Equilibrium expectations for non-Gaussian fluctuations near a QCD critical point

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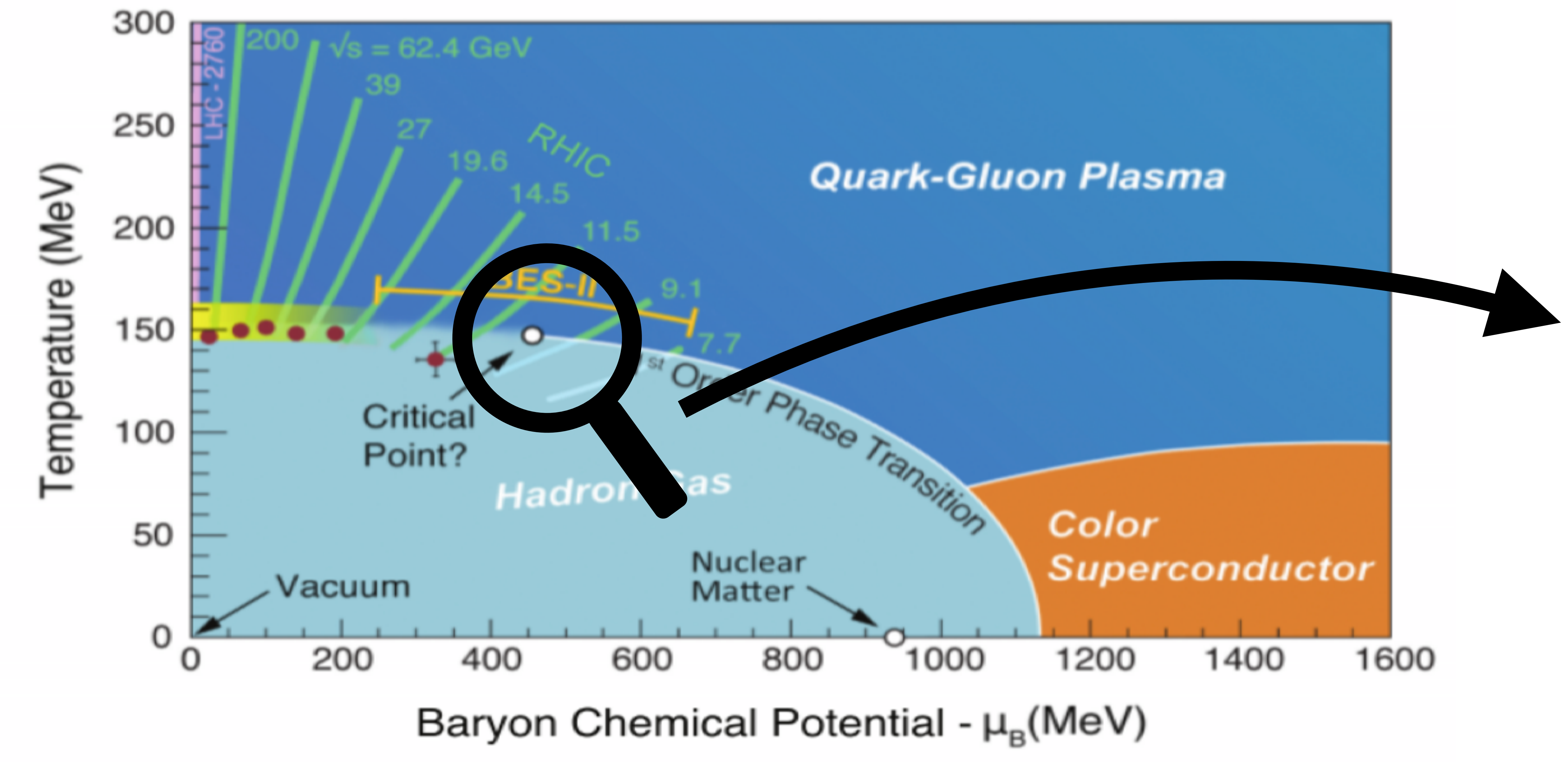
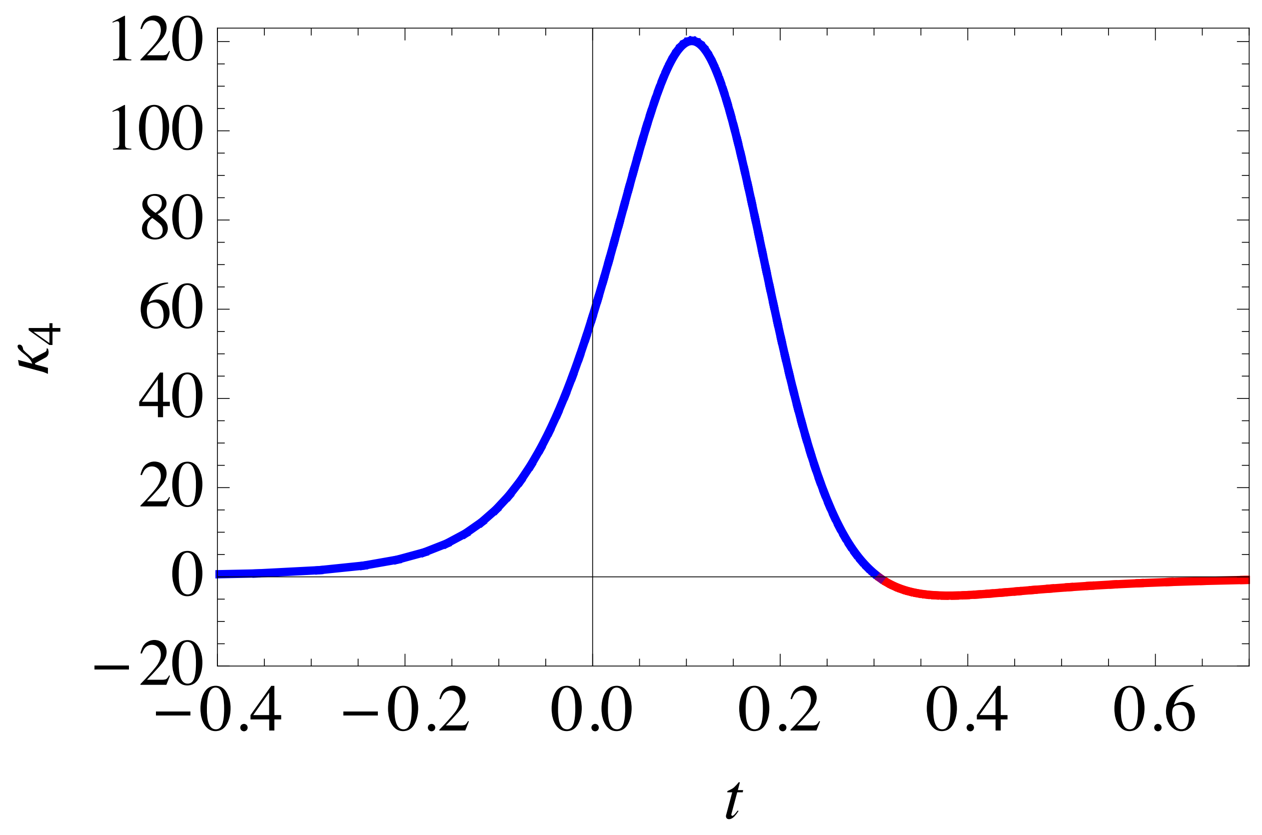
Krishna Rajagopal, and Yi Yin

Search for Criticality

- The open questions on QCD phase structure require support from theory community to provide candidates for criticality-carrying observables





