Spin polarization of fermions at local equilibrium

second order gradient expansion

Xin-Li Sheng

based on:
XLS, F. Becattini, Z.-H. Zhang, X.-G. Huang, in preparation

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Heavy-ion collisions

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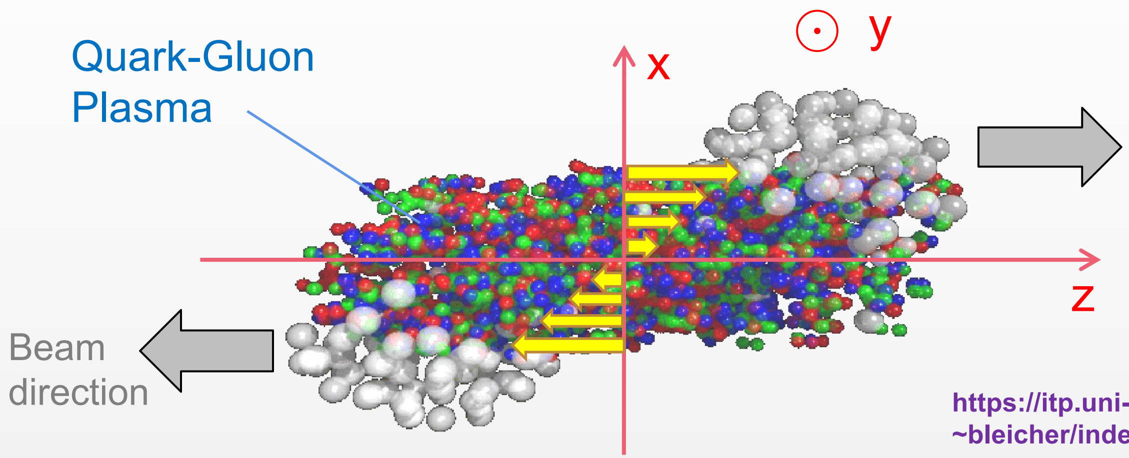
- Non-central heavy-ion collisions generate strongly interacting matter with large Orbital Angular Momentum (OAM)



https://itp.uni-frankfurt.de/~bleicher/index.html?content=urqmd

Non-central collisions → Initial orbital angular momentum → Global spin polarization for spin-1/2 or spin-3/2 baryons, Λ, Σ^0, Δ^{++}, Ω^-, ...

Through parton scattering: Z.-T. Liang, X.-N. Wang, Phys. Rev. Lett. 94, 102301 (2005)

Through vorticity field: F. Becattini, F. Piccinini, Annals Phys. 323, 2452 (2008)

Global spin polarization

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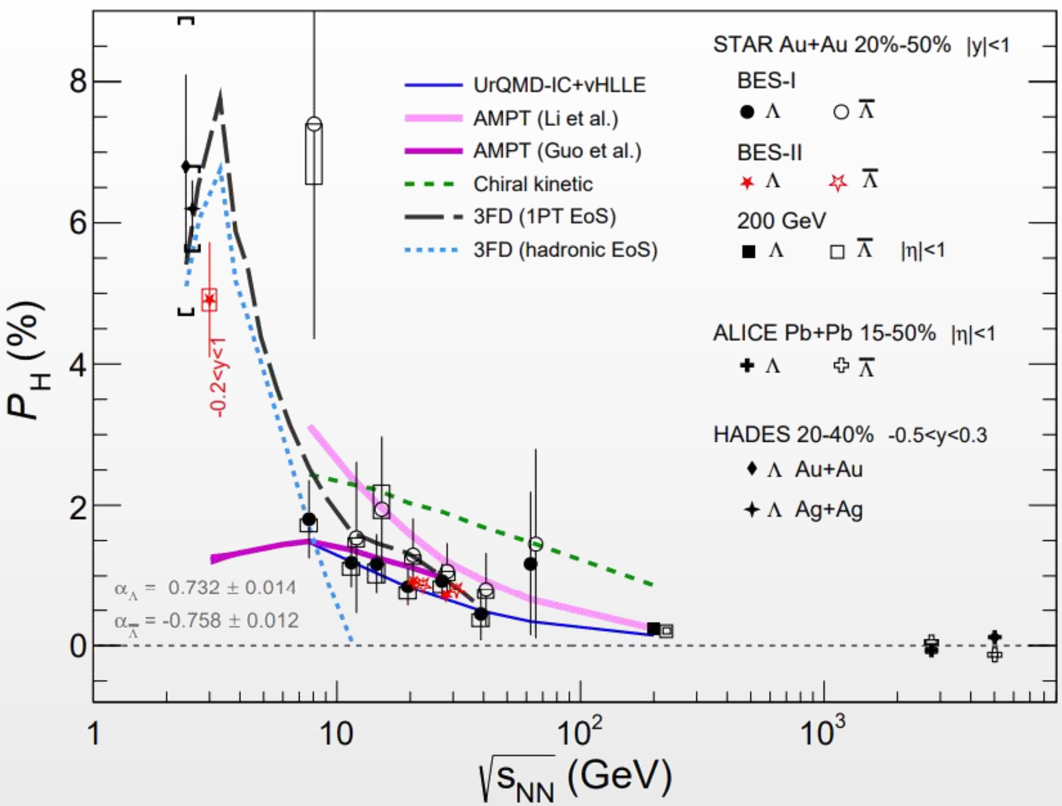


Experimental resutl: STAR, Nature 548, 62 (2017)

Figure from: F. Becattini, M. Buzzegoli, T. Niida, S. Pu, A.-H. Tang, Q. Wang, arXiv: 2402.04540

Λ's global spin polarizations in direction of global OAM

$$\Lambda \to p + \pi^-$$ Weak decay

Relativistic spin polarization induced by thermal vorticity

$$P^\mu = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\nu\varpi_{\rho\sigma}$$

