4D-TExS: A new 4D lattice-QCD equation of state with extended density coverage

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Abstract. Although calculations of QCD thermodynamics from first-principle lattice simulations are limited to zero net-density due to the fermion sign problem, several methods have been developed to extend the equation of state (EoS) to finite values of the B, Q, S chemical potentials. Taylor expansion around $\mu_i = 0$ ($i = B, Q, S$) enables to cover with confidence the region up to $\mu_i/T < 2.5$. Recently, a new method has been developed to compute a 2D EoS in the (T, μ_B) plane. It was constructed through a T-expansion scheme (TExS), based on a resummation of the Taylor expansion, and is trusted up to densities around $\mu_B/T = 3.5$. We present here the new 4D-TExS EoS, a generalization of the TExS to all 3 chemical potentials, expected to offer a larger coverage than the 4D Taylor expansion EoS. After explaining the basics of the T-Expansion Scheme and how it is generalized to multiple dimensions, we will present results for thermodynamic observables as functions of temperature and both finite baryon and strangeness chemical potentials.

1 Introduction

The exploration of the phase diagram of nuclear matter at finite density, to understand how matter behaves in extreme conditions, is a very active field of research nowadays [1]. In order to interpret experimental results and bridge the gap with theory, heavy-ion collision and neutron star merger simulations require a nuclear EoS as input. With the recent development of new frameworks taking into account local charge conservation and diffusion, mainly to study the existence of a critical point in the nuclear phase diagram, or the baryon stopping for instance, the need for a four-dimensional EoS with independent chemical potentials is growing. Since lattice QCD simulations are hindered from going to finite density because of the sign problem [2], one has to use expansion methods to reach this purpose from first-principle QCD calculations. The most successful attempt so far was achieved through a Taylor expansion in several conserved-charge chemical potentials [3, 4], which is, however, limited to $\hat{\mu}_i \ (\equiv \mu_i/T) \leqslant 2 - 2.5$, with $i = B, Q, S$ [5]. We present here the generalization to several dimensions of a new expansion method developed in Refs. [5, 6].

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2 T-Expansion Scheme

The EoS construction we present is based on the T-expansion scheme (TExS), initially proposed in [5] as a 2-dimensional construction, and extended in [6] through the addition of a correction to take into account the Stefan-Boltzmann limit at high temperature. Noticing that $\lim_{\hat{\mu}_X \to 0} \frac{\chi_1^X(T,\hat{\mu}_X)}{\hat{\mu}_X} = \chi_2^X(T,0)$, one can start from the ansatz [6]:

$$\frac{\chi_1^X(T,\hat{\mu}_X)}{\overline{\chi}_1^X(\hat{\mu}_X)} = \frac{\chi_2^X(T',0)}{\overline{\chi}_2^X(0)} \ , \tag{1}$$

with χ_j^X the susceptibilities of conserved charge X, computed on the lattice at $\hat{\mu} = 0$. The rescaled temperature T' is expanded as $T' = T\left(1 + \lambda_2^X(T)\,\hat{\mu}_X^2 + (O^4)\right)$, thus giving to this expansion scheme a dependence on both T and $\hat{\mu}_X = \mu/T$, whereas the Taylor expansion provides only an expansion in $\hat{\mu}_X$ at constant T [4].

3 4D-TExS - Generalization to 4 dimensions

One can generalize the approach defined in Sec. 2 from a two-dimensional phase space $(T, \hat{\mu}_B)$, with one conserved charge X, to a four-dimensional phase space $(T, \hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S)$, with B the baryon number, Q the electric charge and S the strangeness number. By a simple change of variables to spherical coordinates, with the following definitions:

$$\begin{aligned} \hat{\mu}_B &= \hat{\mu}\,.\,\cos(\theta) & \hat{\mu} &= \sqrt{\hat{\mu}_B^2 + \hat{\mu}_Q^2 + \hat{\mu}_S^2}\ , \\ \hat{\mu}_Q &= \hat{\mu}\,.\,\sin(\theta)\cos(\varphi) \quad \Longleftrightarrow & \varphi &= \arccos\left(\hat{\mu}_Q \Big/ \sqrt{\hat{\mu}_Q^2 + \hat{\mu}_S^2}\right)\ , \\ \hat{\mu}_S &= \hat{\mu}\,.\,\sin(\theta)\sin(\varphi) & \theta &= \arccos\left(\hat{\mu}_B/\hat{\mu}\right)\ , \end{aligned} \tag{2}$$