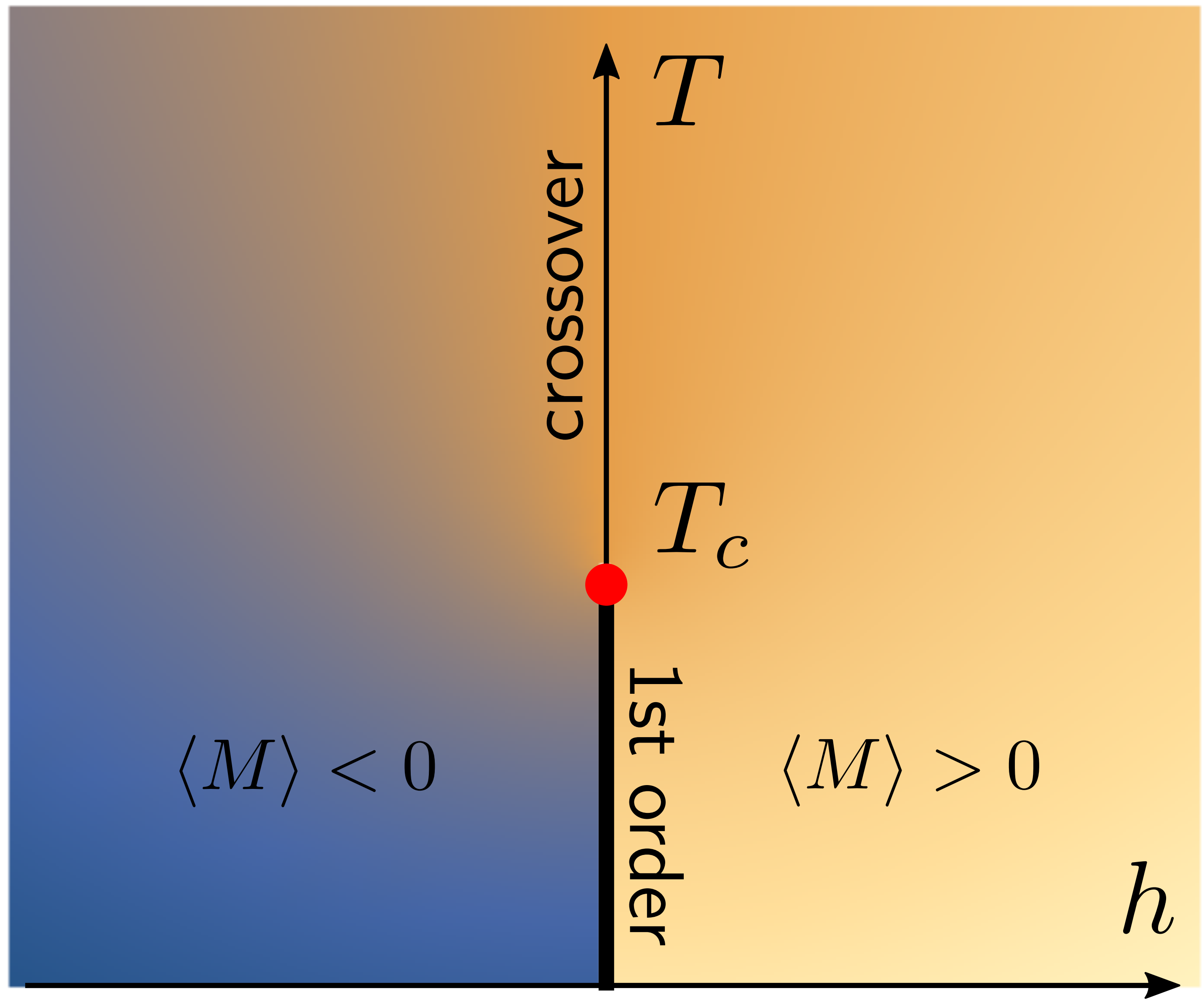
- Higher order susceptibilities diverge with higher power of the correlation length, $\kappa_4 \propto \xi^7$ $\qquad \chi_n^B \equiv \frac{\partial^n (p/T^4)}{\partial(\mu_B/T)^n}$
- Related to moments of the net-proton distribution: can be measured experimentally $\qquad \kappa_4\sigma^2 = \chi_4^B/\chi_2^B$

NSAC 2015 Long Range Plan for Nuclear Physics
M. Stephanov, K. Rajagopal and E. Shuryak, PRD (1999)
M. Stephanov, PRL (2011)

Universal Scaling EOS

- Average critical fluctuations of σ give rise to "magnetization": $M = \langle \sigma \rangle$
- Universal critical scaling behavior given by the 3D Ising model equation of state:
  - Magnetic field: $h = h_0 R^{\beta\delta} H(\theta)$, $H(\theta) = \theta(3 - 2\theta^2)$
  - Reduced temperature: $t = R(1 - \theta^2)$
  - Magnetization: $M = M_0 R^{\beta} \theta$
- Critical fluctuations calculated in 3D Ising EOS

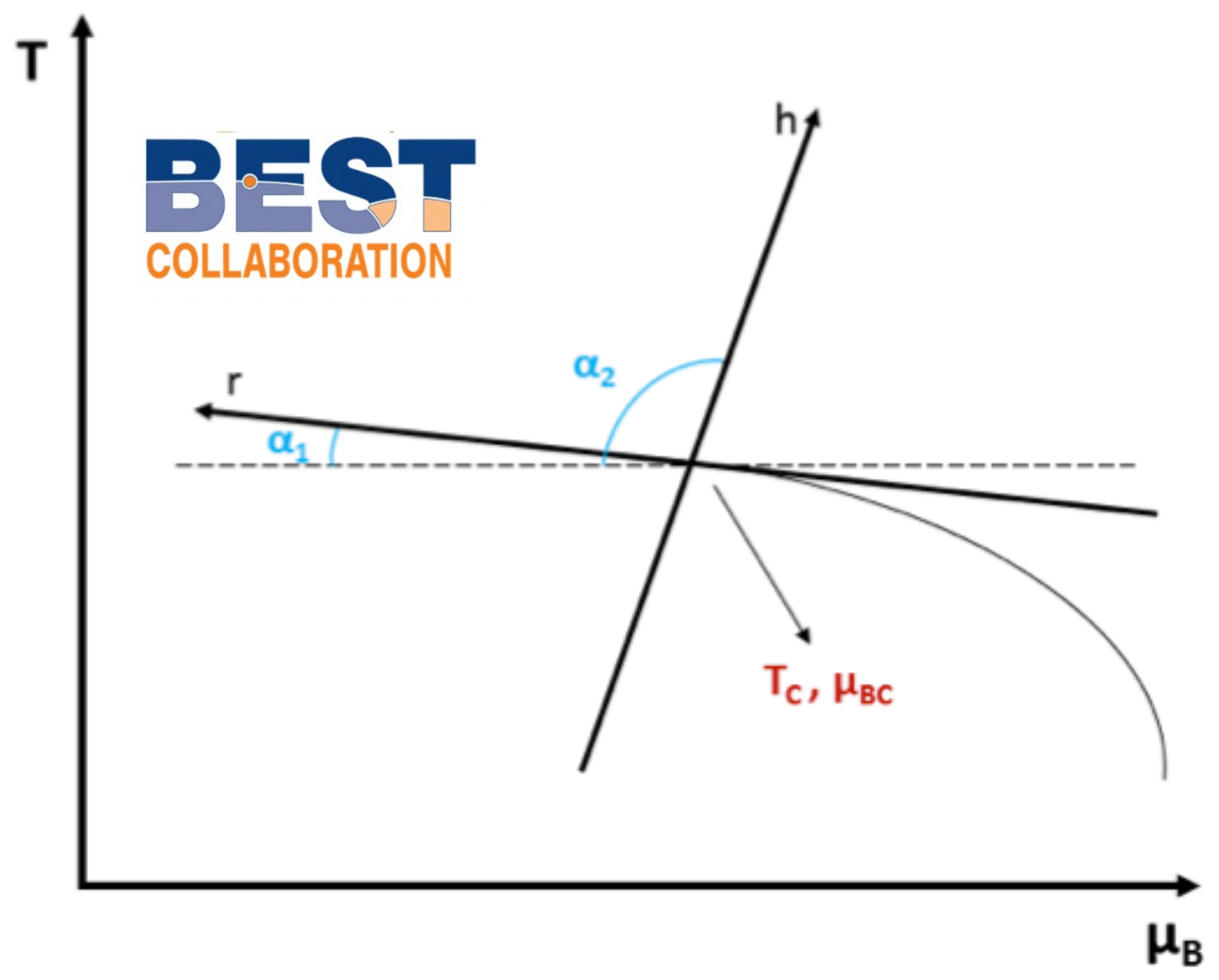
$$\kappa_{n+1}^{\mathrm{eq}} \propto \left(\frac{\partial^n M^{\mathrm{eq}}(t,h)}{\partial h^n} \right)_t$$



K. Rajagopal and F. Wilczek, Nucl. Phys. B (1993)
J. Zinn-Justin, Quantum Field Theory and Critical Phenomena
S. Mukherjee, R. Venugopalan, Y. Yin, PRC (2015)
A. Bzdak et al, Phys. Rep. (2020)

EoS for BES

- Combine Lattice + HRG equation of state and incorporate universal scaling features into the QCD phase diagram from the 3D Ising Model equation of state



$$(\mathbf{r}, \mathbf{h}) \longleftrightarrow (\mathbf{T}, \mu_\mathbf{B}) : \quad \frac{T - \mathbf{T_C}}{\mathbf{T_C}} = \mathbf{w}\,(r\rho\,\sin\alpha_1 + h\,\sin\alpha_2)$$

$$\frac{\mu_B - \mu_\mathbf{BC}}{\mathbf{T_C}} = \mathbf{w}\,(-r\rho\,\cos\alpha_1 - h\,\cos\alpha_2)$$

- Reconstruct the pressure via Taylor expansion coefficients from Lattice QCD

$$T^4 c_n^{\mathrm{LAT}}(T) = T^4 c_n^{\mathrm{Non-Ising}}(T) + c_n^{\mathrm{Ising}}(T)$$

$$P(T, \mu_B) = T^4 \sum_n c_n^{\mathrm{Non-Ising}}(T) \left(\frac{\mu_B}{T}\right)^n + P_{\mathrm{crit}}^{\mathrm{QCD}}(T, \mu_B)$$