Non-relativistic

$$\mathbf{P} = \frac{\boldsymbol{\omega}}{2T}$$

Recent reviews:

Q. Wang, Nucl. Phys. A 967, 225 (2017)

F. Becattini, M. Lisa, Ann. Rev. Nuccle. Part. Sci. 70, 395 (2020)

X.-G. Huang, J. Liao, Q. Wang, X.-L. Xia, Lect. Notes Phys. 987, 281 (2021)

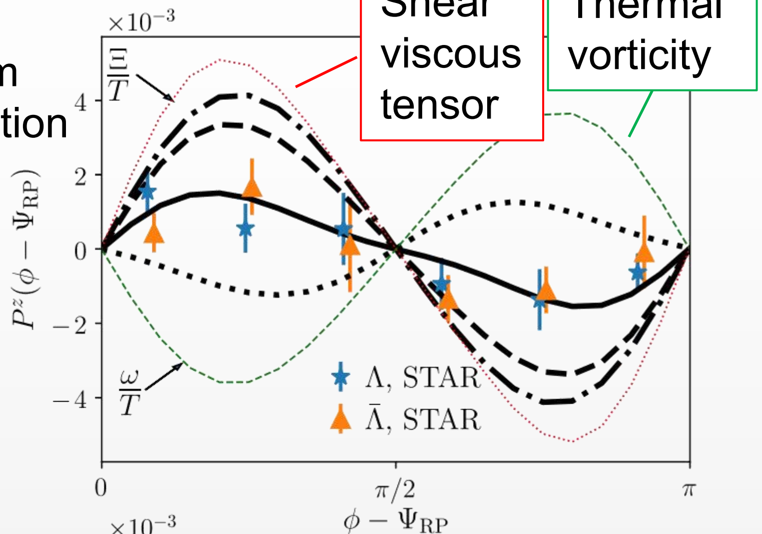
J. Gao, Z.-T. Liang, Q. Wamg, X.-N. Wang, Lect. Notes Phys. 987, 195 (2021)

F. Becattini, Rept. Prog. Phys. 85, 122301 (2022)

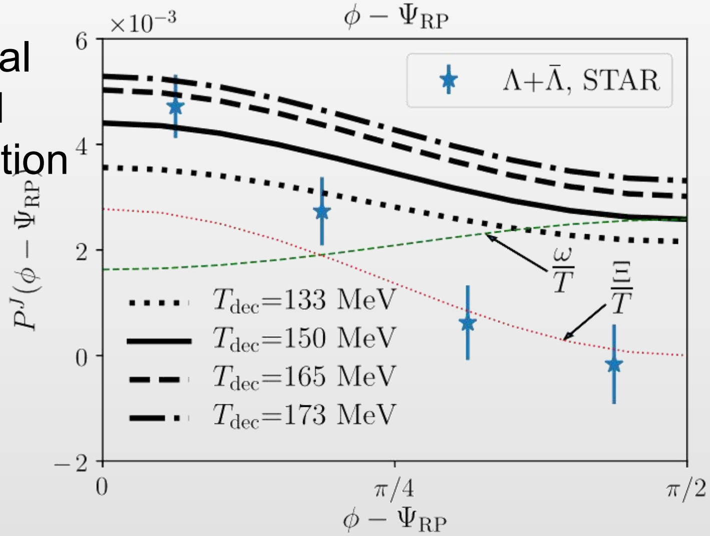
Y. Hidaka, S. Pu, Q. Wang, D.-L. Yang, Part. Nucl. Phys. 127, 103989 (2022)

Local spin polarization

Beam direction



Global OAM direction



F. Becattini, M. Buzzegoli, T. Niida, S. Pu, A.-H. Tang, Q. Wang, arXiv: 2402.04540

Local polarization: polarizations in global OAM direction or beam direction as functions of azimuthal angle

Cannot be explained by thermal vorticity

Additional contribution from thermal shear tensor $\xi_{\mu\nu} = \frac{1}{2}(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu)$

F. Becattini, M. Buzzegoli, et. al., PRL 127, 272302 (2021)
A. Palermo, M. Buzzegoli, F. Becattini, JHEP 10, 077 (2021)
S. Liu, Y. Yin, JHEP 07, 188 (2021)
B. Fu, S. Liu, et. al., PRL 127, 142301 (2021)
C. Yi, S. Pu, D.-L. Yang, PRC 104, 064901 (2021);

Also see talks:
Qiang Hu, June 5th, 9:10 a.m.
Andrea Palermo, Jun 5th, 10:40 a.m.
Chenyan Li, June 5th, 11:00 a.m.
Xu-Guang Huang, June 6th, 9:00 a.m.

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- Spin polarization induced by thermal vorticity/shear are first order in gradient expansion

