GPDs: Combining Experimental and Simulation Data

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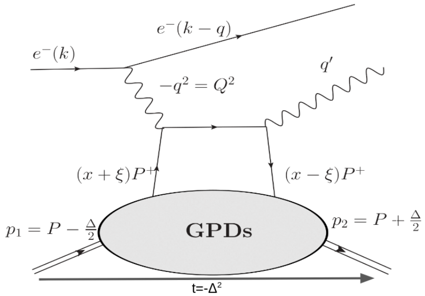

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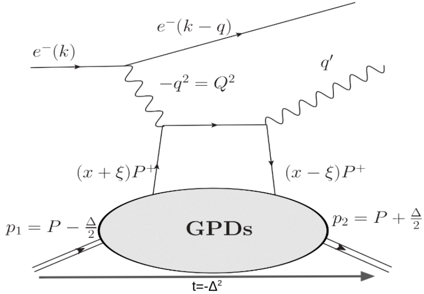

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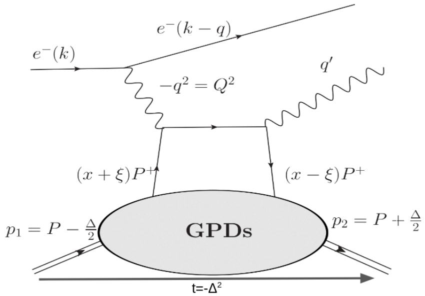

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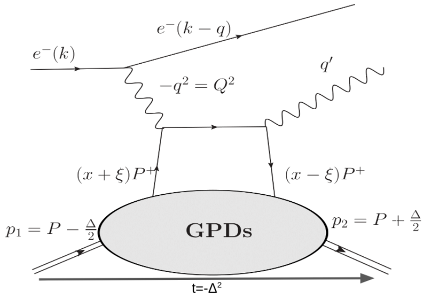

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- Admit a 3D probabilistic interpretation

Forward Limit

- x=$\frac{k^+}{P^+}$ is the average momentum fraction of the struck quark
- $\xi = -\frac{\Delta^+}{2P^+}$ is the skewness, or lightcone "kick"
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- In the so-called 'forward limit' GPDs reproduce the well-known PDFs
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- This is because the GPDs are a generalization of PDFs from matrix elements diagonal in momentum space to analogous matrix elements which are off-diagonal in momentum space

GPD Modeling

- GPDs contribute to DVCS cross sections via Compton Form Factors via a convolution in x. At leading order in the strong coupling

$$\mathcal{H} = \int_{-1}^{1} dx \mathrm{H}(x, \xi, t)\left(\frac{1}{\xi - x - i\epsilon} - \frac{1}{\xi + x - i\epsilon}\right) \tag{1}$$

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- Therefore, there is an inherent deconvolution problem in extracting GPDs from DVCS data (GPDs aren't observables, DVCS is exclusive $\rightarrow$ Low statistics)
- Enter: GPD Modeling using artificial neural networks to
  - Fulfill some theoretical constraints at the level of network architecture
  - Assess systematic uncertainties inherent to this univertible problem

Generation

- The set of GPD replicas as a function of x and ξ (neglecting t dependence) was produced by fitting a set of GK model GPD pseudodata using a set of ANNs [H. Dutrieux et al., 2022]

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- Modeling took place in double distribution space in order to ensure both polynomiality
  $\int dx x^n H^q(x,\xi) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (2\xi)^{2i} A^q_{n+1,2i} + \mathrm{mod}(2,n)(2\xi)^{n+1} C^q_{n+1}$

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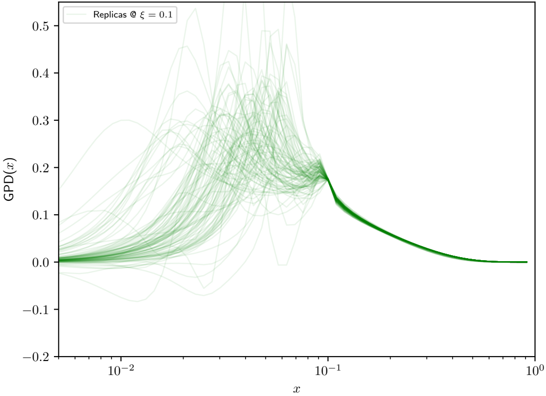
- Positivity was enforced numerically

Example Replica Set

$$\xi = 0.1:$$



- Some replicas deviate greatly from the central value when $x < \xi$ (No ERBL positivity constraint exists)
- How might the replica band be further constrained?

