0_{123}t}\; \delta^0_{123}(\mathbf{k}) \; \prod_{i=1}^{3}\mathrm{d}^2\mathbf{k}_i,$$

where: $\mathbf{L}^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ the interaction coefficient, $\Omega^0_{123} = sk - s_1k_1 - s_2k_2 - s_3k_3$ and $\delta^0_{123}(\mathbf{k}) = \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$. They define the resonant manifold.

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Main objective: to derive the evolution of the statistical mean of the wave action: $n(\mathbf{k}) = \langle a^s(\mathbf{k})a^{-s}(-\mathbf{k})\rangle$.

The multiple time scale method

We introduce a set of time variables: $T_0 = t$, $T_2 = \varepsilon^2 t$, $T_4 = \varepsilon^4 t, \dots$ so that:

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With these definitions, we obtain the following expansion:

$$\begin{aligned}
&\left(\partial_{T_0} + \varepsilon^2 \partial_{T_2} + \varepsilon^4 \partial_{T_4} + \dots\right)\left(a_0^s(\mathbf{k}) + \varepsilon^2 a_2^s(\mathbf{k}) + \varepsilon^4 a_4^s(\mathbf{k}) + \dots\right)\\
&\quad = \varepsilon^2 \int_{\mathbb{R}^6} \sum_{s_1,s_2,s_3} \mathbf{L}^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\left(a_0^{s_1}(\mathbf{k}_1) + \varepsilon^2 a_2^{s_1}(\mathbf{k}_1) + \dots\right)\left(a_0^{s_2}(\mathbf{k}_2) + \varepsilon^2 a_2^{s_2}(\mathbf{k}_2) + \dots\right)\\
&\qquad\qquad \times \left(a_0^{s_3}(\mathbf{k}_3) + \varepsilon^2 a_2^{s_3}(\mathbf{k}_3) + \dots\right) \; \delta^0_{123}(\mathbf{k}) \; e^{i\Omega^0_{123} T_0} \prod_{i=1}^{3} \mathrm{d}^2\mathbf{k}_i.
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$$= \varepsilon^2 \int_{\mathbb{R}^6} \sum_{s_1,s_2,s_3} \mathbf{L}^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \left(a_0^{s_1}(\mathbf{k}_1) + \varepsilon^2 a_2^{s_1}(\mathbf{k}_1) + \ldots\right)\left(a_0^{s_2}(\mathbf{k}_2) + \varepsilon^2 a_2^{s_2}(\mathbf{k}_2) + \ldots\right)$$
$$\times \left(a_0^{s_3}(\mathbf{k}_3) + \varepsilon^2 a_2^{s_3}(\mathbf{k}_3) + \ldots\right)\ \delta^0_{123}(\mathbf{k})\ e^{i\Omega^0_{123} T_0} \prod_{i=1}^{3} \mathrm{d}^2\mathbf{k}_i.$$

We get the evolution of $a_0^s(\mathbf{k})$, $a_2^s(\mathbf{k})$, $\ldots$ by identifying the different power of ε.

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Their time integration introduce seculars drifts (terms $\propto T_0$ or $\propto {T_0}^2$).

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$$\langle a^s(\mathbf{k})a^{-s}(-\mathbf{k})\rangle = \underbrace{\langle a_0^s(\mathbf{k})a_0^{-s}(-\mathbf{k})}_{=n(\mathbf{k})}\rangle + \epsilon^2\langle a_2^s(\mathbf{k})a_0^{-s}(-\mathbf{k}) + a_0^s(\mathbf{k})a_2^{-s}(-\mathbf{k})\rangle + \varepsilon^4(\dots) + \dots$$

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$$\partial_{T_0} n(\mathbf{k}) = 0;$$

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and:

$$\partial_{T_4} n(\mathbf{k}) = 36\pi \int_{\mathbb{R}^6} \sum_{s_1,s_2,s_3} s \left|\mathbf{L}^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right|^2 \left(\frac{s}{n(\mathbf{k})} - \frac{s_1}{n(\mathbf{k}_1)} - \frac{s_2}{n(\mathbf{k}_2)} - \frac{s_3}{n(\mathbf{k}_3)}\right) \times n(\mathbf{k})n(\mathbf{k}_1)n(\mathbf{k}_2)n(\mathbf{k}_3)\delta^0_{123}(\omega)\delta^0_{123}(\mathbf{k}) \prod_{i=1}^{3} \mathrm{d}^2\mathbf{k}_i .$$

The kinetic equation

Using all the symmetries of the interaction coefficient, we find:

$$\partial_t n(\mathbf{k}) = 36\pi\epsilon^4 \int_{\mathbb{R}^6} \left| \mathbf{L}^{s-sss}_{\mathbf{k}-\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \right|^2 \left(\frac{1}{n(\mathbf{k})} + \frac{1}{n(\mathbf{k}_1)} - \frac{1}{n(\mathbf{k}_2)} - \frac{1}{n(\mathbf{k}_3)} \right) \times n(\mathbf{k}) n(\mathbf{k}_1) n(\mathbf{k}_2) n(\mathbf{k}_3) \delta^{01}_{23}(\omega) \delta^{01}_{23}(\mathbf{k}) \prod_{i=1}^{3} \mathrm{d}^2\mathbf{k}_i .$$

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$$\times\, n(\mathbf{k}) n(\mathbf{k}_1) n(\mathbf{k}_2) n(\mathbf{k}_3) \delta^{01}_{23}(\omega) \delta^{01}_{23}(\mathbf{k}) \prod_{i=1}^{3} \mathrm{d}^2\mathbf{k}_i.$$

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$$n(k) \propto (-\zeta)^{1/3}\, k^{-2/3} \quad \text{and} \quad n(k) \propto \epsilon^{1/3} k^{-1}.$$

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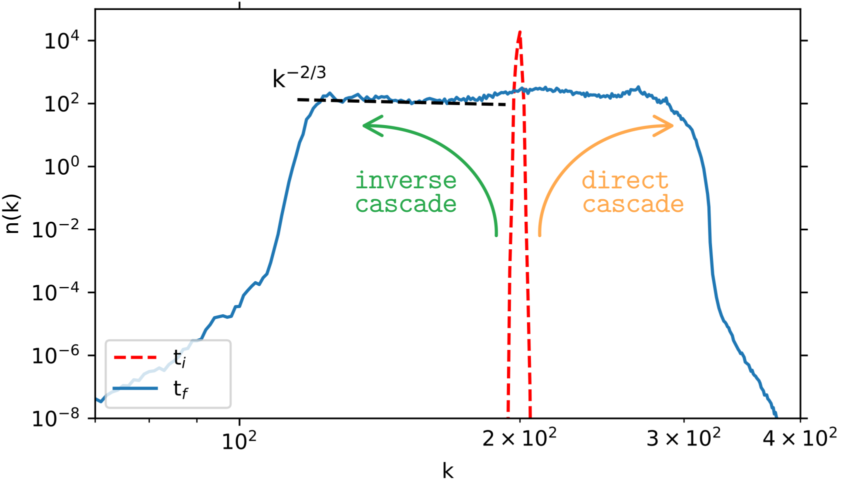
First numerical results on GPU (DNS, 1024×1024):



Conclusion

We aim to describe the weakly non linear regime of gravitational waves using statistical and analytical tools [Gay *et al.*, in prep.], in order to predict the existence of a dual cascade numerically observed.

Further works need to be performed:

- How to generalize this method to a more general model? What about the other polarization?
- What about the strong turbulent regime? Is there a link with inflation ?