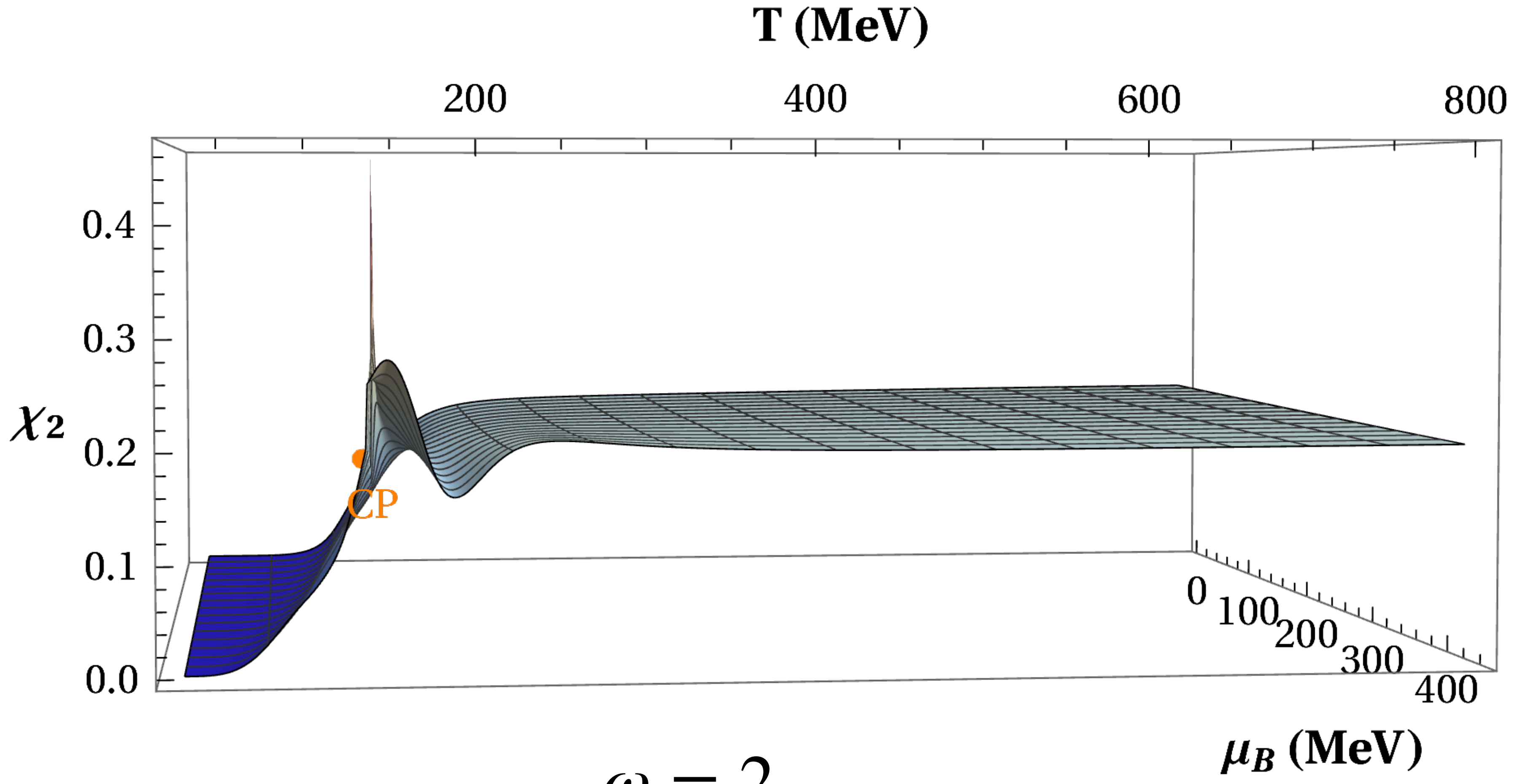
- Reduce free parameters by imposing constraints from Lattice

$$T = T_0 + \kappa T_0 \left(\frac{\mu_B}{T_0}\right)^2 + O(\mu_B^4), \qquad \alpha_1 = \tan^{-1}\left(2\frac{\kappa}{T_0}\mu_{BC}\right)$$

- Further constrain with future experimental data

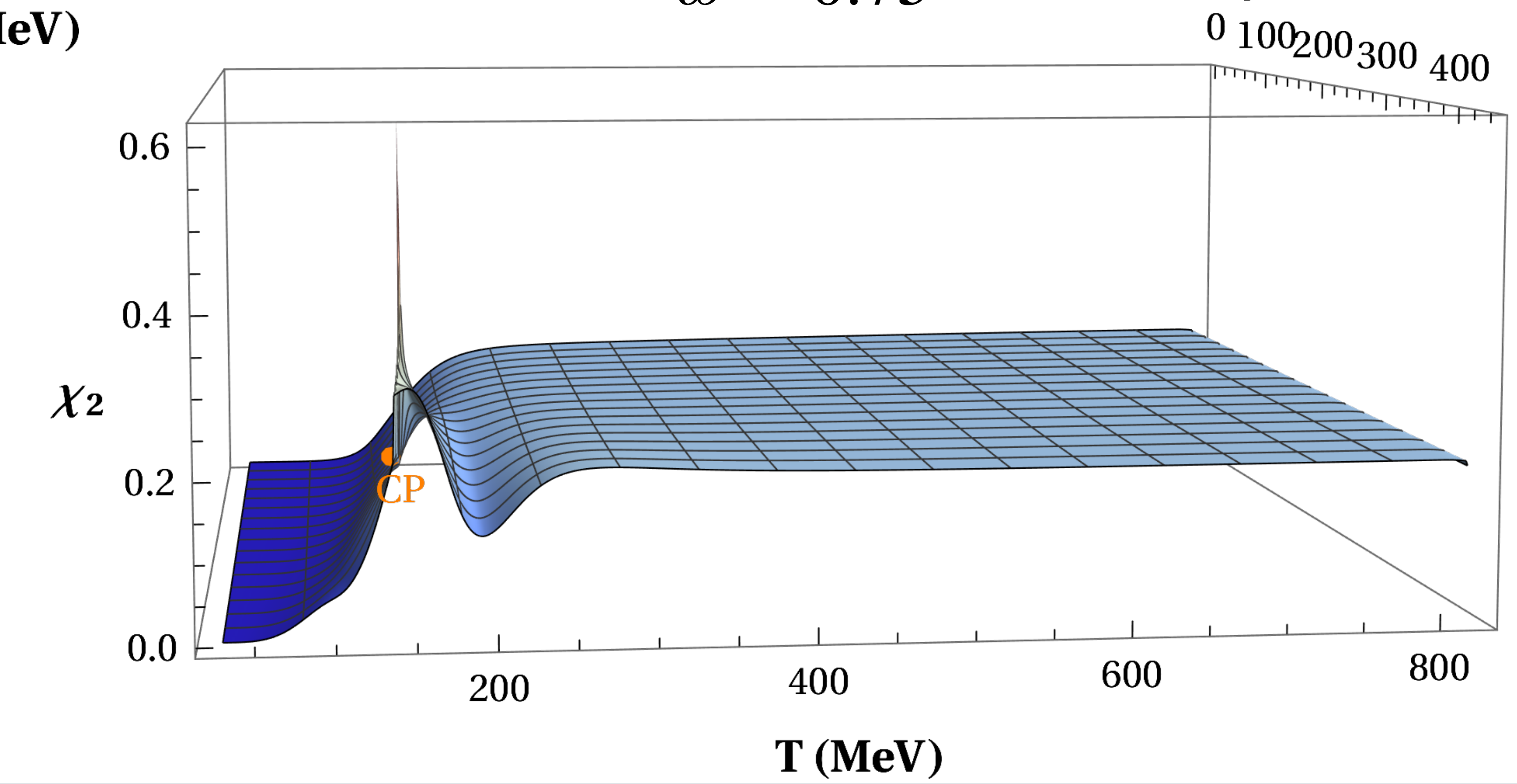
P. Parotto et al, PRC (2020),
***J. M. Karthein** et al, EPJ+ (2021)*

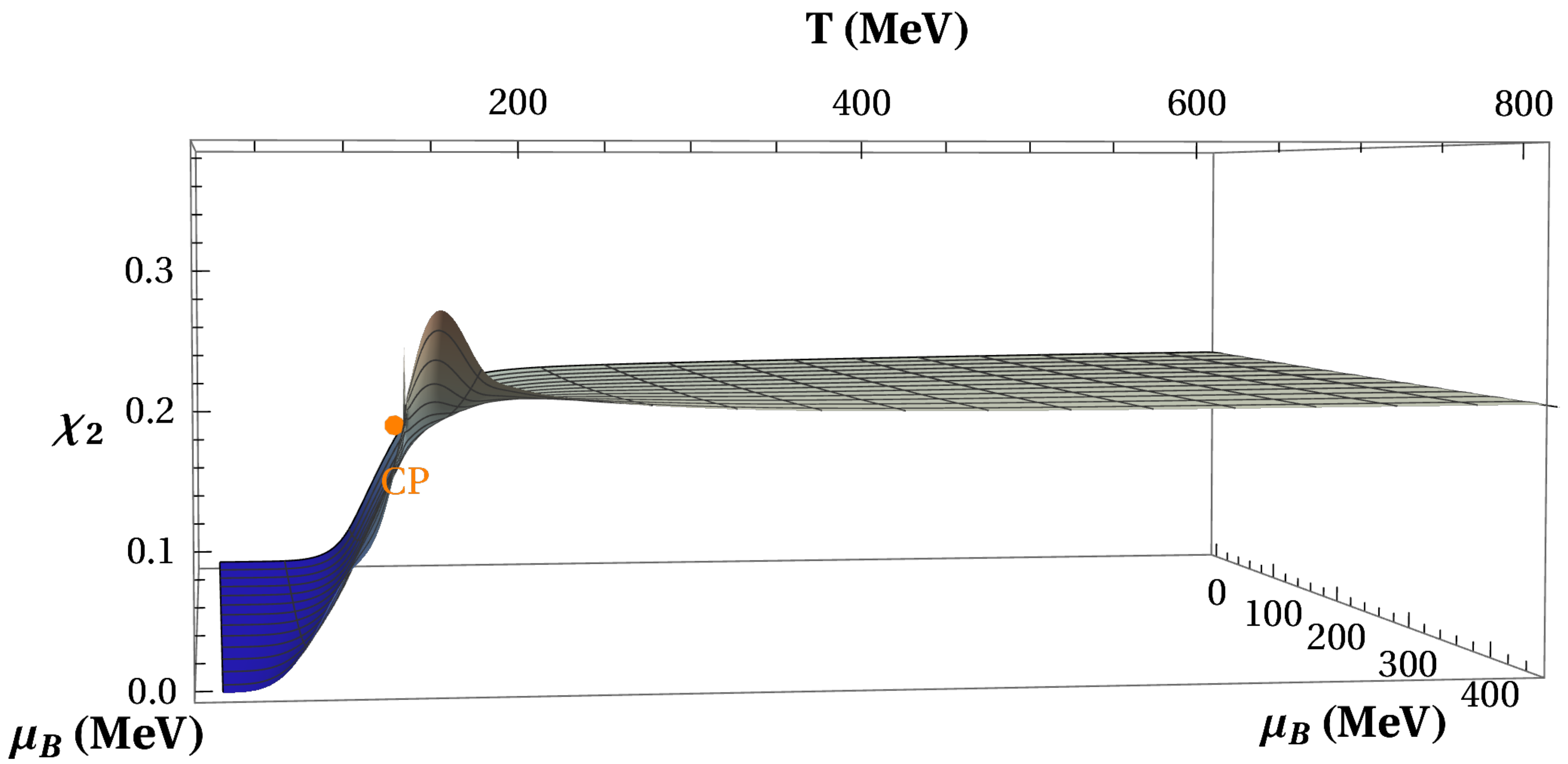
- ➤ By changing the parameters of the mapping we can control the critical contribution to the overall thermodynamics