$$\varpi_{\rho\sigma},\, \xi_{\rho\sigma} \sim \mathcal{O}(\partial)$$

- Hydrodynamic limit

$$K_n \ll 1$$

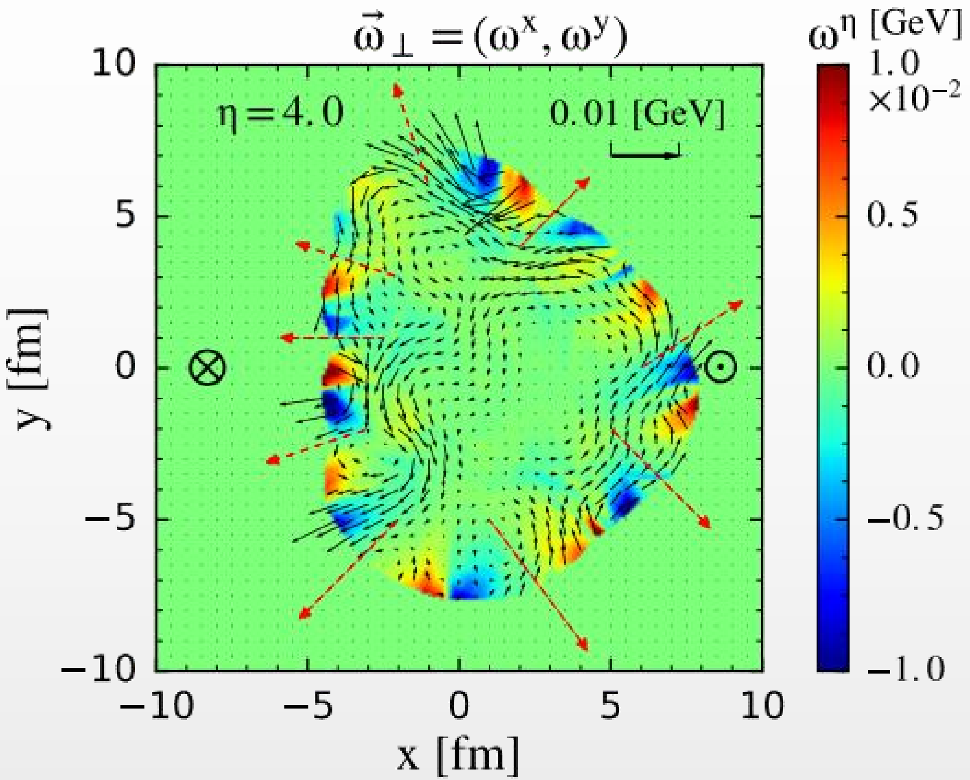
Interaction length $\ll$ Typical scale of inhomogeneity of thermodynamic fields

→ Validity of gradient expansion

- Thermal vorticity/shear could have significant inhomogeneity

→ Spin polarization induced by $\partial_\mu \varpi_{\rho\sigma}$ and $\partial_\mu \xi_{\rho\sigma}$?

Main goal of our work



L.-G. Pang, H. Elfner, Q. Wang, X.-N. Wang, PRL 117, 192301 (2016)

Wigner function

- Wigner function for spin-1/2 fermions

$$W_{ab}(x,p) \equiv \frac{1}{(2\pi)^4} \int \mathrm{d}^4y\, \mathrm{e}^{-ip\cdot y} \left\langle \overline{\psi}_b \left(x+\frac{y}{2}\right) \psi_a \left(x-\frac{y}{2}\right)\right\rangle$$

4×4 matrix in spinor space

Momentum space

$$W_{ab}(x,p) = \frac{1}{(2\pi)^8} \int \mathrm{d}^4k_1 \mathrm{d}^4k_2\, \delta^4 \left(p - \frac{k_1+k_2}{2}\right) \mathrm{e}^{-i(k_1-k_2)\cdot x} \boxed{\langle \overline{\psi}_b(k_2)\psi_a(k_1)\rangle}$$

H. T. Elze, M. Gyulassy, D. Vasak, Nucl. Phys.B 276, 706 (1986)
D. Vasak, M. Gyulassy, H. T. Elze, Annals Phys. 173, 462 (1987)

Key point: mean value of two-field correlator

- Cooper-Frye formula for spin polarization

$$S^\mu(p) = \frac{1}{2} \frac{\int_\Sigma \mathrm{d}\Sigma \cdot p_+ \,\mathrm{tr}[\gamma^\mu\gamma^5 W(x,p)]}{\int_\Sigma \mathrm{d}\Sigma \cdot p_+ \,\mathrm{tr}[W(x,p)]}$$

$p_+^\mu \equiv p^\mu \theta(p^0)$ selects positive energy particles

F. Becattini, Lect. Notes Phys. 987, 15 (2021)

- Local equilibrium density operator

D. N. Zubarev, A. V. Prozorkevich, S. A. Smolyanskii, Theor. Math. Phys. 40, 821 (1979); C. van Weert, Annals of Physics 140, 133 (1982)

$$\widehat{\rho}_{\mathrm{LE}} = \frac{1}{Z_{\mathrm{LE}}} \exp\left[-\int_{\Sigma} \mathrm{d}\Sigma_{\mu} \left(\widehat{T}^{\mu\nu}(y)\beta_{\nu}(y) - \frac{1}{2}\widehat{S}^{\mu\nu\lambda}(y)\Omega_{\nu\lambda}(y) - \widehat{j}^{\mu}(y)\zeta(y)\right)\right]$$

Energy-stress tensor | Spin tensor | Particle current

- Thermodynamical parameters

$$\beta_{\nu} = \frac{u_{\nu}}{T}$$ Thermal velocity = Fluid 4-velocity / Temperature

$$\zeta = \frac{\mu}{T}$$ Chemical potential / Temperature

$\Omega_{\nu\lambda}$ Spin potential, antisymmetic

d.o.f. = 5 + 6

* Spin potential $\Omega_{\nu\lambda}$ could be different from thermal vorticity

→ pseudo-gauge dependence

→ cannonical $\widehat{T}^{\mu\nu}$ and $\widehat{S}^{\mu\nu\lambda}$ are used in our work

Gradient expansion

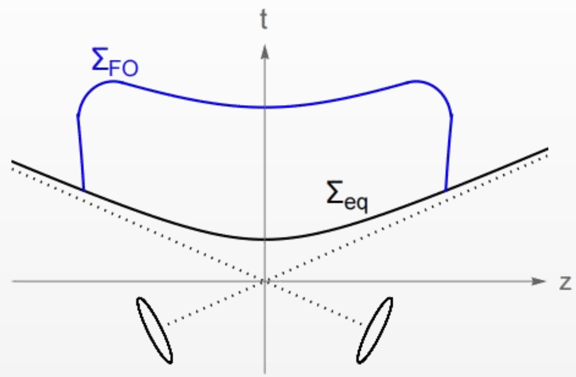
- Local equilibrium density operator

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \frac{1}{2}\widehat{S}^{\mu\nu\lambda}(y)\Omega_{\nu\lambda}(y) - \widehat{j}^\mu(y)\zeta(y) \right) \right]$$