one maps the system to a $(T, \hat{\mu}, \theta, \varphi)$ space, hence using the same two-dimensional TExS approach described in Sec. 2, but now allowing to explore three finite chemical potentials at the same time by varying the angles θ and φ. One needs thus to define the so-called generalized second-order susceptibility:

$$\begin{aligned} X_2^{\theta,\varphi}(T) = \left.\frac{\partial^2 P/T^4}{\partial \hat{\mu}^2}\right|_{\hat{\mu}=0} &= c_\theta^2\,.\,\chi_2^B(T) + s_\theta^2 c_\varphi^2\,.\,\chi_2^Q(T) + s_\theta^2 s_\varphi^2\,.\,\chi_2^S(T) \\ &\quad + 2c_\theta s_\theta c_\varphi\,.\,\chi_{11}^{BQ}(T) + 2c_\theta s_\theta s_\varphi\,.\,\chi_{11}^{BS}(T) + 2s_\theta^2 c_\varphi s_\varphi\,.\,\chi_{11}^{QS}(T)\ , \end{aligned} \tag{3}$$

as a linear combination of the second-order charge susceptibilities, where $c_{\theta/\varphi} \equiv \cos(\theta/\varphi)$ and $s_{\theta/\varphi} \equiv \sin(\theta/\varphi)$. In the same way, one obtains by further derivation the generalized fourth-order cumulant $X_4^{\theta,\varphi}(T)$.

This leads to rewrite the main identity from Eq. (1) using $X_1^{\theta,\varphi}$ and $X_2^{\theta,\varphi}$, where the expanded temperature is written $T'^{\theta,\varphi}(T,\hat{\mu}) = T\left(1 + \lambda_2^{\theta,\varphi}(T) \times \hat{\mu}_B^2 + ...\right)$. The second-order expansion coefficient is expressed as:

$$\lambda_2^{\theta,\varphi}(T) = \frac{1}{6T}\frac{1}{X_2'^{\theta,\varphi}(T)} \times \left(X_4^{\theta,\varphi}(T) - \frac{\overline{X}_4^{\theta,\varphi}}{\overline{X}_2^{\theta,\varphi}} X_2^{\theta,\varphi}(T)\right)\ . \tag{4}$$

4 Results and thermodynamics

In this section, we show as a proof of concept some preliminary results obtained for an expansion along a trajectory with $(\theta, \varphi) = (90°, 135°)$, where $\mu_B = -\mu_S$. All the lattice QCD data used in our construction have been computed by the Wuppertal-Budapest collaboration, and have been matched with hadron resonance gas (HRG) calculation for $T < 130$ MeV [7].

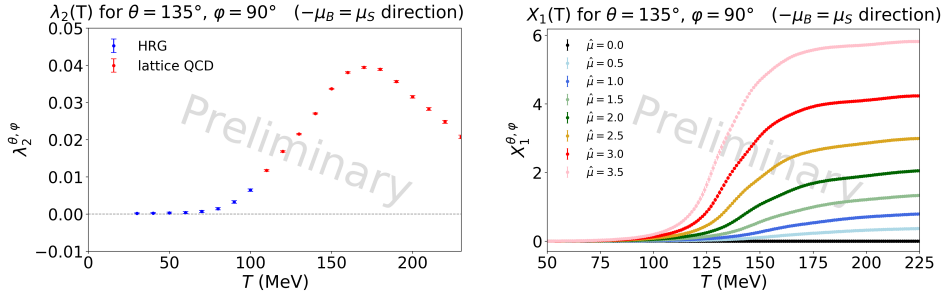

Figure 1. *Left panel:* T' expansion coefficient $\lambda_2^{\theta,\varphi}$ as a function of T, for $\theta = 135°$ and $\varphi = 90°$. We display in blue the range calculated with HRG data, and in red the range from lattice QCD calculations. *Right panel:* generalized charge density X_1 as a function of T, for different values of $\hat{\mu}$ from 0 to 3.5.