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- assigning each a weight ω_k using a Bayesian reweighting procedure based on the introduction of mock lattice data
- assessing the reduction of uncertainty in both x and ν spaces by using the weights ω_k to calculate "Reweighted" central values and error bars
- Weights are robust against transformations of replicas

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- that is the region in which lattice data may be provided given the current state of the arts
- the lattice signal vanishes around $\nu = 10$ [Egerer et al., 2021]

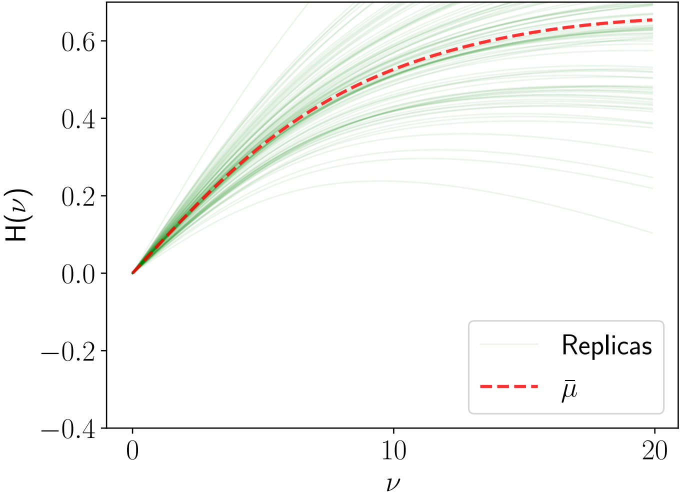
Why Reweight in Blocks at Low ν?

Goal

Reweighting at low values of ν may then be used to constrain them in the high ν region

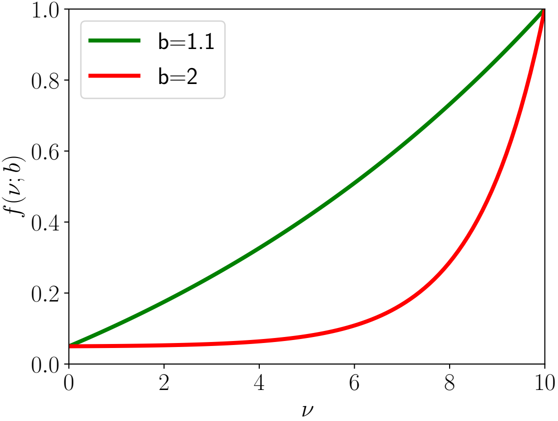
Procedure

- 1: Calculate the central value $\bar{\mu}_i$ of the set of replicas at each value ν_i

