Conformal Bootstrap and beyond

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Outline:

1. Conformal field theories
2. The numerical bootstrap
3. An S-matrix bootstrap?

Conformal field theories

(examples: $[\phi^4]_3$, $\mathcal{N}=4$ SYM, some 4d gauge thies)

(d-dim., Euclidean) (operator - algebraic perspective)

① symmetries:

Poincaré	$\hat{P}_\mu, \hat{M}_{\mu\nu}$	$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + b^\mu$
Scale	$\hat{D}$	$x^\mu \mapsto \lambda x^\mu$
Special Conf.	$\hat{K}_\mu$	$\leftarrow x^\mu \mapsto x^\mu / x^2$

(commutators ... e.g. $[\hat{D}, \hat{P}_\mu] = \hat{P}_\mu$)

② local operators $\mathcal{O}_\bullet(x)$ obey $[\hat{D}, \mathcal{O}_\bullet(0)] = \Delta\, \mathcal{O}_\bullet(0)$

- Lorentz index
- Dilatation weight

($\Delta >$ unitarity bound)

and are either derivatives: $\mathcal{O}_\bullet(x) = \partial_\mu \partial_\nu .. \partial_\rho \tilde{\mathcal{O}}_\bullet(x)$ or not.

- "descendant"
- "primary"

(3) correlation functions are constrained by symmetry for primaries:

* 1-pt. functions are trivial: $\langle \mathbb{1} \rangle = 1$, $\langle \mathcal{O}_\bullet(x) \rangle = 0$

tensor structure:

* 2-pt. functions are fixed:

$$\langle \mathcal{O}^I_\bullet(x)\, \mathcal{O}^J_\bullet(y) \rangle = \frac{\delta_{IJ}\, t_{\bullet\bullet}}{|x-y|^{2\Delta_I}}$$

$t_{\bullet\bullet} \to 1$ for scalar

$t_{\alpha\dot\alpha} \to \dfrac{\sigma^\mu_{\alpha\dot\alpha}(x-y)_\mu}{|x-y|}$ for spinor

* 3-pt. functions are almost fixed:

3-pt. coeffs $\in \mathbb{R}$

$$\langle \mathcal{O}^I_\bullet(x)\, \mathcal{O}^J_\bullet(y)\, \mathcal{O}^K_\bullet(z) \rangle = \frac{\sum_\alpha \lambda^{(\alpha)}_{IJK}\, t^{(\alpha)}_{\bullet\bullet\bullet}}{|x-y|^{\Delta_I+\Delta_J-\Delta_K}\, |x-z|^{\Delta_I+\Delta_K-\Delta_J}\, |y-z|^{\Delta_J+\Delta_K-\Delta_I}}$$

* 4- and higher-pt. functions have "cross ratios" $\sim \dfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$

for descendants: just take derivatives ...

4 $\exists$ convergent operator product expansion

$$\mathcal{O}_I(x)\,\mathcal{O}_J(y) = \sum_K f_{IJK} \frac{\mathcal{O}_K(y)}{|x-y|^{\Delta_I+\Delta_J-\Delta_K}}$$

(drop t. from now on)

all operators

$$= \sum_K{}' f_{IJK} \frac{1}{|x-y|^{\Delta_I+\Delta_J-\Delta_K}} \left[\mathcal{O}_K(y) + \#\,(x-y)^\mu \partial_\mu \mathcal{O}_K(y) + \ldots\right]$$

all primary operators

known!

$$= \sum_K{}' f_{IJK}\, \underbrace{C\left[\begin{matrix}\Delta_I & \Delta_J \\ & \Delta_K\end{matrix}, (x-y)^\mu, \partial_\mu\right]}_{\text{known!}} \mathcal{O}_K(y)$$

OPE

$$\mathcal{O}_I(x)\,\mathcal{O}_J(y) = \sum_K{}' f_{IJK}\,\underbrace{C\left[\begin{smallmatrix}\Delta_I\ \Delta_J\\ \Delta_K\end{smallmatrix}, (x-y)^\mu, \partial^y_\mu\right]^\bullet}_{\text{Known!}}\,\mathcal{O}_{K\bullet}(y)$$

in 3-pt. fns:

$$\langle \mathcal{O}_I(x)\,\mathcal{O}_J(y)\,\mathcal{O}_{\bullet L}(z)\rangle = \frac{\lambda_{IJL}\,t_\bullet}{|x-y|^{\#}\,|x-z|^{\#}\,|y-z|^{\#}}$$

$$\|$$

$$\left\langle \sum_K{}' f_{IJK}\,C[\,\ldots\ldots\partial_\mu]^\bullet\,\mathcal{O}_{\bullet K}(y)\,\mathcal{O}_{\bullet L}(z)\right\rangle$$

$$\|$$

$$f_{IJL}\,C[\ldots\partial_\mu]^\bullet\,t_\bullet\,|y-z|^{-2\Delta_L}$$

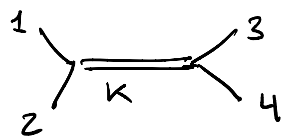
(recall: $\langle \mathcal{O}_K \mathcal{O}_L\rangle \propto \delta_{KL}$)

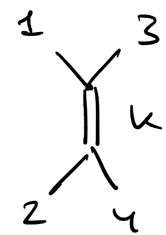
$\Longrightarrow$ $\boxed{f_{IJL} = \lambda_{IJL}}$ and a way to fix $C[\ldots]$.