- Mean value of a local operator

$$\langle \widehat{O}(x) \rangle_{\rm LE} = {\rm Tr}\left(\widehat{O}(x) \widehat{\rho}_{\rm LE} \right)$$

Not only depends on fields at point x, but also depends on fields at point y on hypersurface



- Gradient expansion

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[\widehat{A}_x + \widehat{B}_x \right]$$

Zeroth order

$$\widehat{A}_x \equiv -\beta_\nu(x)\widehat{P}^\mu + \zeta(x)\widehat{Q}$$

Momentum / charge operators

All higher order terms, depends on Σ

$$\widehat{B}_x \equiv - \int_\Sigma {\rm d}\Sigma_\mu \, \widehat{T}^{\mu\nu}(y) \left[\beta_\nu(y) - \beta_\nu(x) \right] + \int_\Sigma {\rm d}\Sigma_\mu \, \widehat{j}^\mu(y) \left[\zeta(y) - \zeta(x) \right] + \frac{1}{2} \int_\Sigma {\rm d}\Sigma_\mu \, \widehat{S}^{\mu\nu\lambda} \Omega_{\nu\lambda}(y)$$

$$\beta_\nu(y) \approx \beta_\nu(x) + (y-x)^\lambda [\partial_\lambda \beta_\nu(x)] + \mathcal{O}(\partial^2)$$

$$\zeta(y) \approx \zeta(x) + (y-x)^\lambda [\partial_\lambda \zeta(x)] + \mathcal{O}(\partial^2)$$

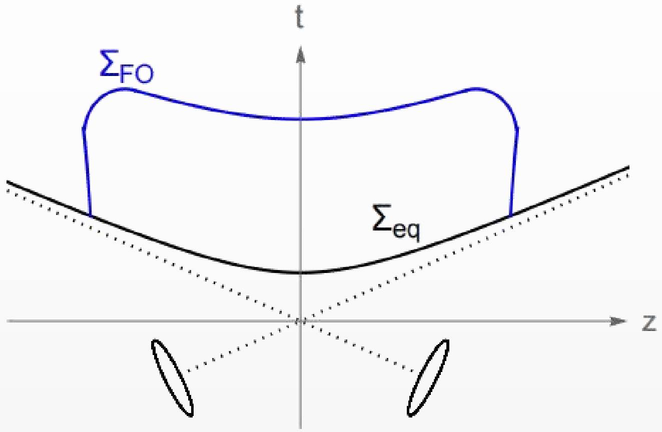
First order in gradient

$$\Omega_{\nu\lambda}(y) \approx \Omega_{\nu\lambda}(x) + \mathcal{O}(\partial^2)$$

Spin potential is treated as a first-order quantity

- Cannonical energy-stress. spin, and current operators

$$\widehat{T}^{\mu\nu}(x) = \frac{i}{2}\overline{\psi}(x)\gamma^{\nu}(\overrightarrow{\partial}^{\mu} - \overleftarrow{\partial}^{\mu})\psi(x)\,,$$
$$\widehat{S}^{\mu\nu\lambda}(x) = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\overline{\psi}(x)\gamma_{\rho}\gamma^{5}\psi(x)\,,$$
$$\widehat{j}^{\mu}(x) = \overline{\psi}(x)\gamma^{\mu}\psi(x)\,.$$



- Express operator as a bilinear combination of Dirac field operators

$$\widehat{B}_0 \equiv e^{-ix\cdot\widehat{P}}\widehat{B}_x e^{ix\cdot\widehat{P}} = \frac{1}{(2\pi)^5}\int d^4k_1 d^4k_2 \overline{\psi}(k_2)\mathcal{B}(k_2,k_1)\psi(k_1)$$

$$\mathcal{B}(k_2,k_1) = -\frac{1}{(2\pi)^3}\Big\{\gamma^{\mu}k^{\nu}[\beta_{\nu}(x+i\partial_q) - \beta_{\nu}(x)] + \frac{1}{4}\epsilon^{\mu\nu\lambda\rho}\gamma_{\rho}\gamma^{5}\Omega_{\nu\lambda}(x+i\partial_q) - \gamma^{\mu}[\zeta(x+i\partial_q) - \zeta(x)]\Big\}\left[\int_{\Sigma} d\Sigma_{\mu} e^{-iq\cdot(y-x)}\right]$$

Contains $n \geq 1$ orders in gradient expansion

Derivative w.r.t. relative momentum

- Integral over freeze-out hypersurface can be expressed as sum of an integral over flat 3-d hypersurface and a 4D-integral (Gauss' theorem)

$$\int_{\Sigma_{\rm FO}} d\Sigma_\mu e^{-iq\cdot(y-x)} = \left[\int_{\Sigma_B} d\Sigma_\mu + \int d\Omega_B \partial^y_\mu\right] e^{-iq\cdot(y-x)}$$