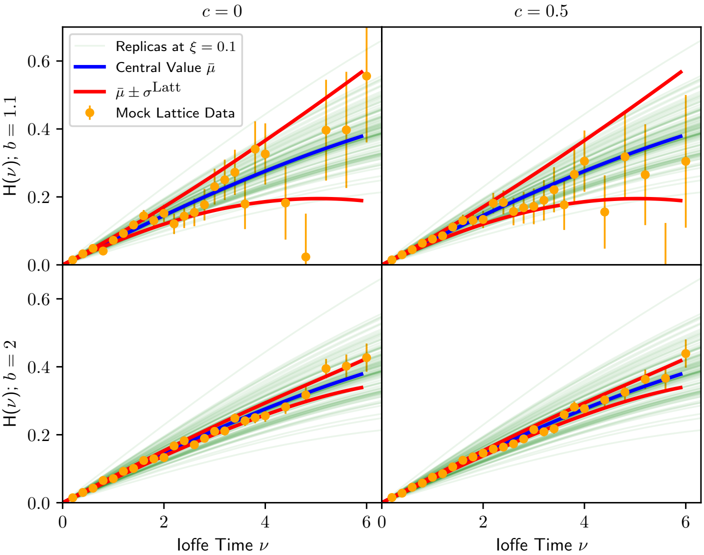


Procedure

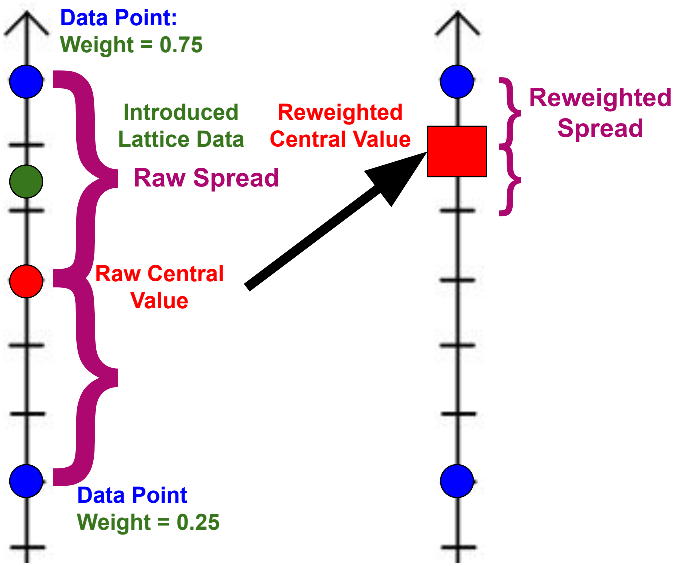
- 2: Assign a corresponding standard deviation to each mock lattice point defined as $\sigma_i \equiv \bar{\mu}_i f(\nu_i, b)$ where b determines the base of an exponential function f constrained by $f(0, b) = 0.05$, $f(10, b) = 1$



Mock Lattice Data Fabrication: An Example



Bayesian Reweighting



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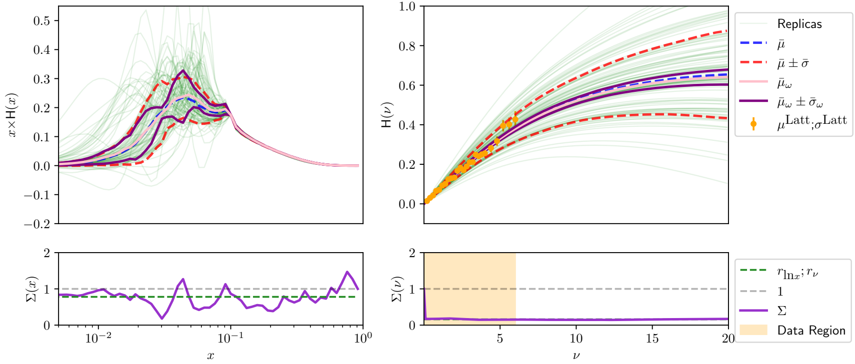
- Global Uncertainty Retainment:
  x: $r_{\ln x} = \int_{\log(d)} \frac{dx}{\log(D)} \frac{\Sigma(x)}{x}$; ν: $r_\nu = \int_d \frac{d\nu}{D} \Sigma(\nu)$



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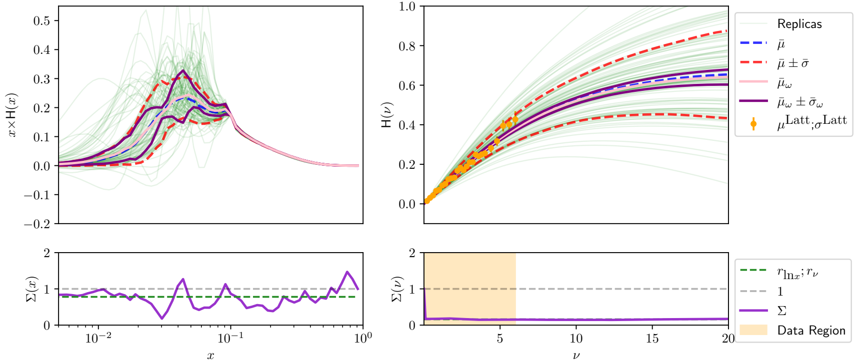

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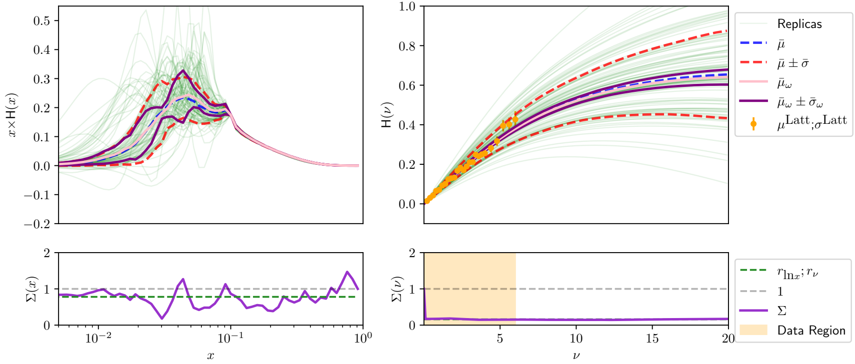