$\boxed{\text{in 4-pt. fns:}}$

$$\langle O(x_1) \dots O(x_4) \rangle = \sum{}' \lambda_k^2 \; C[\dots]\cdot C[\dots]\cdot \frac{t_{\cdot\cdot}}{|x_2 - x_4|^{2\Delta_k}}$$

$$= \sum{}' \lambda_k^2 \underbrace{G[\Delta_O, \Delta_k, \ell_k ; x_i]}_{\text{conformal block}} =$$

$\sum' \lambda_k^2$ 

$= \sum' \lambda_k^2$ 

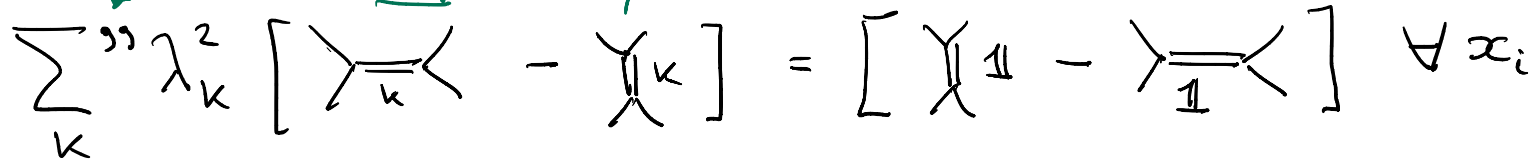
Crossing symmetry:

∞-ly many consistency conditions

for OPE data $\lambda_{IJK}, \{\Delta_I, \ell_I\}$.

The numerical bootstrap

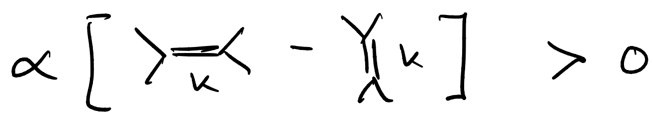
✓ primaries except identity

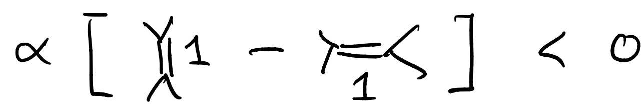


Let functional $\alpha[\ldots]$ be:

$$\alpha\left[\, f(x_i)\,\right] := \sum_{mnpq} \alpha_{mnpq} \frac{\partial}{\partial x_1^m} \cdots \frac{\partial}{\partial x_4^q} f(x_i)\Big|_{x_i^*}$$

Suppose:

 $\forall$ non-id. operators

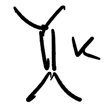
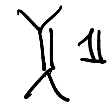
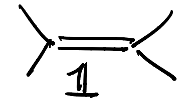


then: no CFT with operator $\mathcal{O}(x)$ exists!

(in reality: no such $\alpha[\ldots]$ exists)

The numerical bootstrap

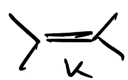
primaries except identity

$\sum_{k}{}^{\prime\prime}\, \lambda_k^2$ [ − ] = [ − ] $\forall x_i$!

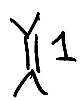
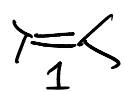
Let functional $\alpha[\ldots]$ be:

$$\alpha[f(x_i)] := \sum_{mnpq} \alpha_{mnpq} \frac{\partial}{\partial x_1^m} \cdots \frac{\partial}{\partial x_4^q} f(x_i)\Big|_{x_i^*}$$

Suppose:

α [ − ] > 0 $\quad \forall$ operators with $\ell > 0$

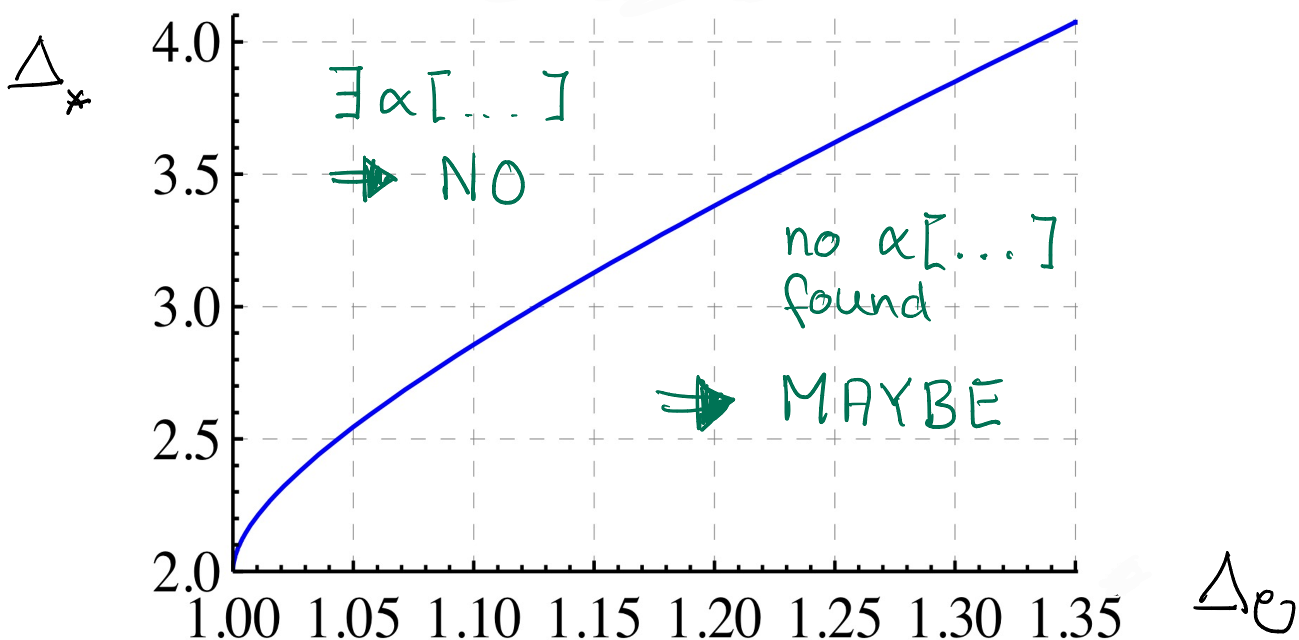
operators with $\ell = 0$ and $\Delta > \Delta_*$