$\omega = 2$

$\omega = 0.75$





$\omega = 4$

Framework allows for a range of critical signals due to uncertainty in the EoS mapping

M. Pradeep and M. Stephanov, PRD (2019)
P. Parotto et al, PRC (2020)
D. Mroczek et al, PRC (2021)
Wei-jie Fu et al, arXiv: 2101.06035
***J. M. Karthein** et al, EPJ+ (2021)*

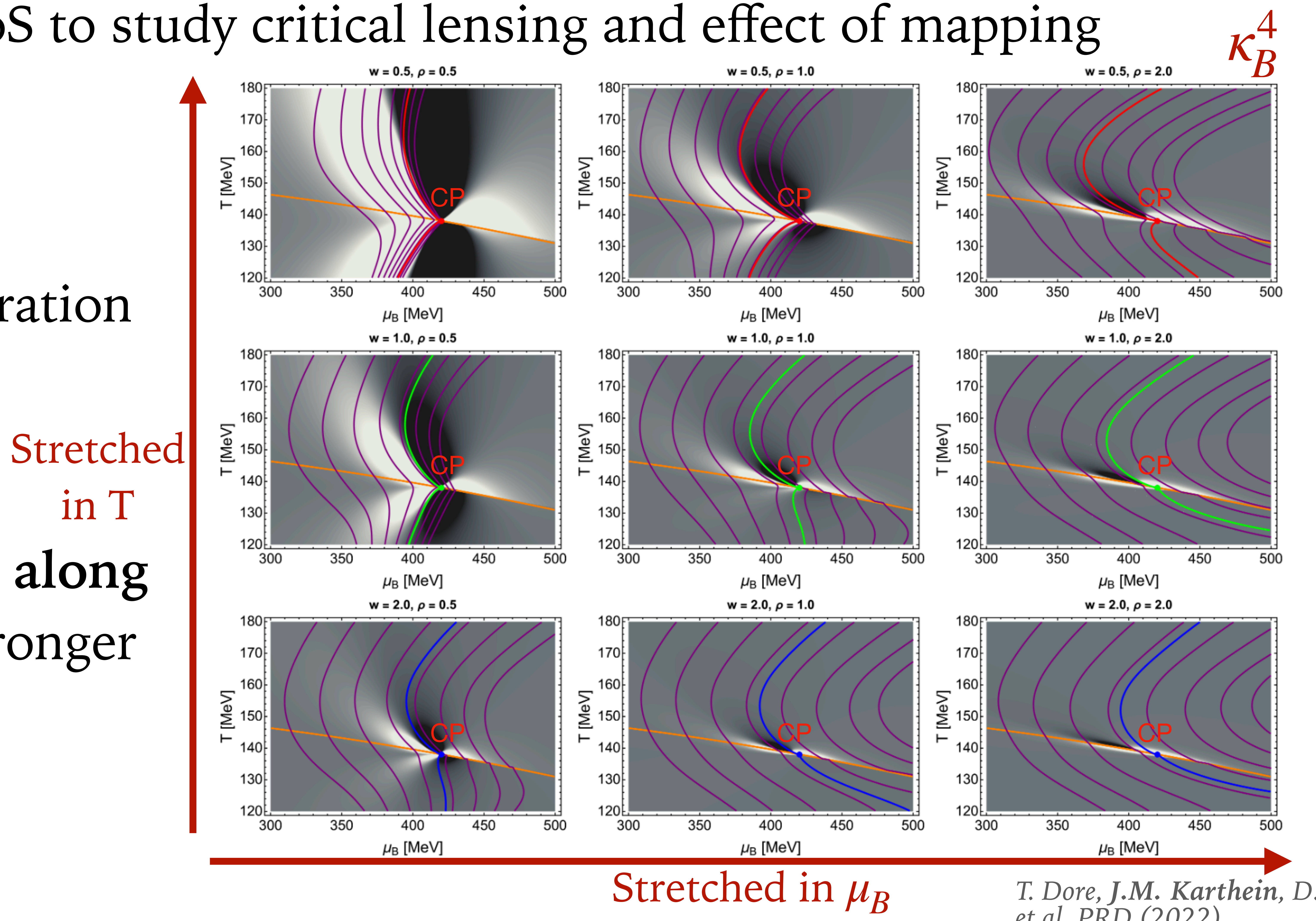
BEST EoS Used to Calculate in Equilibrium: κ_B

- Calculate κ_B^4 from BEST EoS to study critical lensing and effect of mapping parameters

- Small $w, \rho \rightarrow$ smaller separation

$$\frac{d\mu_B}{d(s/n)} \sim (w\rho)r$$

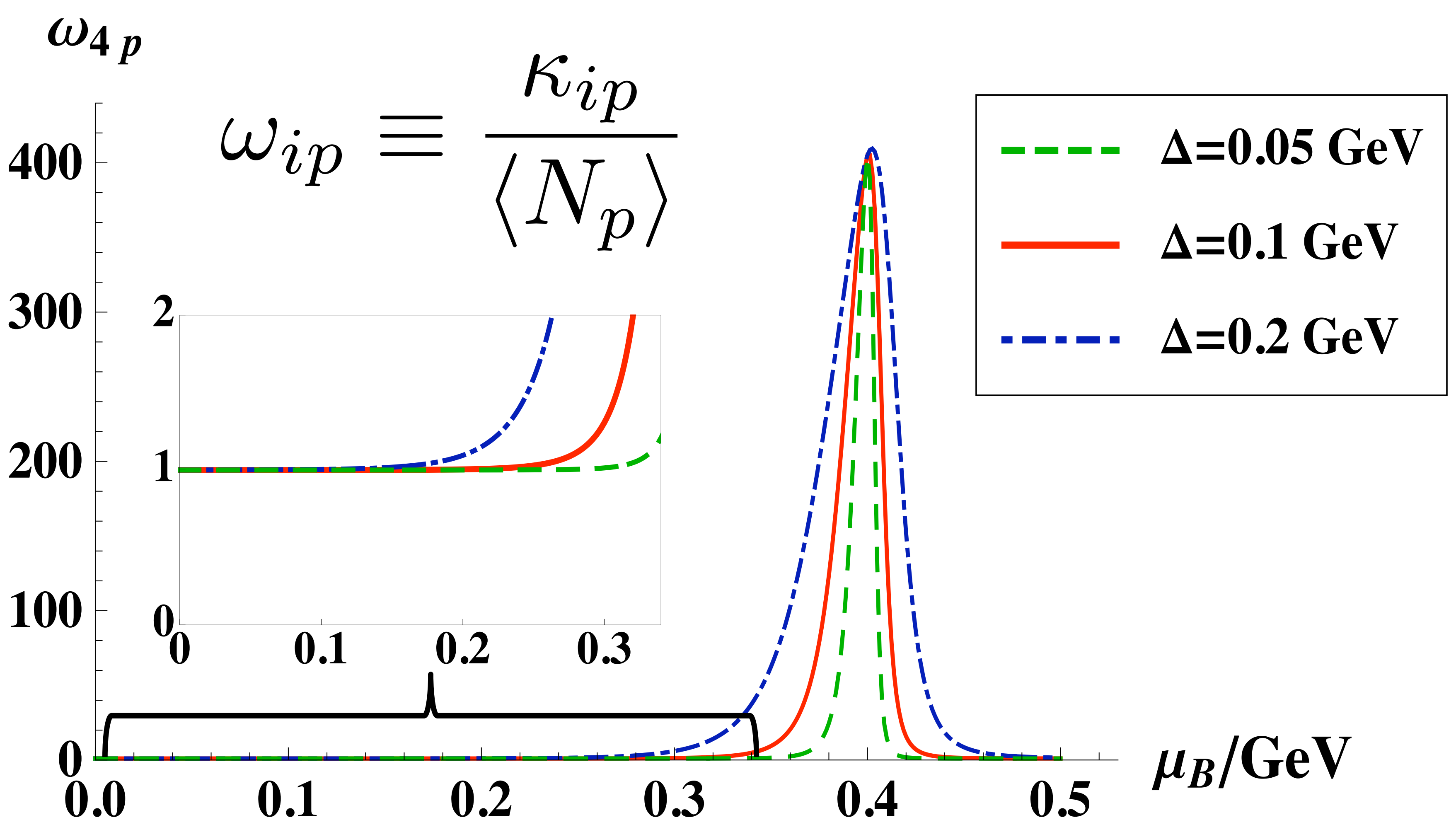
- Critical regions **extending along the T-direction** show a stronger signal and lensing effect