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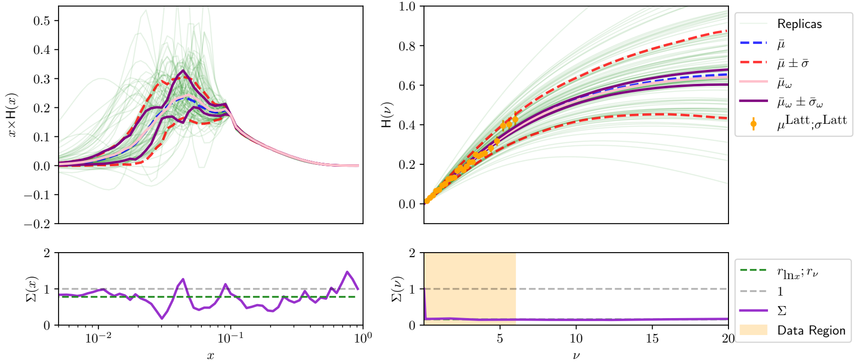

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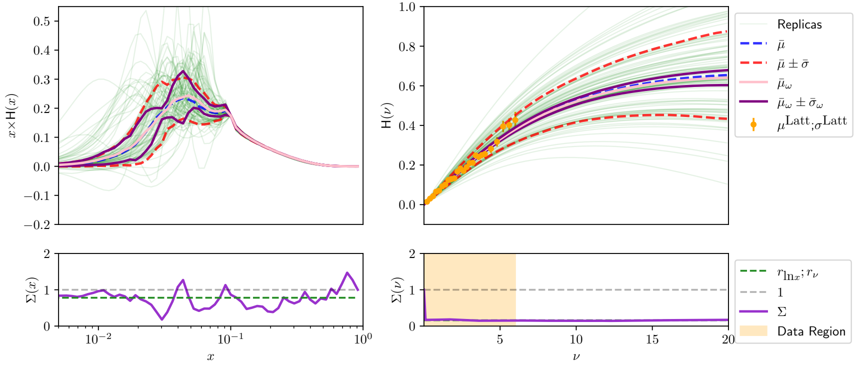

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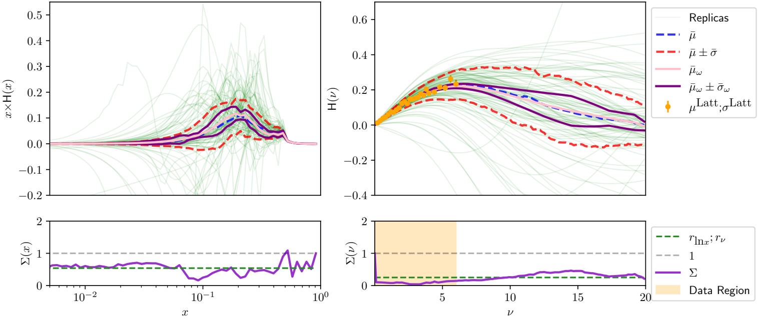
- $\Sigma(\nu)$ is flat and r_ν is low as replicas are coherent
- $\Sigma(x)$ Peaks above 1 because mock lattice data is used to prioritize replicas based on their low ν behaviour, and highly weighted replicas may decohere at high ν (τ is relatively small)



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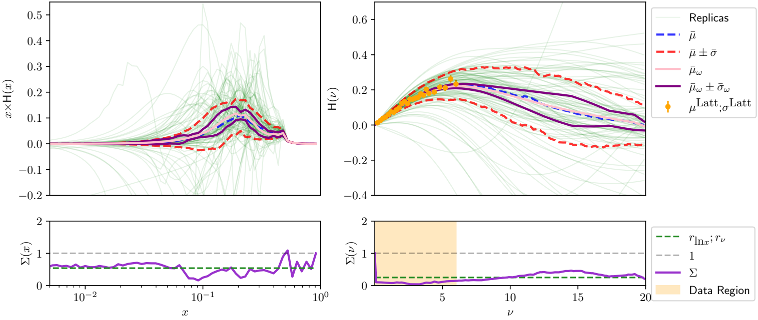

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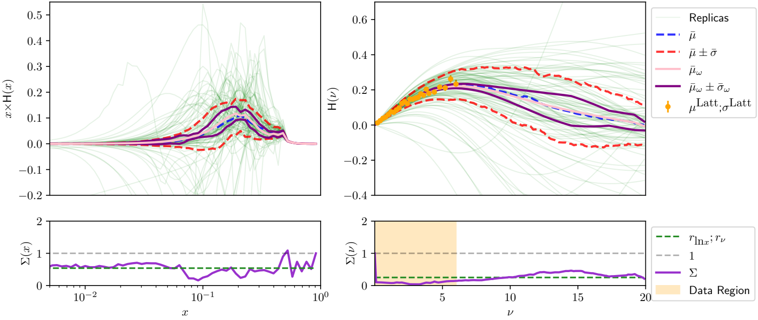