α [ − ] < 0

then: <u>every</u> CFT with $\mathcal{O}(x)$ must also have a scalar operator with $\Delta < \Delta_*$!

The numerical bootstrap – 1st result

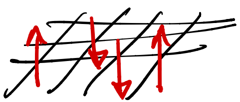
analyze $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$, suppose $\mathcal{O} \times \mathcal{O} = 1 + \begin{pmatrix} \text{scalars } \Delta > \Delta_* \\ \text{tensors any } \Delta \end{pmatrix}$
(in $d = 4$)



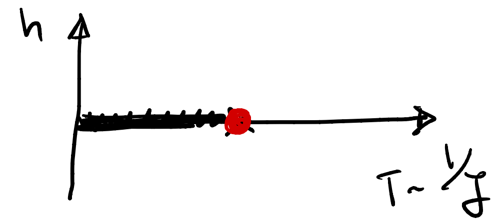
[0807.0004 Rattazzi Rychkov Tonvi Vichi]

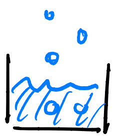
The numerical bootstrap - 3d Ising

- fixed point of $\int d^3x \left[-\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 - \frac{1}{4!}\lambda\Phi^4\right]$
- describes the critical behavior of many systems:
  - uniaxial ferromagnets

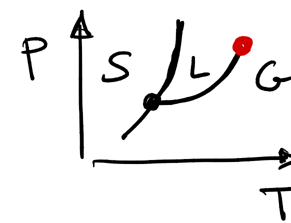


$$H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j + h\sum_i \sigma_i$$



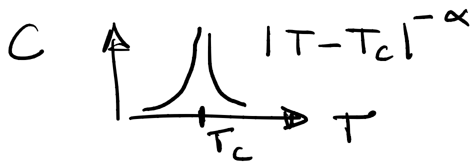


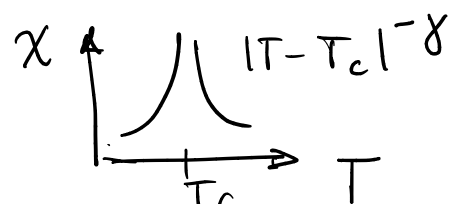
  - critical liquid-vapor points



  - ....

where we see power laws:





Q: $\alpha, \gamma, \dots$?

The numerical bootstrap - 3d Ising

From our perspective:

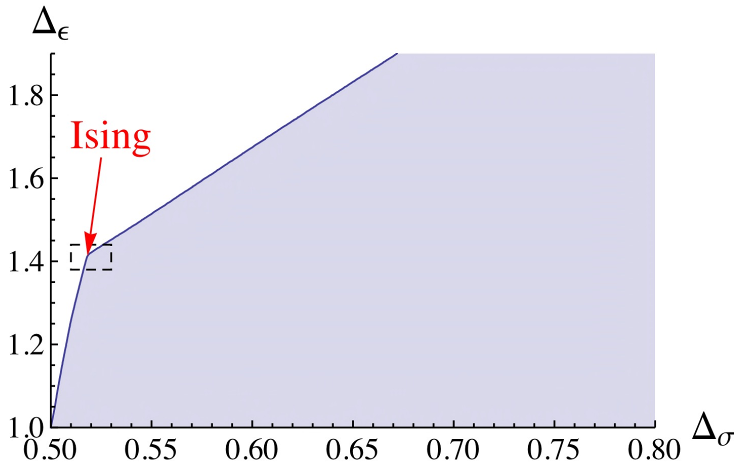
* reflection positive CFT
* $\mathbb{Z}_2$ symmetry $(\Phi \to -\Phi, \ \sigma \to -\sigma)$
* 1 $\mathbb{Z}_2$ odd scalar operator σ, $\Delta_\sigma \sim 0.52$
* 1 $\mathbb{Z}_2$ even " " ε, $\Delta_\varepsilon \sim 1.41$
* all other scalars $\Delta > 3$
* all other tensor operators $\Delta > \begin{pmatrix}\text{unitarity}\\ \text{bound}\end{pmatrix}$