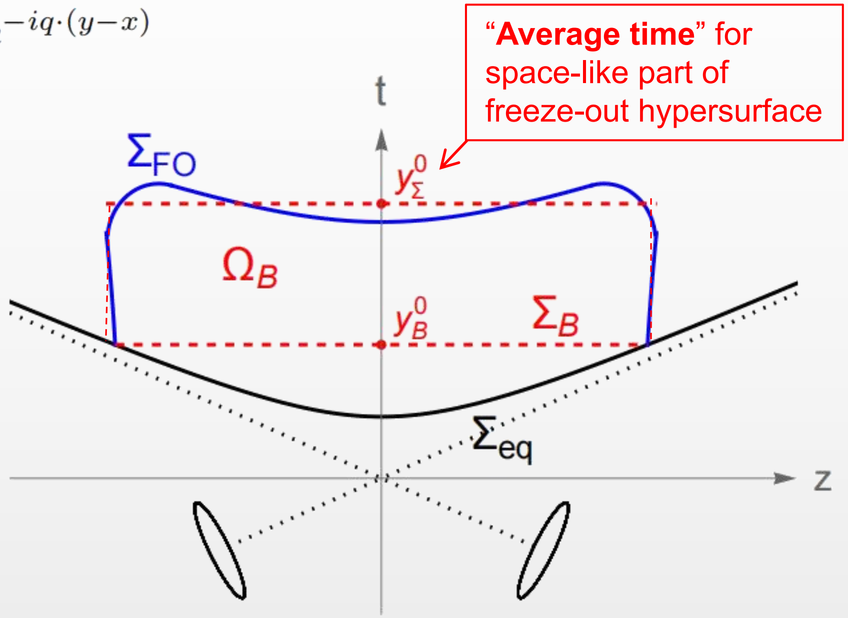
- Ω_B 4D region enclosed by $\Sigma_{\rm FO}$ and Σ_B

Assume that boundary in time direction independent to $\mathbf{y}$

$$y^0_B \le y^0 \le y^0_\Sigma \qquad \int d\Omega_B \approx \int_{y^0_B}^{y^0_\Sigma} dy^0 \int d^3\mathbf{y}$$

- Matching condition is needed

$$\left[\int d\Omega_B - \int_{y^0_B}^{y^0_\Sigma} dy^0 \int d^3\mathbf{y}\right] e^{-iq\cdot(y-x)} = 0$$



$$\int_{\Sigma_{\rm FO}} d\Sigma_\mu e^{-iq\cdot(y-x)} \approx (2\pi)^3 e^{-iq^0(y^0_\Sigma - x^0)}\delta^{(3)}(\mathbf{q})\hat{t}_\mu$$

Unit vector in time direction $\hat{t}_\mu = (1,0,0,0)$

Independent to Σ_B

Spatial extents of Σ_B and Ω_B are infinitely large

Zeroth and first orders

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- Wigner function at zeroth order

$$W^{(0)}(x,p) = \frac{\delta(p^2-m^2)\mathrm{sgn}(p^0)}{(2\pi)^3}(\not{p}+m)n_F(x,p) \qquad n_F(x,p) \equiv \frac{1}{1+\exp\left[\beta(x)\cdot p-\zeta(x)\right]}$$

Free Wigner function, no spin polarization $S^{(0)\mu}(p) = 0$

- Spin polarization at first order

Y.-C. Liu, X.-G. Huang, Sci. China Phys. Mech. Astron. 65, 272011 (2022)

M. Buzzegoli, PRC 105, 044907 (2022)

$$S^{(1)\mu}(p) = \frac{\int d\Sigma\cdot p_+ \,\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(1)}(x,p)\right]}{2\int d\Sigma\cdot p_+ \,\mathrm{tr}[W^{(0)}(x,p)]}$$

$$= -\frac{1}{8mN}\int d\Sigma\cdot p_+ \, n_F(x,p)\left[1-n_F(x,p)\right]$$

$$\times\left\{\epsilon^{\mu\nu\lambda\sigma}(\varpi_{\nu\lambda}+\Delta\Omega_{\nu\lambda})p_\sigma - \frac{2}{p^0}\hat{t}_\nu\epsilon^{\mu\nu\lambda\sigma}p_\lambda\left[(\xi_{\sigma\rho}+\Delta\Omega_{\sigma\rho})p^\rho-\partial_\sigma\zeta\right]\right\}$$

Particle number $N \equiv \int d\Sigma\cdot p_+ \, n_F(x,p)$

Vorticity-induced polarization

Polarization induced by $\Delta\Omega_{\nu\lambda} \equiv \Omega_{\nu\lambda} - \varpi_{\nu\lambda}$

Shear-induced polarization

Spin Hall effect

Vanish at global equilibrium

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- Spin polarization at second order in gradient

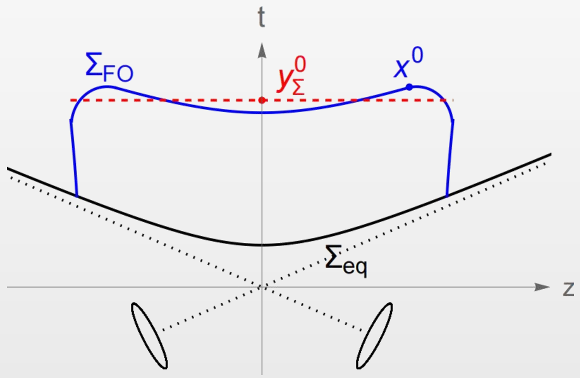
$$S^{(2)\mu}(p) = \frac{\int d\Sigma\cdot p_+ \,\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(2)}(x,p)\right]}{2\int d\Sigma\cdot p_+\,\mathrm{tr}[W^{(0)}(x,p)]} - \frac{\left\{\int d\Sigma\cdot p_+\,\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(1)}(x,p)\right]\right\}\left\{\int d\Sigma\cdot p_+\,\mathrm{tr}[W^{(1)}(x,p)]\right\}}{2\left\{\int d\Sigma\cdot p_+\,\mathrm{tr}[W^{(0)}(x,p)]\right\}^2}$$