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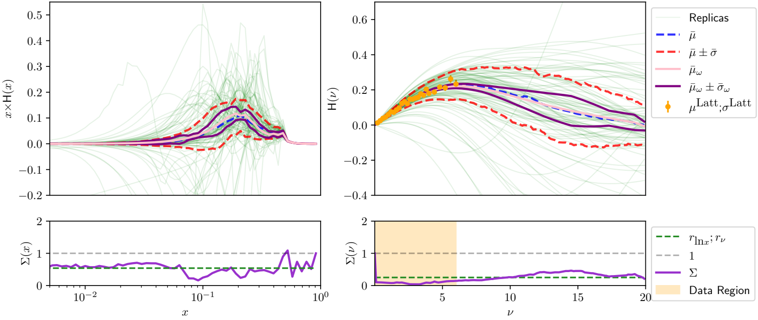

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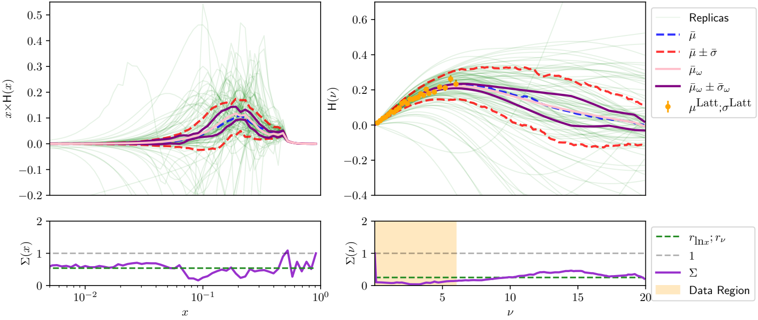

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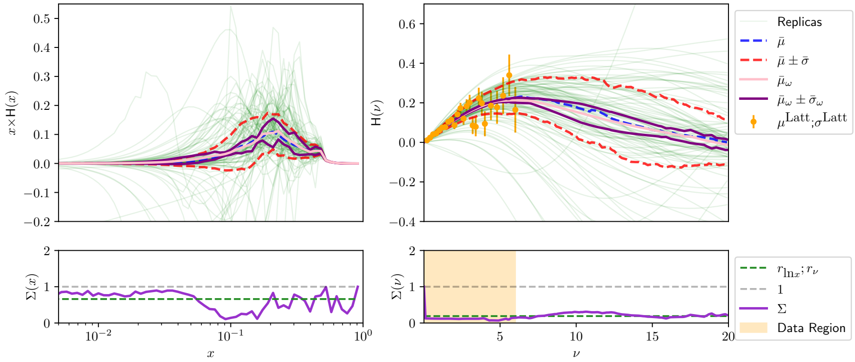
- Local x space uncertainty retainment is decreased around $x = 0.1$ due to presence of mock lattice data, but is less drastic at high x due to the positivity constraint



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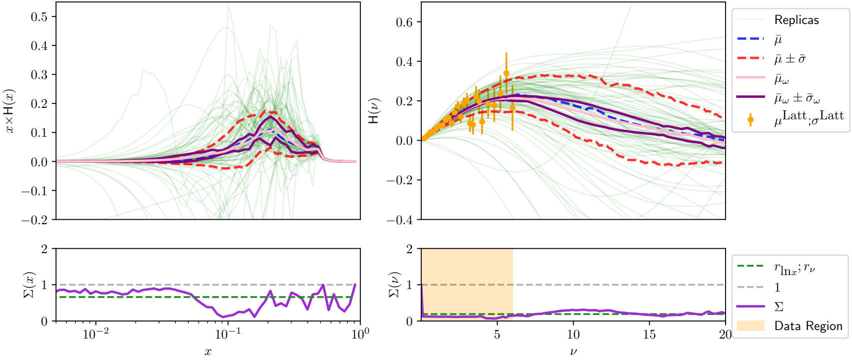