  (example: stress tensor $T_{\mu\nu}$ has $\Delta_T = 3$)

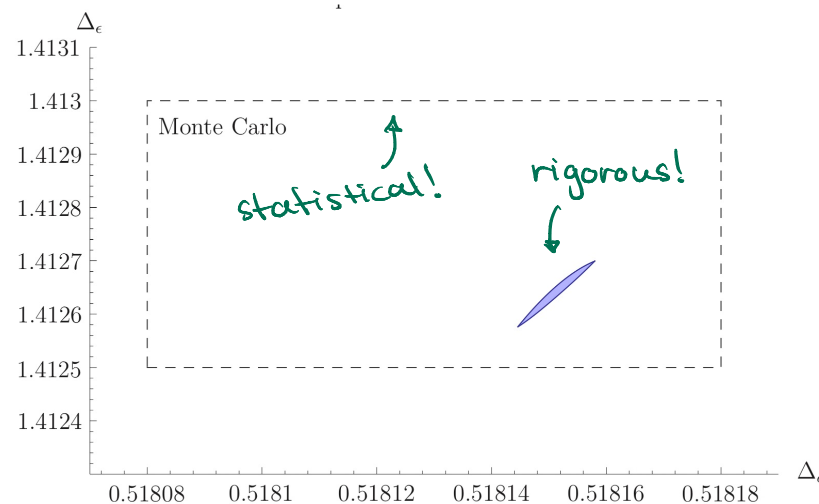
physics: $\alpha = 2 - \dfrac{3}{3-\Delta_\varepsilon}$, $\gamma = \dfrac{3-2\Delta_\sigma}{3-\Delta_\varepsilon}$, ...

The numerical bootstrap - 3d Ising

1203.6064: analyze
$\langle \sigma\sigma\sigma\sigma \rangle$



1502.02033: analyze
$\langle \sigma\sigma\sigma\sigma \rangle$, $\langle \sigma\sigma\varepsilon\varepsilon \rangle$, $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$
and assume no more relevant scalars



1603.04436:
$\Delta_\sigma = 0.518149(10)$
$\Delta_\varepsilon = 1.412625(10)$

(not shown: peninsula)

The numerical bootstrap - numerics

- computations of conformal blocks
  (nowadays: various software packages)
- finding $\alpha[\ldots]$ (using polynomial approximations)
  is a semidefinite programming problem:

  $$\text{find } Y \succcurlyeq 0 \quad \text{s.t.} \quad \mathrm{tr}[\vec{A}\cdot Y] = \vec{c}.$$

  (nowadays: dedicated solver "SDPB", under active development, arbitrary precision)
- 2104.09518: promote "NO"/"MAYBE" to smooth continuous function $\Rightarrow$ "navigate" to extremal pts. (e.g. max Δ_σ s.t. $\alpha[\ldots]$ exists)

1 The O(2) model

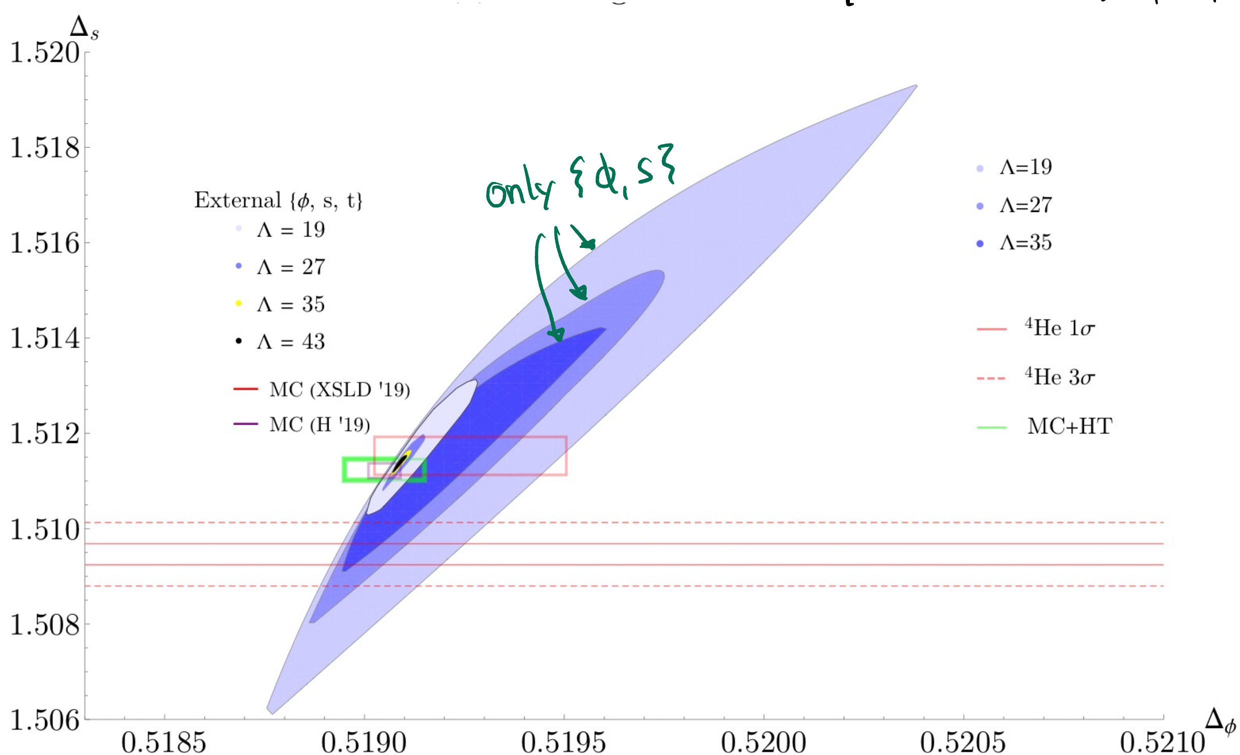
Superfluid ^{4}He transition

assume: no relevant scalars besides ϕ_i, s, t_{ij} for $q = 1, 0, 2$, respectively.