We start the construction of the EoS by calculating the generalized second-order and fourth-order susceptibilities for a given direction (θ, φ) in the phase diagram. From there, one obtains the expansion parameter $\lambda_2^{\theta,\varphi}$ as a function of T, shown on the left panel of Fig.1 for the chosen direction. Note that, for consistency, we stop the expansion of T' at second order, since the $\lambda_4^{\theta,\varphi}$ coefficient relies on sixth-order susceptibilities which have not all been computed yet for mixed-charges B, Q, S. The generalized charged density $X_1^{\theta,\varphi}$, thus obtained using Eq. (1), is shown on the right panel of Fig.1 for different values of $\hat{\mu}$. From there, one can compute pressure by integrating $X_1^{\theta,\varphi}$ using $P^{\theta,\varphi}(T,\hat{\mu}) = P(T,0) + \int_0^{\hat{\mu}} X_1^{\theta,\varphi}(T,\hat{\mu}')d\hat{\mu}'$, with the zero-density pressure $P(T,0)$ obtained from lattice simulations. One can obtain entropy density directly from pressure as well, using $s^{\theta,\varphi}(T,\hat{\mu}) = s(T,0) + \left[\partial P^{\theta,\varphi}(T,\hat{\mu})/\partial T\right]_{\hat{\mu}}$. Both quantities are displayed in Fig.2 as functions of T, for different values of $\hat{\mu}$. Although pressure looks smooth up to $\hat{\mu} = 3.5$, one can observe that the entropy density starts to display oscillations as a function of T from $\hat{\mu} = 2.5$. Even though it remains an increasing function of temperature, as physically expected, these oscillations are most likely due to a problem in the way lattice susceptibilities used to construct the EoS are fitted.

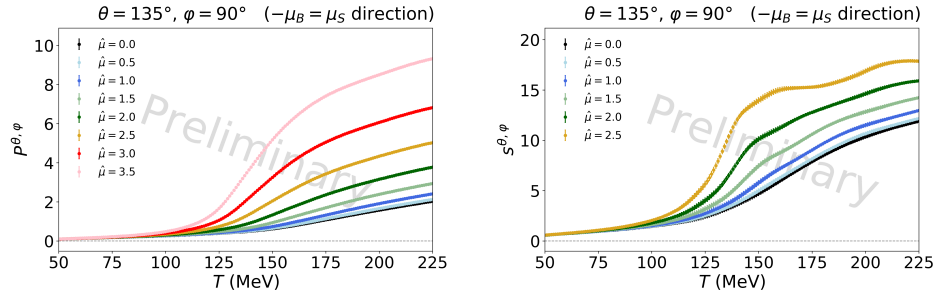

Figure 2. Pressure (*left panel*) and entropy density (*right panel*) as a function of T, for $\theta = 135°$ and $\varphi = 90°$. Different colors correspond to different values of $\hat{\mu}$ used, ranging from 0 to 3.5.

5 Summary & Outlook

We present a formalism to construct a four-dimensional EoS based on a generalization of the T-expansion scheme introduced in Refs. [5, 6], by means of a change of coordinates. This new expansion method, which consist of a resummation of the Taylor expansion, has been used to compute an equation of state at finite (T, μ_B, μ_S), for which thermodynamics quantities has been shown. Although this expansion scheme has been shown to be trusted up to $\hat{\mu}_B = 3.5$ when other chemical potentials are 0 [5], the values it can reach in other direction is still left to be determined. In the future, we will make calculation to finite chemical potentials for all three conserved charges B, Q and S. We will work on the modeling of the susceptibilities, as we need to ensure a smooth merging between HRG and lattice QCD regimes. At the same time, we will work as well on extending the range of the susceptiblities by extrapolating them to higher temperature, using lattice QCD data as a guidance and by ensuring a convergence to their respective Stefan-Boltzmann limits at $T \to \infty$. Finally, it is worth mentioning that all the work presented here being preliminary, no proper error estimate has been performed yet.

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