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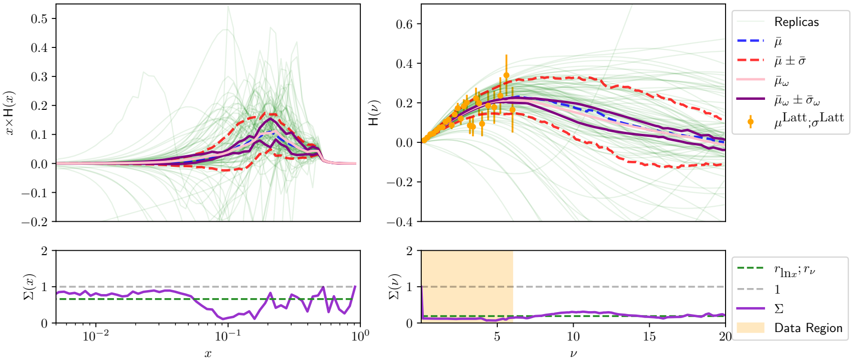

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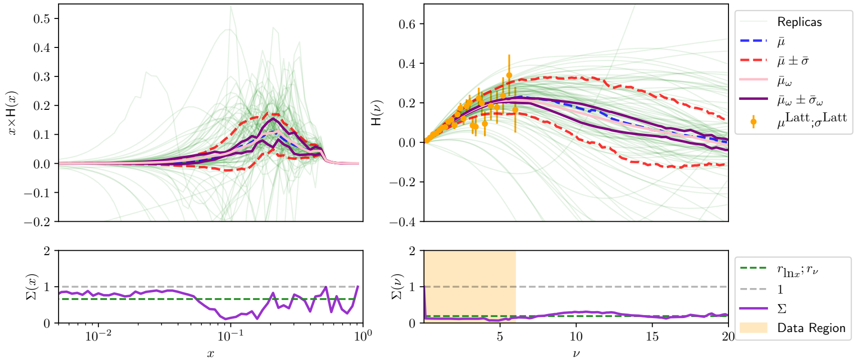

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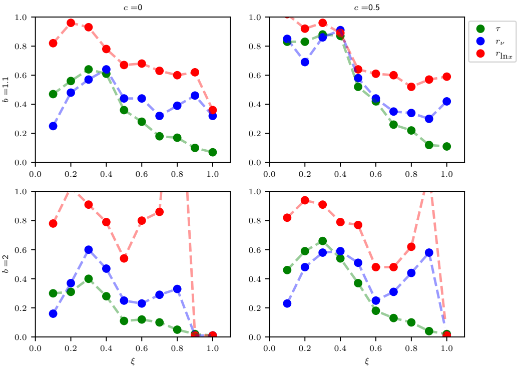
- Approximate replication of uncertainty retainment of one highly constraining data set by the current 5 sets of moderate constraining ability data in $\xi \leq 0.5$

Monokinematic reweighting at 10 value of ξ



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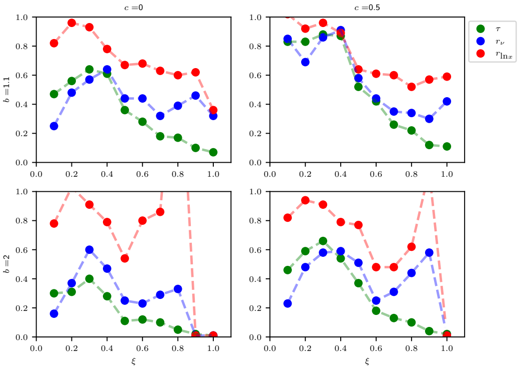

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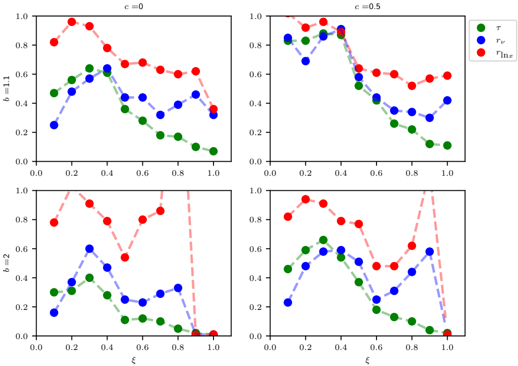

- Low precision, low correlation (top left) and high precision, high correlation (bottom right) reweightings yield similar uncertainty reductions with similar values of τ ($\rightarrow$ similar ANN replica generation costs)
- Lattice facility of each of these two compromisory options is to be further investigated

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- **We now have a consistent way to combine experimental and lattice data**
  - Lattice data help to reduce the deconvolution uncertainties in momentum space by 25-50% at 0th order of the strong coupling.

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- Lattice data would be more than welcome!