Linear response to second order functions

$$\mathcal{O}(\partial^2):\ \partial_\lambda\partial_\mu\beta_\nu,\ \partial_\lambda\partial_\mu\zeta,\ \partial_\mu\Omega_{\nu\lambda}$$

$$S^{(2)\mu}_{\rm lin}(p) = \frac{1}{4m(p^0)^2N}\int d\Sigma\cdot p_+\, n_F(x,p)\left[1-n_F(x,p)\right](y^0_\Sigma(0)-x^0)$$
$$\times\hat{t}_\alpha p_\rho\Big[\epsilon^{\mu\sigma\alpha\rho}p^\lambda p^\nu\partial_\sigma\xi_{\nu\lambda} + \left(\frac{1}{2}p^\alpha\epsilon^{\mu\nu\lambda\rho} - \epsilon^{\mu\alpha\lambda\rho}p^\nu\right)p^\sigma\partial_\sigma\varpi_{\nu\lambda}$$
$$-\epsilon^{\mu\sigma\alpha\rho}p^\lambda\partial_\sigma\partial_\lambda\zeta + \frac{1}{2}\epsilon^{\alpha\nu\lambda\sigma}\partial^\rho(\Omega_{\nu\lambda}-\varpi_{\nu\lambda})(p^\mu p_\sigma - m^2 g^\mu_\sigma)\Big].$$

$y^0_\Sigma - x^0$ is difference between "average time" and time of a specified point



Vanishes if freeze-out hypersurface is a 3D surface with constant-time

XLS, F. Becattini, Z.-H. Zhang, X.-G. Huang, in preparation

- Spin polarization at second order in gradient

$$S^{(2)\mu}(p) = \frac{\int d\Sigma\cdot p_+ \,\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(2)}(x,p)\right]}{2\int d\Sigma\cdot p_+ \,\mathrm{tr}[W^{(0)}(x,p)]} - \frac{\left\{\int d\Sigma\cdot p_+ \,\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(1)}(x,p)\right]\right\}\left\{\int d\Sigma\cdot p_+ \,\mathrm{tr}[W^{(1)}(x,p)]\right\}}{2\left\{\int d\Sigma\cdot p_+ \,\mathrm{tr}[W^{(0)}(x,p)]\right\}^2}$$

Quadratic response to first order functions

$$\mathcal{O}(\partial^2):\ (\partial_\lambda\beta_\sigma)(\partial_\mu\beta_\nu),\ (\partial_\lambda\zeta)(\partial_\mu\zeta),\ \Omega_{\lambda\sigma}\Omega_{\mu\nu}$$
$$(\partial_\mu\beta_\nu)(\partial_\lambda\zeta),\ (\partial_\mu\beta_\nu)\Omega_{\lambda\sigma},\ (\partial_\lambda\zeta)\Omega_{\mu\nu}$$

$$S^{(2)\mu}_{\mathrm{quad}}(p) = \frac{1}{2\int d\Sigma\cdot p_+ \mathrm{tr}\left[W^{(0)}(x,p)\right]}\int d\Sigma\cdot p_+ \frac{[1-2n_F(x,p)]\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(1)}(x,p)\right]\mathrm{tr}\left[W^{(1)}(x,p)\right]}{[1-n_F(x,p)]\mathrm{tr}\left[W^{(0)}(x,p)\right]}$$
$$-\frac{1}{2}\left\{\frac{\int d\Sigma\cdot p_+ \mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(1)}(x,p)\right]}{\int d\Sigma\cdot p_+ \mathrm{tr}\left[W^{(0)}(x,p)\right]}\right\}\left\{\frac{\int d\Sigma\cdot p_+ \mathrm{tr}\left[W^{(1)}(x,p)\right]}{\int d\Sigma\cdot p_+ \mathrm{tr}\left[W^{(0)}(x,p)\right]}\right\}$$

$$\mathrm{tr}\left[W^{(1)}(x,p)\right] = -\frac{\delta(p^2-m^2)}{(2\pi)^3|p^0|} n_F(x,p)\left[1-n_F(x,p)\right](y^0_\Sigma - x^0)4mp^\lambda\partial_\lambda[p^\sigma\beta_\sigma(x)-\zeta(x)],$$

$$\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(1)}(x,p)\right] = -\frac{\delta(p^2-m^2)}{(2\pi)^3|p^0|} n_F(x,p)\left[1-n_F(x,p)\right]$$
$$\times\left\{2\epsilon^{\mu\nu\rho\lambda}p_\nu\hat{t}_\rho\partial_\lambda\left[p^\tau\beta_\tau(x)-\zeta(x)\right] + (p^\mu p_\tau - g^\mu_\tau m^2)\hat{t}_\rho\epsilon^{\rho\nu\lambda\tau}\Omega_{\nu\lambda}(x)\right\}.$$

- Calculated spin polarization of spin-1/2 fermions at local equilibrium upto second order in gradient expansion

- Spin polarization at second order $\propto y_\Sigma^0 - x^0$ (Non-dissipative!)

  - Vanish if freezeout hypersurface has constant-t, which agree with result from an exact analytical calculation
    A. Palermo and F. Becattini, Eur. Phys. J. Plus 138, 547 (2023)

  - Non-vanish if hypersurface has non-trivial space-time structure

  - Magnitude needs numerical simulations

Thanks for your attention!