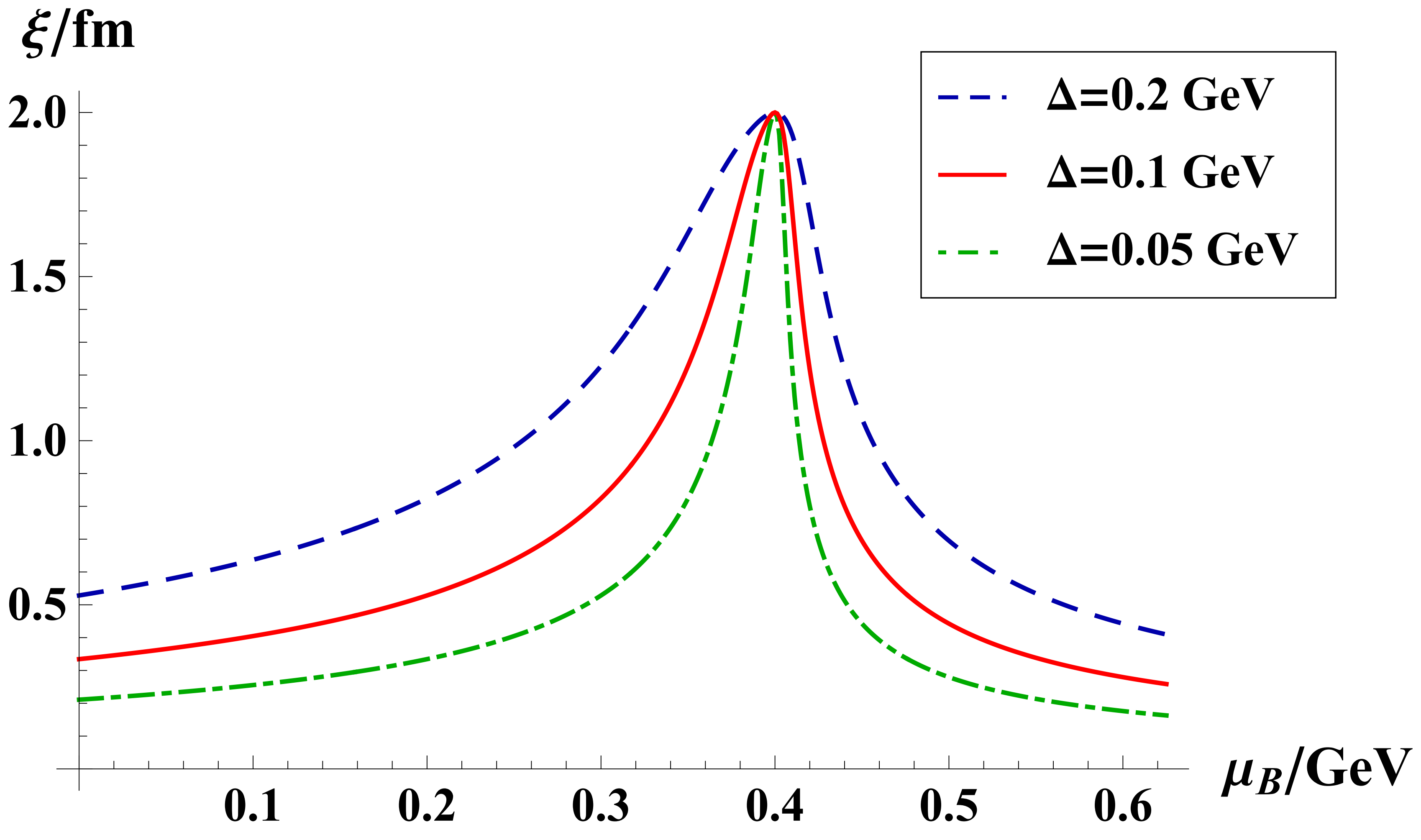
*T. Dore, **J.M. Karthein**, D. Mroczek et al, PRD (2022)*

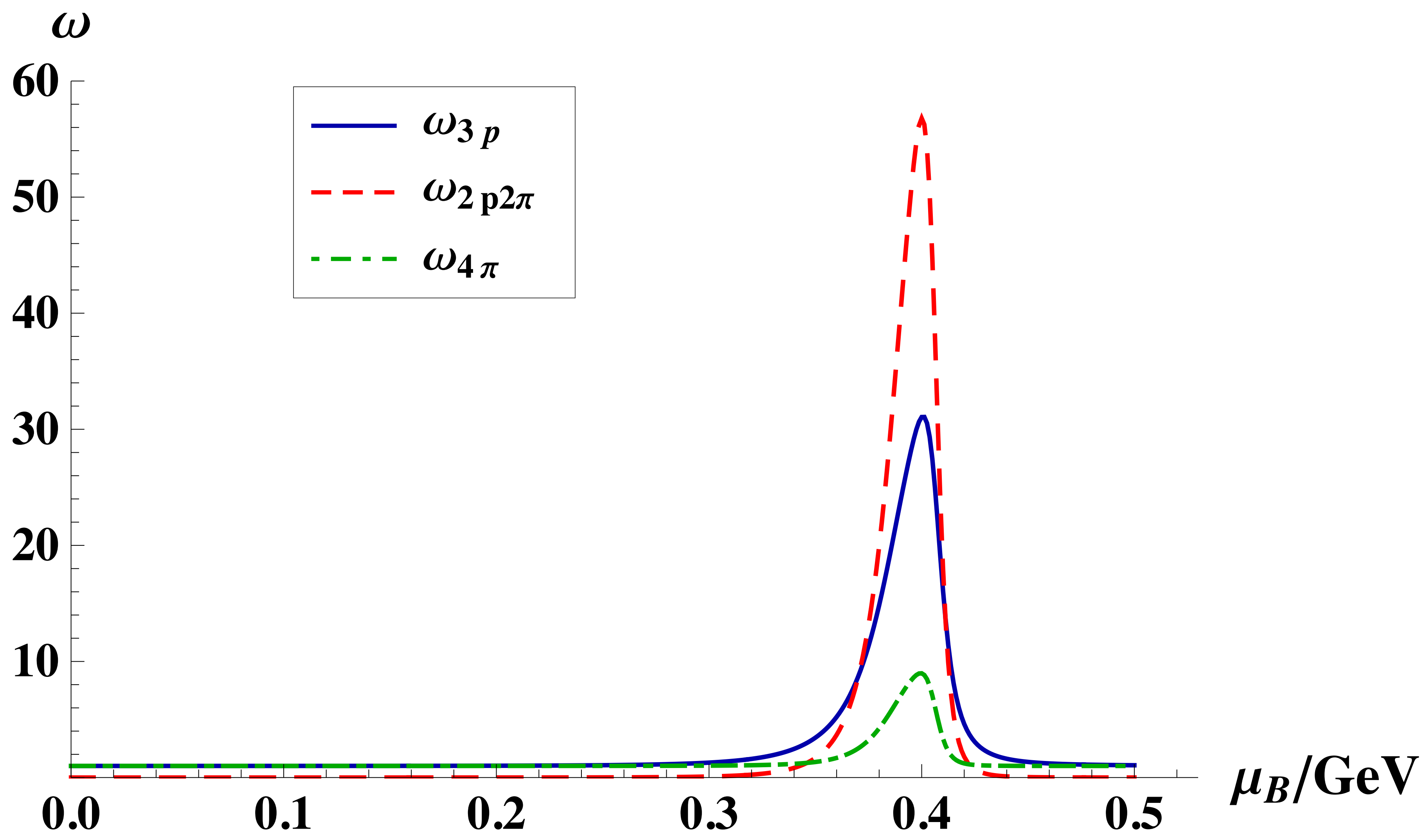
Early Estimates of Equilibrium Fluctuations

- Order-of-magnitude predictions of volume-independent normalized cumulants from 2010 relied on ansätze
  - Original estimates used parametrized correlation length with width Δ



$$\xi(\mu_B) = \frac{\xi_{\max}}{\left[1 + \frac{(\mu_B - \mu_B^c)^2}{W(\mu_B)^2}\right]^{1/3}}$$