Thank You

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- The blocks correspond to the three regions in ν:
  - $0.2 \le \nu \le 2$, $\Delta\nu = 0.2$
  - $2.2 \le \nu \le 4$, $\Delta\nu = 0.2$
  - $4.4 \le \nu \le 6$, $\Delta\nu = 0.4$

Replica Weights ω_i & Effective Fraction of Replicas τ

- A corresponding set of weights ω_k are then calculated from the χ_k^2 and the number of (mock) lattice data values introduced N as $\omega_k \equiv \frac{(\chi_k^2)^{\frac{N-1}{2}}}{Z} e^{-\frac{\chi_k^2}{2}}$ where Z is a normalization factor

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- We also define $\tau \equiv \frac{\exp(\sum_k \omega_k \ln(\omega_k))}{\mathrm{N_{rep}}}$ as the effective fraction of replicas retained after the reweighting is completed, where $\mathrm{N_{rep}}$ is the range of the index k

The Trouble with Outliers

We began by calculating the reweighted central value $\mu_R(\nu; x)$ and uncertainties $\sigma_R(\nu; x)$ as a function of ν or x as

- $\mu_R(\nu; x) = \sum_k \omega_k R_k(\nu; x)$
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However, this method of estimation of the uncertainty associated with the reweighted central value was extremely sensitive to replicas far from the central value.

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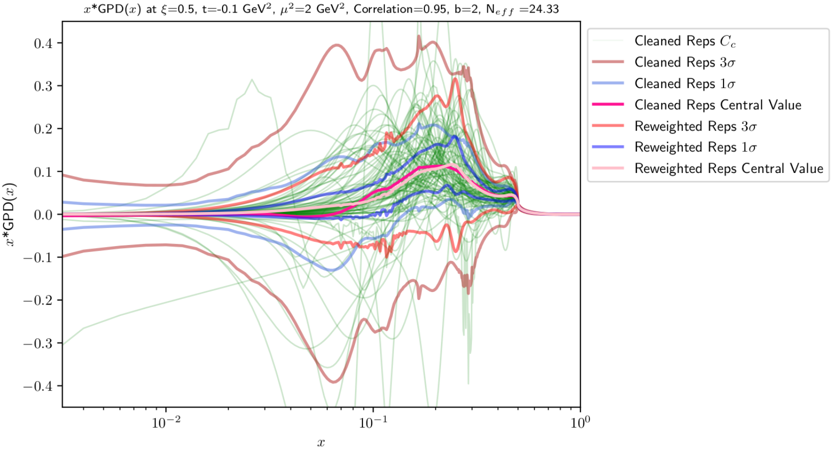

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- However, the definition of "outlier" is not very obvious and is ultimately arbitrary
- We decided to locally employ the MAD (Median Absolute Deviation) to compute uncertainty bands

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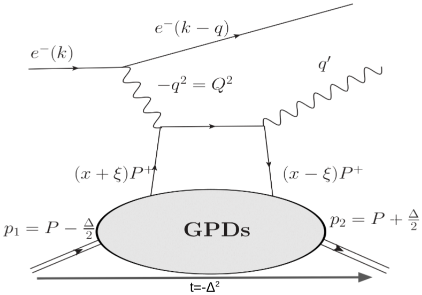
- At a value of ξ called ξ_{shown} use the weights to plot uncertainty bands. ξ_{shown} may or may not be present in the set ξ_{used}

Forward Limit



- In the so-called 'forward limit' GPDs reproduce the well-known PDFs
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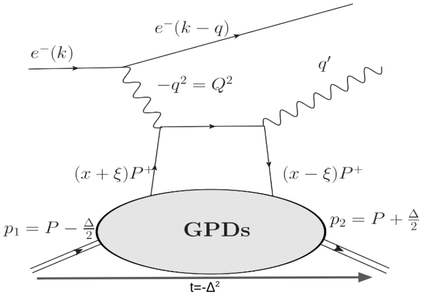


- In the so-called 'forward limit' GPDs reproduce the well-known PDFs
  - $\lim_{t\to 0}\lim_{\xi\to 0} \mathrm{GPD}(x,\xi,t) = \mathrm{PDF}(x_{\mathrm{BJ}})$
- This is because the GPDs are a generalization of PDFs from matrix elements diagonal in momentum space to analogous matrix elements which are off-diagonal in momentum space

Nucleon Tomography

When $\xi \to 0$:

- $|\vec{b}_\perp|$ and $\sqrt{-t}$ are Fourier Conjugates
- One recovers a Probabilistic Interpretation