analyze: all ϕ_i, s, t_{ij} correlators

1912.03324

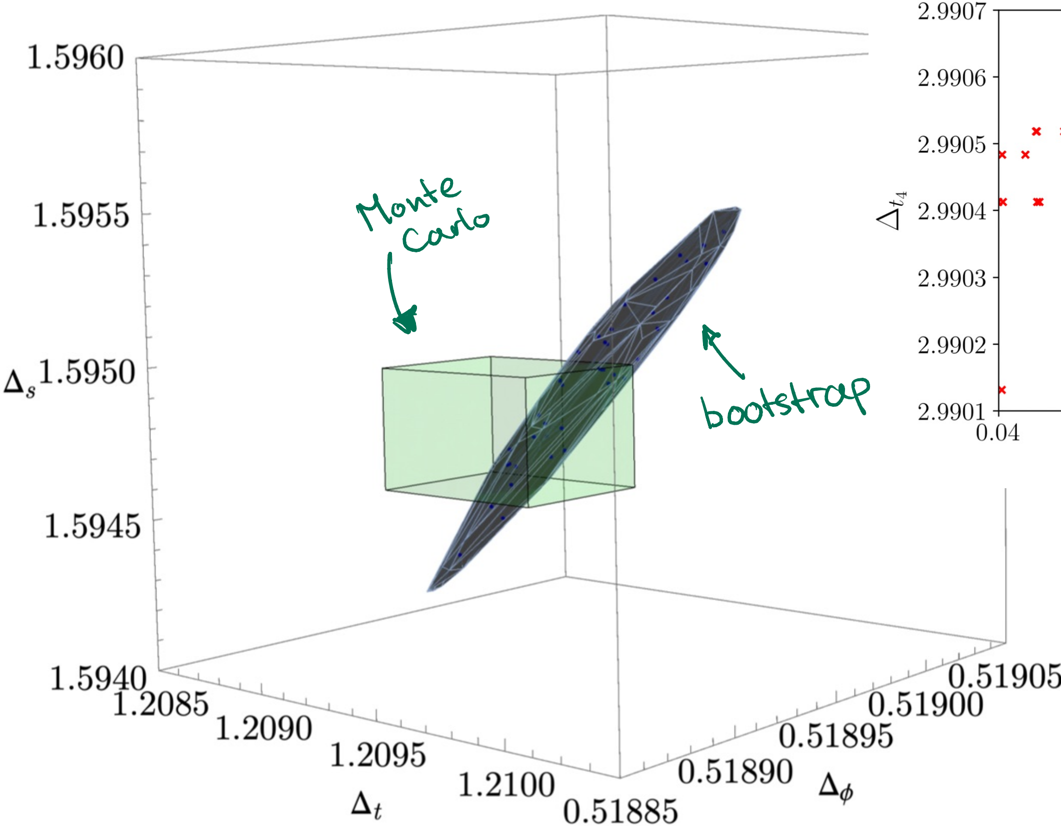
Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi

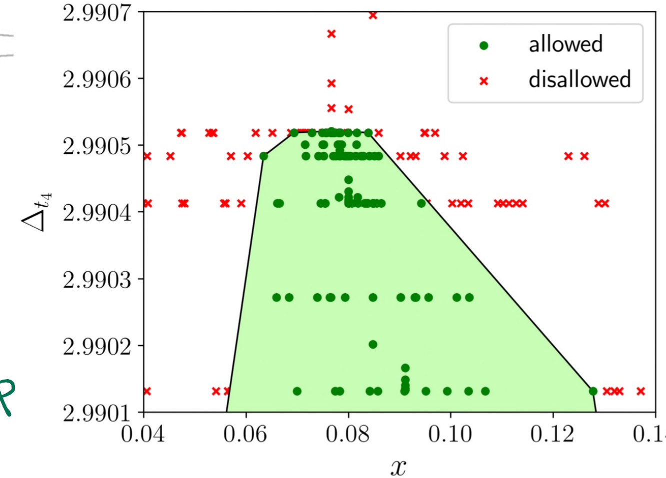


2 The O(3) model

2011.14647

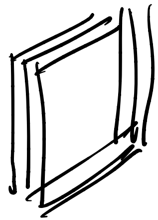
assume + analyze:
as in O(2) model





another relevant scalar!