Backup slides

Global spin polarization

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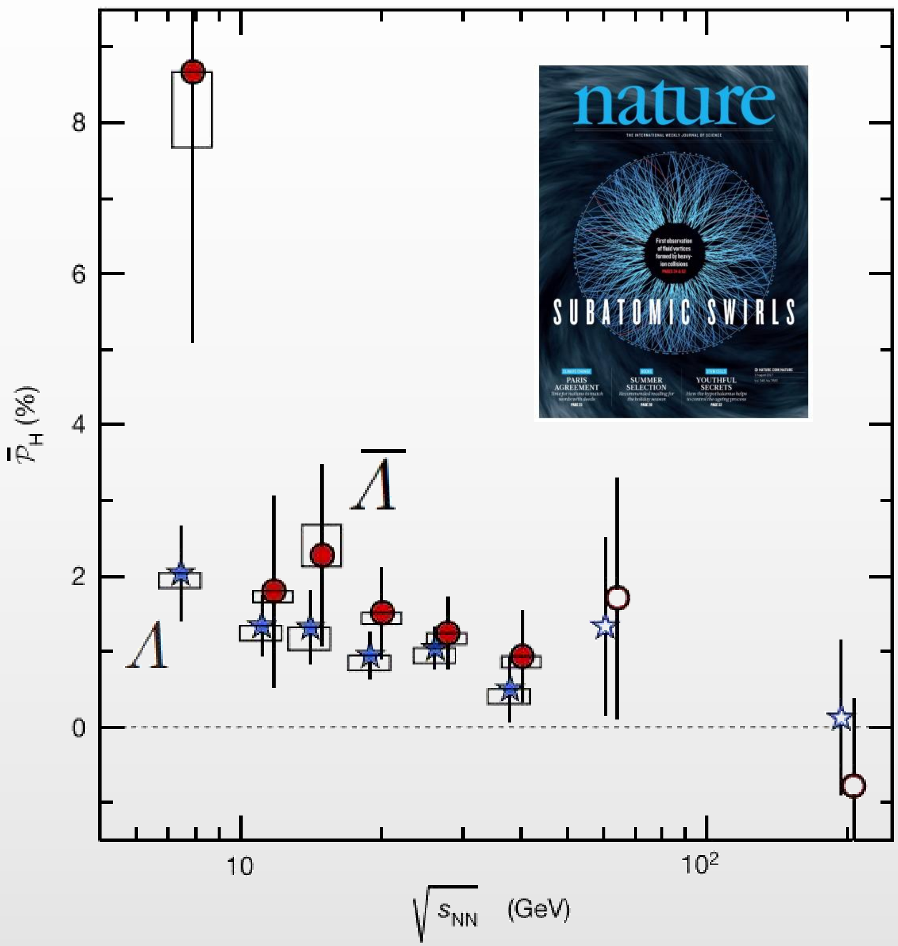


STAR, Nature 548, 62 (2017)

Λ's global spin polarizations in direction of global OAM

$$\Lambda \rightarrow p + \pi^-$$

Parity-violating weak decay

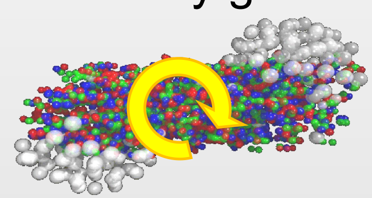
$$\frac{dN}{d\Omega} = \frac{1}{4\pi}(1 + \alpha_\Lambda \mathbf{P}_\Lambda \cdot \hat{\mathbf{p}})$$

decay parameter (constant)

Λ's spin polarization

unit vector along proton's momentum

Mostly induced by global rotation



S. A. Voloshin, arXiv:nucl-th/0410089.
Z.-T. Liang, X.-N. Wang, PRL 94 039901, (2004)
F. Becattini, F. Piccinini, Annals Phys. 323, 2452 (2008)
F. Becattini, V. Chandra, et. al., Annals Phys. 338, 32 (2013)

Gradient expansion

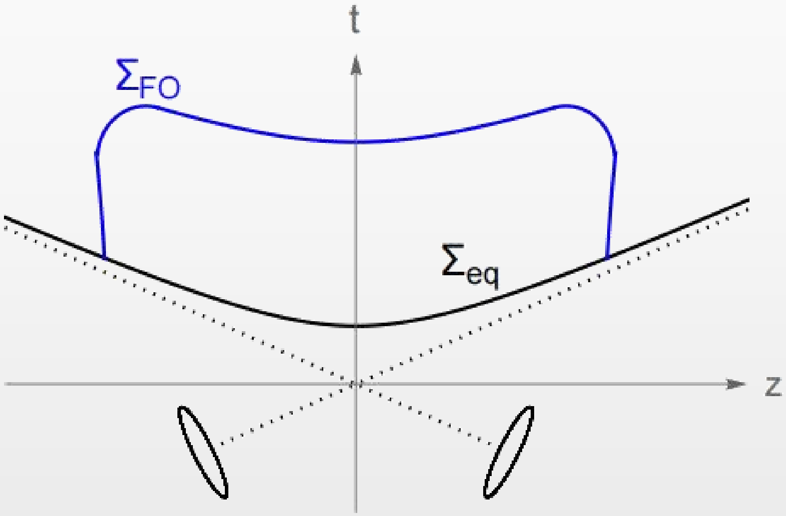
- Local equilibrium density operator

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \frac{1}{2}\widehat{S}^{\mu\nu\lambda}\Omega_{\nu\lambda}(y) - \widehat{j}^\mu(y)\zeta(y) \right) \right]$$

- Mean value of a local operator

$$\langle \widehat{O}(x) \rangle_{\rm LE} = {\rm Tr}\left(\widehat{O}(x) \widehat{\rho}_{\rm LE} \right)$$

Not only depend on fields at point x, but also depends on fields at point y on hypersurface



- Gradient expansion

$$\beta_\nu(y) \approx \beta_\nu(x) + (y-x)^\lambda [\partial_\lambda \beta_\nu(x)] + \mathcal{O}(\partial^2)$$

$$\zeta(y) \approx \zeta(x) + (y-x)^\lambda [\partial_\lambda \zeta(x)] + \mathcal{O}(\partial^2)$$

~~~~~~~~~~~~~
First order in gradient

$$\Omega_{\nu\lambda}(y) \approx \Omega_{\nu\lambda}(x) + \mathcal{O}(\partial^2)$$

~~~~~~
Spin potential is treated as a first-order quantity

1. At global equilibrium, spin potential equals thermal vorticity, which is first order
2. Global spin polarization observed in experiments is small

Density operator and mean value

- Local equilibrium density operator

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[\widehat{A}_x + \widehat{B}_x\right] \qquad \widehat{A}_x \equiv -\beta_\nu(x)\widehat{P}^\mu + \zeta(x)\widehat{Q}$$