C. Athanasiou, K. Rajagopal, M. Stephanov, PRD (2010)

Estimating Equilibrium Fluctuations with BEST EoS

- Re-evaluate equilibrium estimates for factorial cumulants $\omega_{ip} \equiv \frac{\kappa_{ip}}{\langle N_p \rangle}$ with realistic critical EOS
  - Updates: $\xi, \lambda_3, \lambda_4$ (dimensionless: $\tilde{\lambda}_3 = \lambda_3 T^{1/2} \xi^{3/2}$, $\tilde{\lambda}_4 = \lambda_4 T \xi$)
  - Remaining dependence on coupling: g_p

$$\omega_{4p,\sigma} = \frac{6(2\tilde{\lambda}_3^2 - \tilde{\lambda}_4)}{T^2 n_p} \xi^7 \left(d_p \, g_p \int_k \frac{v_k^2}{\gamma_k} \right)^4 \xrightarrow{\textit{generalize}} \omega_{ip} = 1 + \omega_{ip}^{\text{prefactor}} \left(\frac{n_p}{n_0} \right)^{i-1} \left(\frac{\xi}{\xi_{\max}} \right)^{\frac{5}{2}i-3}$$

$$\omega_{ip}^{\text{prefactor}} = \frac{\tilde{\lambda}_i'(i-1)!\,\xi_{\max}^{\frac{5}{2}i-3}}{T^{i/2} n_p} \left(\int_k d_p \, g_p \frac{v_k^2}{\gamma_k} \right)^i \left(\frac{n_0}{n_p} \right)^{i-1}$$

$$\tilde{\lambda}_3' \equiv \tilde{\lambda}_3 \quad \text{and} \quad \tilde{\lambda}_4' \equiv 2\tilde{\lambda}_3^2 - \tilde{\lambda}_4$$

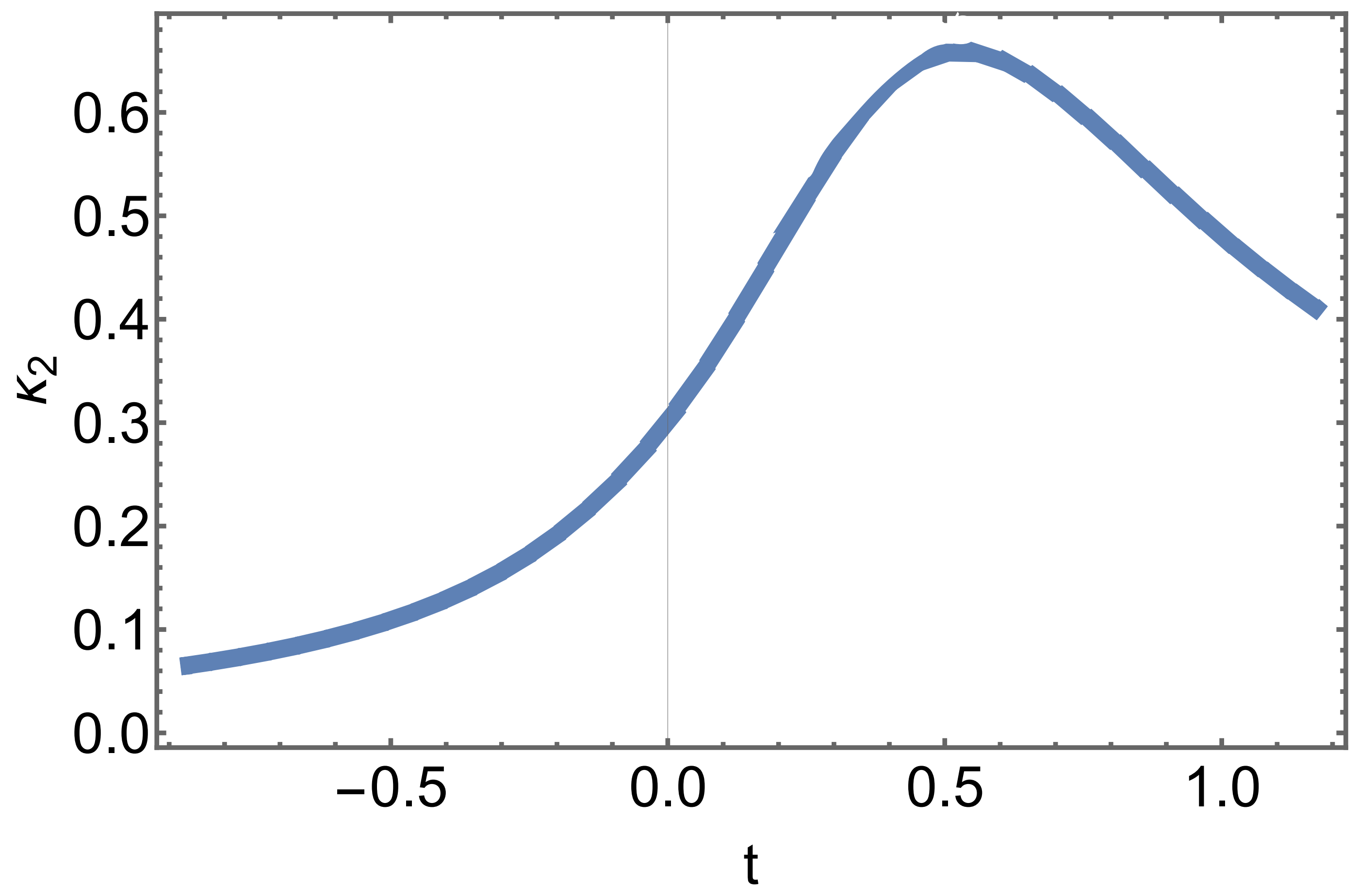
C. Athanasiou, K. Rajagopal, M. Stephanov, PRD (2010)

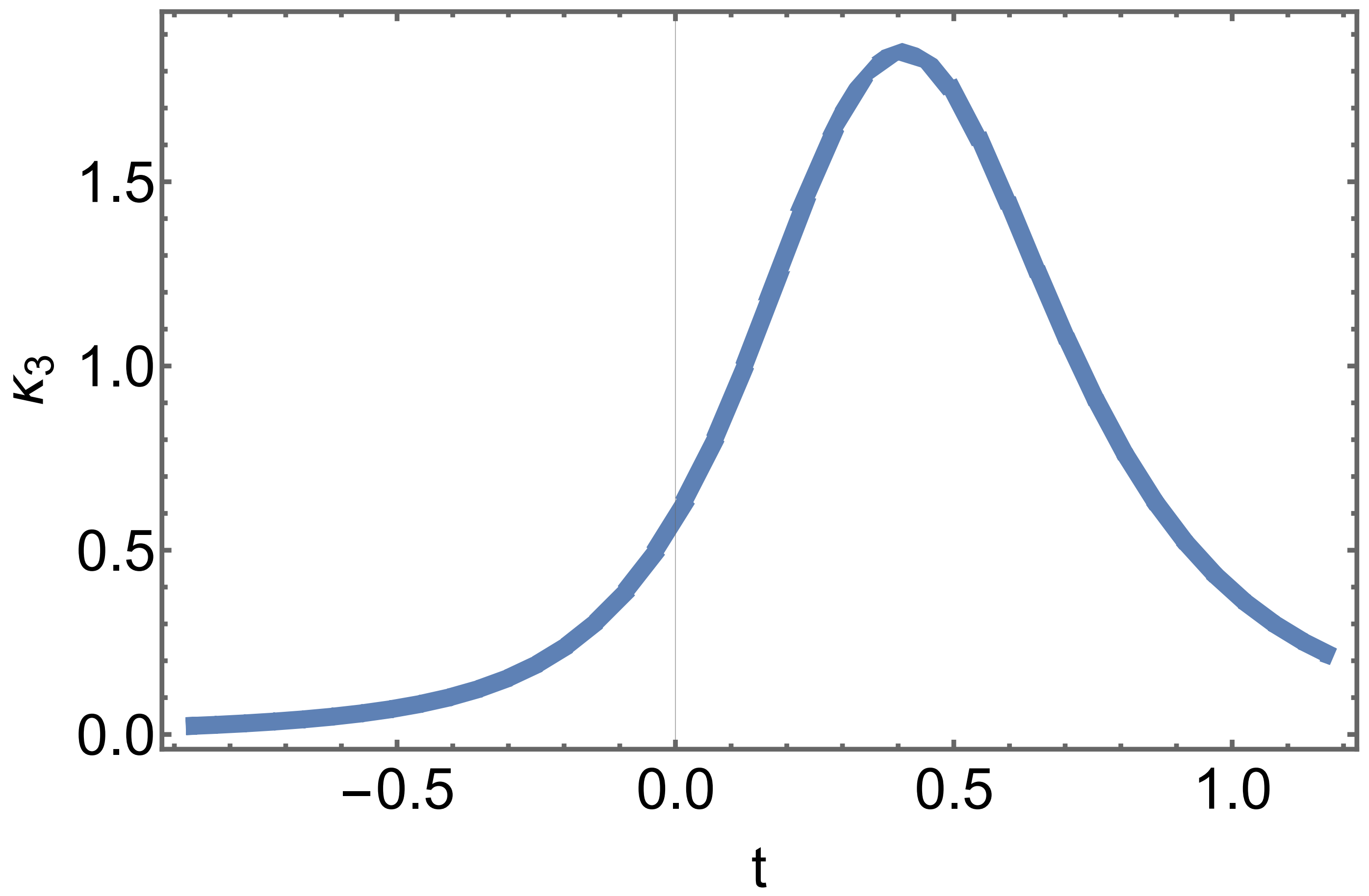
Equilibrium Fluctuations in 3D Ising Model