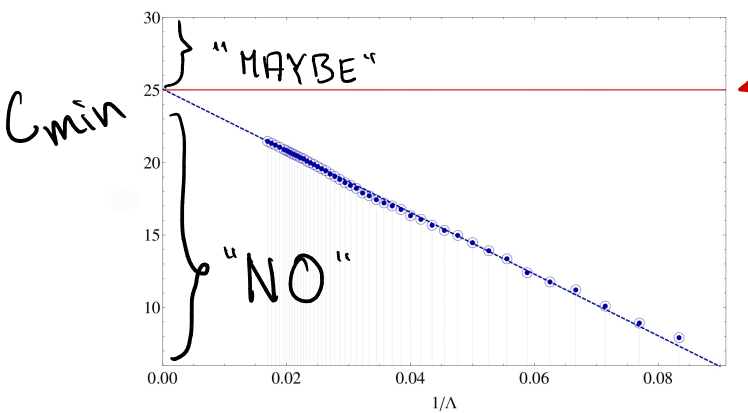
3 The (2,0) theories   1507.05637



- 6d, maximally supersymmetric CFTs
- low-energy physics of N M5 branes
- no Lagrangian!

assume : $\exists$ stress tensor T, no higher spin currents

analyze : $\langle TTTT \rangle$ with "up to Λ derivatives"



$C = 25$ for 2 M5 branes!

[spinoff : VOA structure in $d=6$ and $d=4$ SCFT
$\Rightarrow$ analytic results.
1312.5344]

The numerical bootstrap

Not discussed:

* GNY models (fermions!)
* 3d QED analyses
* $\langle T_{\mu\nu} \cdots T_{\rho\sigma} \rangle$ in 3d
* other SUSY theories
* ...

Not done yet:

* Ising with σ, ε, $T_{\mu\nu}$
* Virasoro conformal blocks
* ...

Important targets:
conformal windows

* $O(N) \times O(M)$
* 3d (scalar) QED
* 3d YM theories
* 4d YM theories

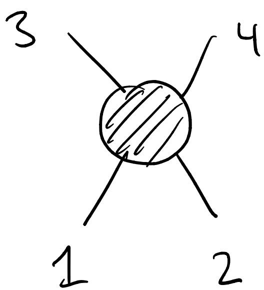
⋮

Outline:

1. Conformal field theories
2. The numerical bootstrap
3. An S-matrix bootstrap?

Spinoff: the S-matrix bootstrap

Scattering amplitude for 2-2 scalars:



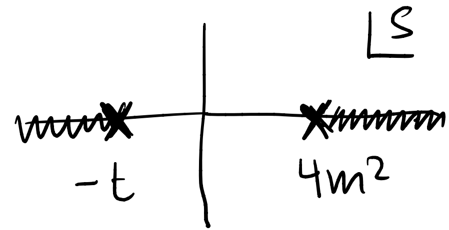
$T(s,t)$

$s = -(p_1+p_2)^2$

$t = -(p_1+p_3)^2$

$p_i^2 = -m^2$

① analyticity (how much?)



② unitarity

$$2\,\mathrm{Im}[\hat{T}] = \hat{T}^\dagger \hat{T}$$

Q: what are the possibilities?

Spinoff: the S-matrix bootstrap

Q: what are the possibilities?

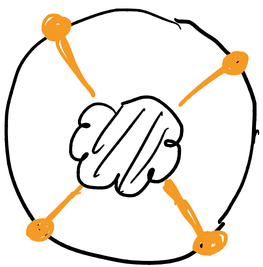
1607.06109 ++ : numerically explore spaces of amplitudes, extremize e.g. $T(\frac{4}{3}, \frac{4}{3})$ or scattering lengths.
(Paulos, Penedones, Toledo, BvR, Vieira)

2011.02957 ++ : bounds on EFT Wilson coeffs using positivity.
(Caron-Huot, Van Duong)

2102.08951 : bounds on gravitational couplings, using positivity
(Caron-Huot, Mazac, Rastelli, Simmons-Duffin)

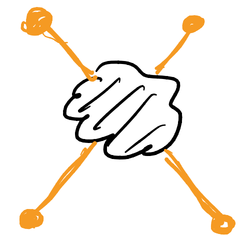
Connecting amplitudes and correlators

using AdS/CFT or QFT in AdS



d-dim. boundary correlator

flat-space limit $R \to \infty$



scattering amplitude

$$ds^2 = dr^2 + R^2 \sinh^2(r/R)\, d\Omega_d^2$$

$$m^2 R^2 = \Delta(\Delta - d)$$

conformal!