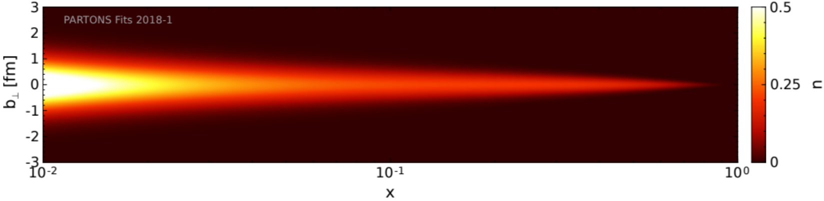
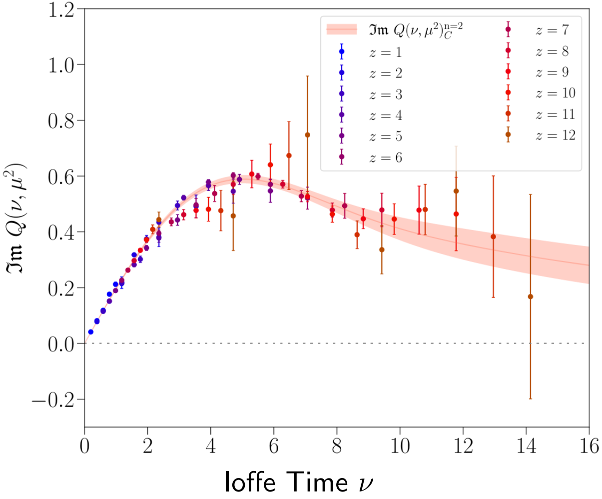


Figure: [Moutarde, Sznajder, and Wagner, 2018] Transverse position $|\vec{b}_\perp|$ of quarks in an unpolarized proton as a function of the longitudinal momentum fraction x. Based on joint fit of CFFs to Hall A, CLAS, HERMES and COMPASS data.

Lattice Errors

Figure 9.a. of [Egerer et al., 2021]:



MAD (Median Absolute Deviation) Estimator

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- We first calculate the central value $\mu_R(\nu; x)$ as the median of the set of replicas weighted by the weights ω_k
- We then estimate the uncertainty $\sigma_R(\nu; x)$ as proportional to the median of a correspondingly weighted distribution given by $|\mu_R(\nu; x) - R_k(\nu; x)|$

Ioffe Time

- $\rightarrow$ The longitudinal momentum fraction x, proportional to a quark's plus momentum is assigned a lightcone distance proportionality fraction $\nu \propto z^-$ as a Fourier conjugate

$$\mathrm{GPD}(\nu, \xi) \equiv -\mathrm{i} \int_{-1}^{1} \mathrm{d}x \mathrm{GPD}(x, \xi) \sin(x\nu) \tag{2}$$

Results

$\xi_{j'}$	ξ_j	c	b	$r_{\ln x}$	τ	r_ν
0.1	0.1	0	1.1	0.82	0.47	0.25
0.1	0.1	0.5	1.1	1.02	0.83	0.85
0.1	0.1	0	2	0.78	0.3	0.16
0.1	0.1	0.5	2	0.82	0.46	0.23
0.5	0.5	0	1.1	0.67	0.36	0.44
0.5	0.5	0.5	1.1	0.64	0.52	0.58
0.5	0.5	0	2	0.54	0.11	0.25
0.5	0.5	0.5	2	0.77	0.37	0.51
0.5	0.1	0	1.1	1.24	0.47	0.92
0.5	0.1	0.5	1.1	1.15	0.83	0.93
0.5	0.1	0	2	1.08	0.3	0.9
0.5	0.1	0.5	2	1.23	0.46	0.91
0.5	0.1 0.2 0.3	0	1.1	0.95	0.3	0.62
0.5	0.1 0.2 0.3	0.5	1.1	1.0	0.77	0.82
0.5	0.1 0.2 0.3	0	2	0.54	0.1	0.34
0.5	0.1 0.2 0.3	0.5	2	0.73	0.3	0.61
0.5	0.1 0.2 0.3 0.4 0.5	0	1.1	0.66	0.16	0.19
0.5	0.1 0.2 0.3 0.4 0.5	0.5	1.1	0.75	0.57	0.65
0.5	0.1 0.2 0.3 0.4 0.5	0	2	0.45	0.03	0.13
0.5	0.1 0.2 0.3 0.4 0.5	0.5	2	0.77	0.18	0.25

Table: Results as a function of the reweighting parameters

$$f(\nu; b) = \frac{0.05(b^{\nu} - b^{10}) + 1 - b^{\nu}}{1 - b^{10}} \tag{3}$$

Double Distribution Definition

$$H(x, \xi, t) = \int \Omega F(\beta, \alpha, t), \quad d\Omega = d\beta d\alpha \delta(x - \beta - \alpha\xi), \quad |\alpha| + |\beta| \leq 1$$