Zeroth order terms

Independent to Σ

Momentum / charge operators

$$\widehat{B}_x \equiv -\int_\Sigma \mathrm{d}\Sigma_\mu\, \widehat{T}^{\mu\nu}(y)\,[\beta_\nu(y)-\beta_\nu(x)] + \frac{1}{2}\int_\Sigma \mathrm{d}\Sigma_\mu\, \widehat{S}^{\mu\nu\lambda}\Omega_{\nu\lambda}(y) + \int_\Sigma \mathrm{d}\Sigma_\mu\, \widehat{j}^\mu(y)\,[\zeta(y)-\zeta(x)]$$

All higher order terms

Dependent on Σ

- Mean value of a local operator

$$\widehat{O}(0) \equiv \mathrm{e}^{-ix\cdot\widehat{P}}\widehat{O}(x)\mathrm{e}^{ix\cdot\widehat{P}} \qquad \widehat{B}_0 \equiv e^{-ix\cdot\widehat{P}}\widehat{B}_x e^{ix\cdot\widehat{P}}$$

$$\left\langle \widehat{O}(x)\right\rangle_{\rm LE} \approx \left\langle \widehat{O}(0)\right\rangle_A + \int_0^1 \mathrm{d}z\, \left\langle \widehat{O}(0),\, \mathrm{e}^{z\widehat{A}_x}\widehat{B}_0\mathrm{e}^{-z\widehat{A}_x}\right\rangle_{A,C} + \int_0^1 \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 \left\langle \widehat{O}(0), \mathrm{e}^{z_2\widehat{A}_x}\widehat{B}_0\mathrm{e}^{-z_2\widehat{A}_x},\, \mathrm{e}^{z_1\widehat{A}_x}\widehat{B}_0\mathrm{e}^{-z_1\widehat{A}_x}\right\rangle_{A,C} + \mathcal{O}(\widehat{B}^3)$$

Homogeneous global equilibrium

$$\left\langle \widehat{O}\right\rangle_A \equiv \frac{\mathrm{tr}\left[\widehat{O}e^{\widehat{A}_x}\right]}{\mathrm{tr}\left[e^{\widehat{A}_x}\right]}$$

Connected mean values

$$\left\langle \widehat{O}_1, \widehat{O}_2\right\rangle_{A,C} \equiv \left\langle \widehat{O}_1\widehat{O}_2\right\rangle_A - \left\langle \widehat{O}_1\right\rangle_A\left\langle \widehat{O}_2\right\rangle_A ,$$

$$\left\langle \widehat{O}_1, \widehat{O}_2, \widehat{O}_3\right\rangle_{A,C} \equiv \left\langle \widehat{O}_1\widehat{O}_2\widehat{O}_3\right\rangle_A - \left\langle \widehat{O}_1\widehat{O}_2\right\rangle_A\left\langle \widehat{O}_3\right\rangle_A - \left\langle \widehat{O}_2\widehat{O}_3\right\rangle_A\left\langle \widehat{O}_1\right\rangle_A - \left\langle \widehat{O}_1\widehat{O}_3\right\rangle_A\left\langle \widehat{O}_2\right\rangle_A + 2\left\langle \widehat{O}_1\right\rangle_A\left\langle \widehat{O}_2\right\rangle_A\left\langle \widehat{O}_3\right\rangle_A ,$$

Density operator and mean value

- Local equilibrium density operator

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[\widehat{A}_x + \widehat{B}_x\right] \qquad \widehat{A}_x \equiv -\beta_\nu(x)\widehat{P}^\mu + \zeta(x)\widehat{Q}$$

Momentum / charge operators

Zeroth order terms

Independent to Σ

$$\widehat{B}_x \equiv -\int_\Sigma {\rm d}\Sigma_\mu\, \widehat{T}^{\mu\nu}(y)\left[\beta_\nu(y) - \beta_\nu(x)\right] + \frac{1}{2}\int_\Sigma {\rm d}\Sigma_\mu\, \widehat{S}^{\mu\nu\lambda}\Omega_{\nu\lambda}(y) + \int_\Sigma {\rm d}\Sigma_\mu\, \widehat{j}^\mu(y)\left[\zeta(y) - \zeta(x)\right]$$

All higher order terms

Dependent on Σ

- Mean value of a local operator

$$\left\langle \widehat{O}(x)\right\rangle_{\rm LE} \approx \left\langle \widehat{O}(0)\right\rangle_A + \int_0^1 {\rm d}z\, \left\langle \widehat{O}(0), {\rm e}^{z\widehat{A}_x}\widehat{B}_0{\rm e}^{-z\widehat{A}_x}\right\rangle_{A,C} + \int_0^1 {\rm d}z_1 \int_0^{z_1} {\rm d}z_2 \left\langle \widehat{O}(0), {\rm e}^{z_2\widehat{A}_x}\widehat{B}_0{\rm e}^{-z_2\widehat{A}_x}, {\rm e}^{z_1\widehat{A}_x}\widehat{B}_0{\rm e}^{-z_1\widehat{A}_x}\right\rangle_{A,C} + \mathcal{O}(\widehat{B}^3)$$

$\mathcal{O}(1) : \beta_\mu, \zeta$

$\mathcal{O}(\partial) : \partial_\mu\beta_\nu, \partial_\mu\zeta, \Omega_{\mu\nu}$

$\mathcal{O}(\partial^2) : \partial_\lambda\partial_\mu\beta_\nu, \partial_\lambda\partial_\mu\zeta, \partial_\mu\Omega_{\nu\lambda}$

$\mathcal{O}(\partial^2) : (\partial_\lambda\beta_\sigma)(\partial_\mu\beta_\nu), (\partial_\lambda\zeta)(\partial_\mu\zeta), \Omega_{\lambda\sigma}\Omega_{\mu\nu}$
$(\partial_\mu\beta_\nu)(\partial_\lambda\zeta), (\partial_\mu\beta_\nu)\Omega_{\lambda\sigma}, (\partial_\lambda\zeta)\Omega_{\mu\nu}$