Connecting amplitudes and correlators

numerics in 1+1d

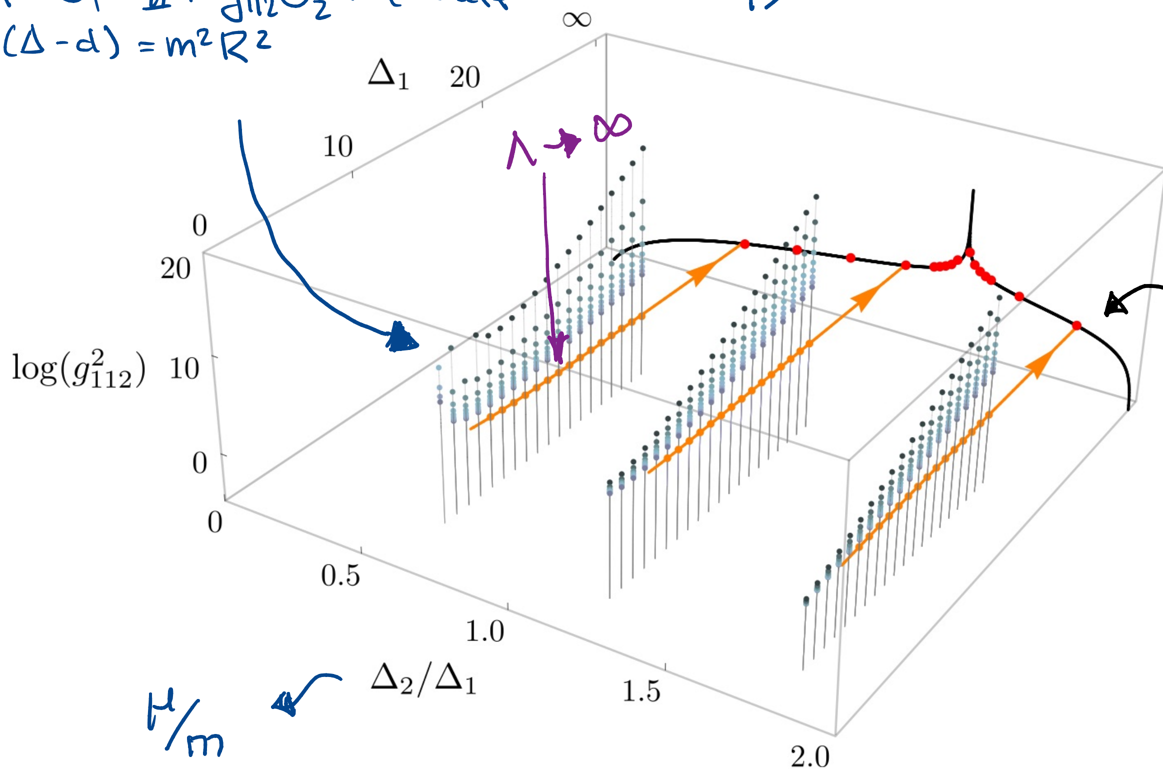
conformal $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle$ with

$\mathcal{O}_1 \times \mathcal{O}_1 = \mathbb{1} + g_{112}\mathcal{O}_2 + (\text{stuff with } \Delta > 2\Delta_1)$

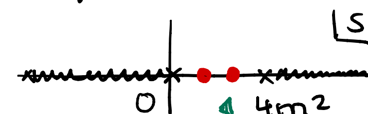
$\Delta(\Delta - d) = m^2 R^2$

1607.06109

Paulos, Penedones, Toledo, BvR, Vieira



analytic S-matrices

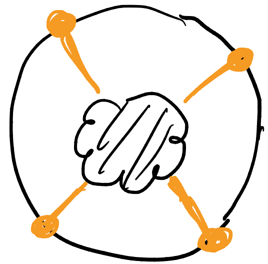


$$-\frac{g_{112}^2}{s-\mu^2}, \quad 0<\mu<2m$$

target: $\max[g_{112}^2]$

Connecting amplitudes and correlators

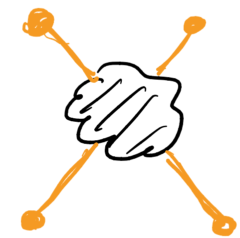
analytic results



conformal correlator

flat-space limit

$R \to \infty$



scattering amplitude

analyticity? ✓ ?

unitarity? $\lambda^2 > 0\ ,\ \Delta > ..$ $2\,\mathrm{Im}[\hat{T}] = \hat{T}^\dagger \hat{T}$

"QFT in AdS instead of LSZ" 2210.15683 BvR, Zhao

S-matrix "bootstrap" ?

promising directions:

- new <u>universal</u> constraints
- new <u>general</u> analyticity results
- consistency of low-energy data / lattice data

Apologies

* modular bootstrap
* bootstrap for boundaries and defects
* SUSY results
* applications to quantum gravity
* ..
* ..

Spinoff : positivity

general idea : unitarity + linear constraints

$\Rightarrow$ semidef. programming problem

for example: positive measure $d\mu_x$, $x \in \mathbb{R}$ $\Rightarrow \int d\mu_x \begin{pmatrix} 1 & x & \cdots \\ x & x^2 & \\ \vdots & & \ddots \end{pmatrix} \succcurlyeq 0$

and eqns of motion may relate coeffs.

- lattice Ising
- quantum mechanics
- matrix models
- ...

[todo: problems with scaling, e.g. turbulence]

Conclusions

... and beyond!

space of
solutions?

rethinking QFT:

axioms + positivity $\Rightarrow$ constraints

how to define
CFT / QFT?

analytical?
numerical?

↳ algorithms!