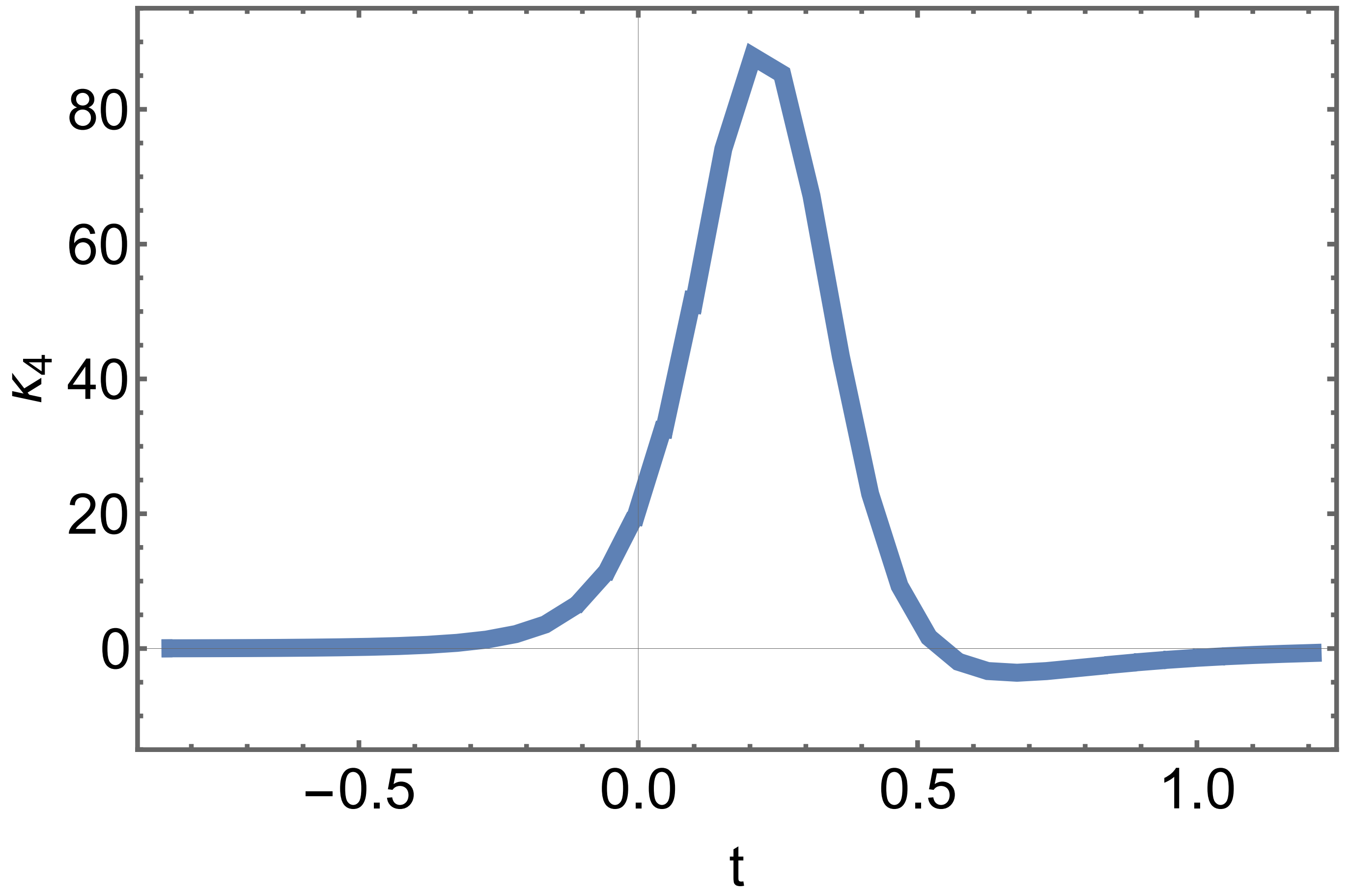
- Calculate critical fluctuations as parametric derivatives of universal EOS utilizing approximate rational critical exponents: $\beta \sim 1/3, \delta \sim 5, \nu \sim 2/3$

$$\kappa_{n+1}^{\mathrm{eq}} \propto \left(\frac{\partial^n M^{\mathrm{eq}}(t,h)}{\partial h^n} \right)_t$$

$$\kappa_2 = \frac{M_0}{h_0} \frac{1}{R^{4/3}\left(3+2\theta^2\right)} \qquad \kappa_3 = \frac{-M_0}{h_0^2} \frac{4\theta\left(9+\theta^2\right)}{R^3\left(3-\theta^2\right)(3+2\theta)^3} \qquad \kappa_4 = \frac{-M_0}{h_0^3} \frac{12\left(2\theta^8 - 5\theta^6 + 105\theta^4 - 783\theta^2 + 81\right)}{R^{14/3}\left(3-\theta^2\right)^3\left(3+2\theta^2\right)^5}$$







M. Stephanov, PRL (2011)
S. Mukherjee, R. Venugopalan, Y. Yin, PRC (2015)

Equilibrium Correlation Length in 3D Ising Model

- 3D Ising EOS also provides a parametrization of the correlation length in the ϵ-expansion

$$\xi^2(M,t) = R^{-2\nu} g_\xi(\theta)$$

  - New (!) equilibrium calculation to $\mathcal{O}(\epsilon^2)$

$$g_\xi(\theta) = g_\xi(0)\left(1 - \frac{5}{18}\epsilon\theta^2 + \left[\frac{1}{972}(24I - 25)\theta^2 + \frac{1}{324}(4I + 41)\theta^4\right]\epsilon^2 \right)$$

$$\text{where:} \quad I \equiv \int_0^1 \frac{\ln[x(1-x)]}{1 - x(1-x)} dx \sim -2.3439$$

- Now with the true critical EOS determine the higher order couplings

$$\kappa_2 = \langle \sigma_V^2 \rangle = VT\xi^2\,; \qquad \kappa_3 = \langle \sigma_V^3 \rangle = 2\lambda_3 VT^2\xi^6$$

$$\kappa_4 = \langle \sigma_V^4 \rangle_c \equiv \langle \sigma_V^4 \rangle - 3\langle \sigma_V^2 \rangle^2 = 6VT^3\left[2(\lambda_3\xi)^2 - \lambda_4\right]\xi^8$$