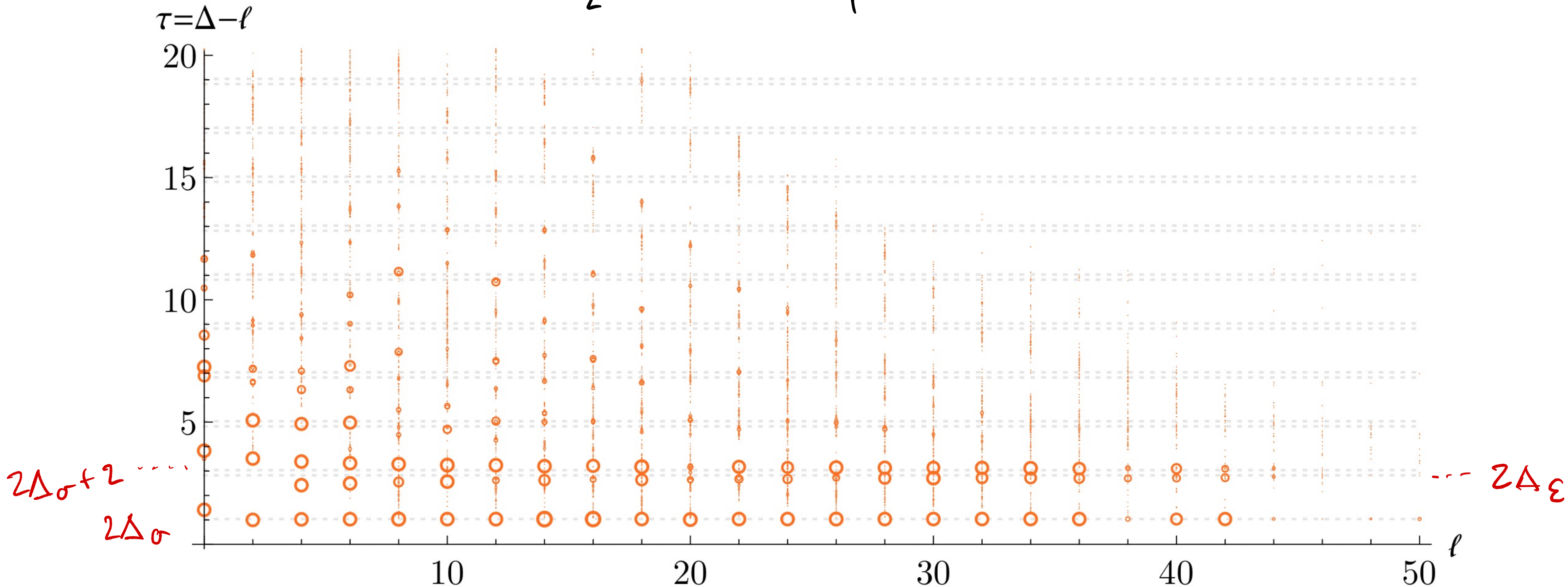
Thank you!

Spinoffs

1. The analytic bootstrap
2. The S-matrix bootstrap
3. Amplitudes from correlators
4. Positivity

Spinoff: the analytic bootstrap

1612.08471: "extremal spectrum" of 3d Ising $\mathbb{Z}_2$ even operators

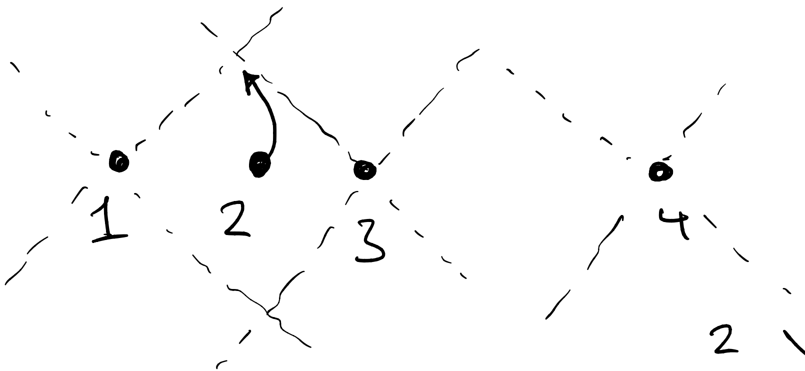


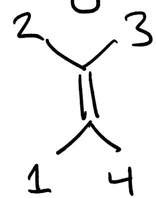
pictorially: $\sigma \overleftrightarrow{\partial}_{\mu_1} \cdots \overleftrightarrow{\partial}_{\mu_\ell} \overleftrightarrow{\Box}^n \sigma$ has spin ℓ and dimension approx. $2\Delta_\sigma + \ell + 2n$

Spinoff: the analytic bootstrap

Claim: in any CFT, if $\exists\ \mathcal{O}$ with twist τ then $\exists$ ops. with twist approx. 2τ (and $\ell \to \infty$).

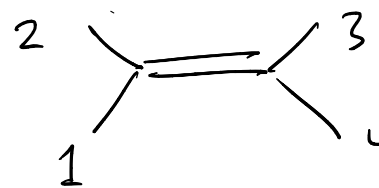
intuition: necessary in lightcone limit:





: $(x_2-x_3)^2 \to 0$

identity op. dominates



: large spin dominates

Alday-Maldacena '08, Komargodski-Zhiboedov '14
Fitzpatrick-Kaplan-Poland-Simmons-Duffin '14

Spinoff : the analytic bootstrap

"Analyticity in spin in conformal theories"

Caron-Huot 1703.00278

claim: CFT spectrum (to some extent) organizes itself in Regge trajectories:

$\Delta(J)$ analytic.

applications: AdS/CFT, lightray operators, ANEC, (more) large spin pert. thy, ...

highlight: exact functionals