J. Zinn-Justin, Quantum Field Theory and Critical Phenomena

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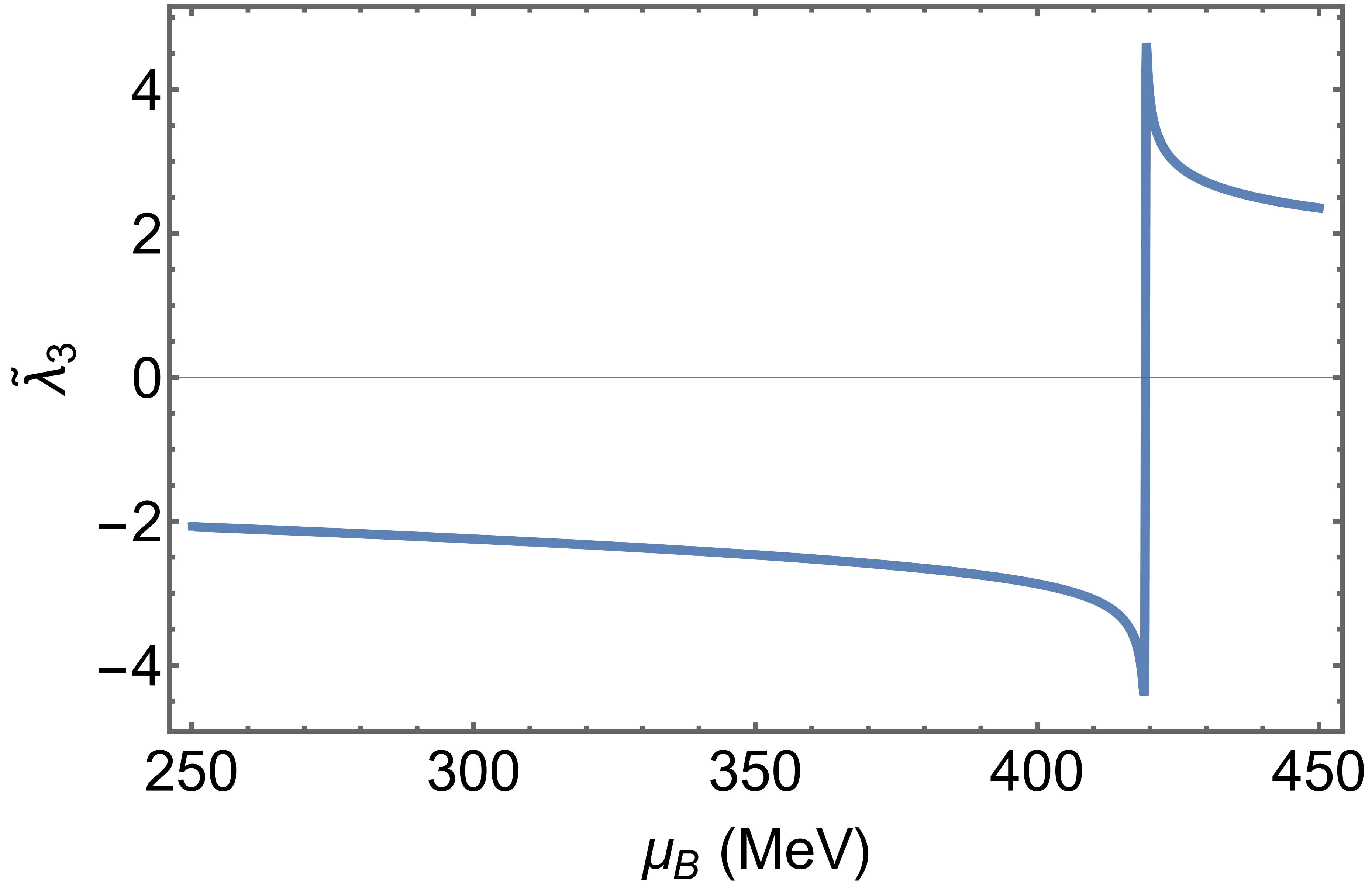
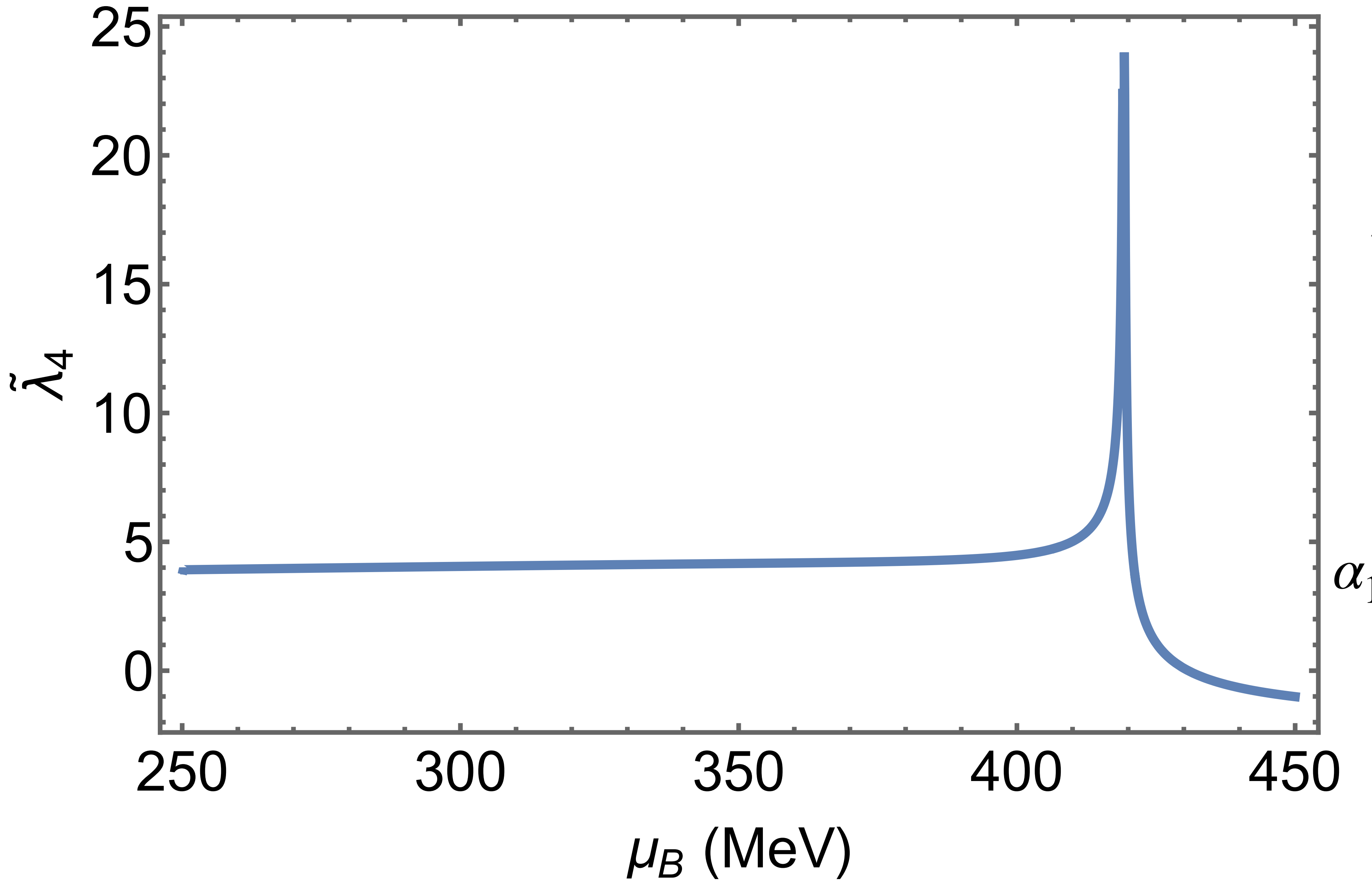
J. Zinn-Justin, Quantum Field Theory and Critical Phenomena

Extracting Higher-point Couplings

- Determine dimensionless couplings and their μ_B-dependence along the crossover as a further constraint from universality: $\tilde{\lambda}_n^{h=0,\, r\to 0^+}$



$|\tilde{\lambda}_3| \lesssim 8$



$|\tilde{\lambda}_4| \lesssim 20$

$\mu_{B,C} = 420\,\mathrm{MeV}$

$T_C = 141\,\mathrm{MeV}$

$w = 8,\ \rho = 0.2$

$\alpha_1 = 4.6^{\mathrm{o}} \quad \alpha_2 = -5.4^{\mathrm{o}}$

M. Stephanov, PRL (2011)

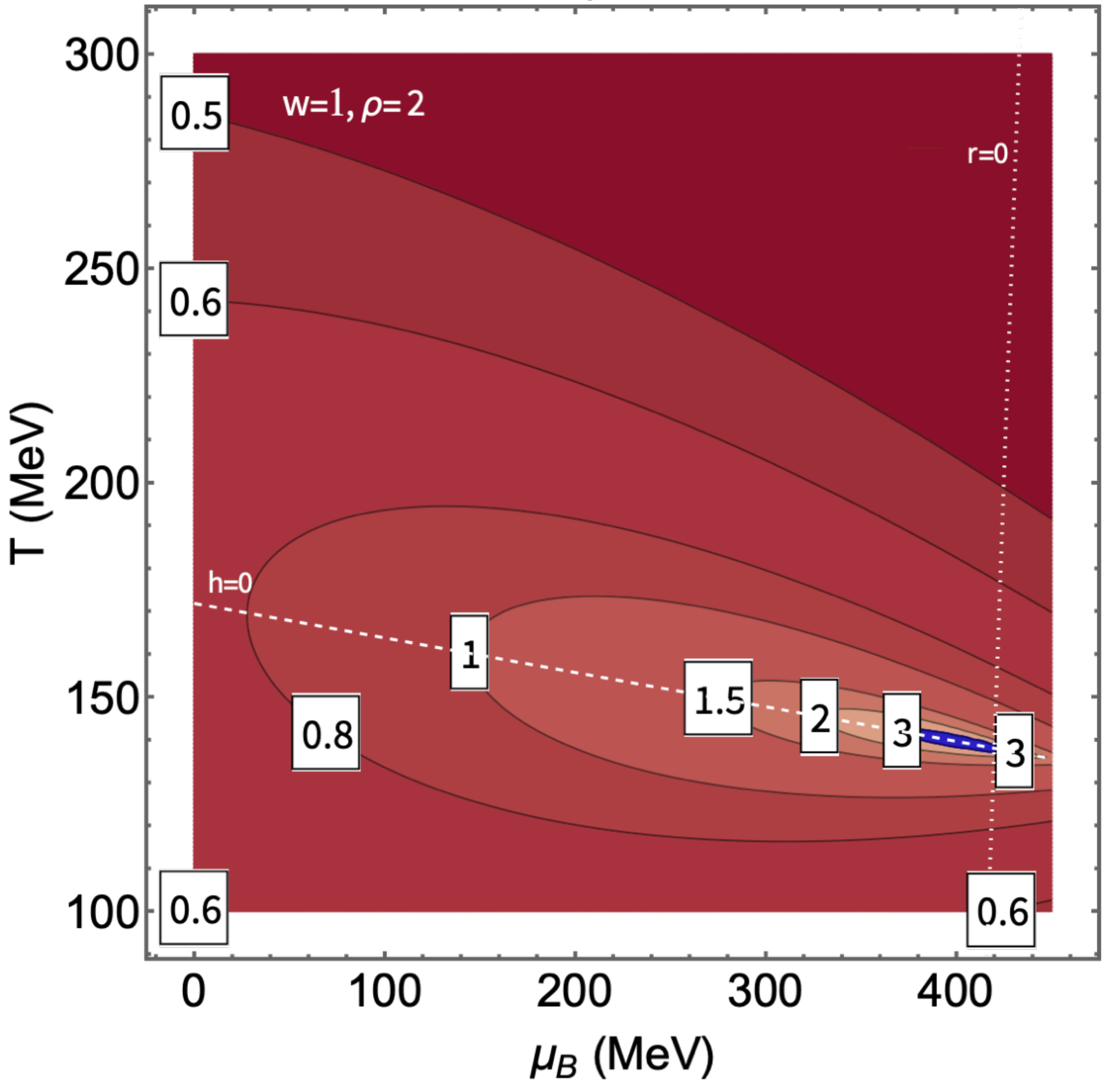
Parameter Constraints & Choices

- Additional universal consideration: angles constrained by smallness of physical quark mass, $\Delta\alpha_{1,2} \propto m_q^{2/5}$
- Non-universal choices remain: scaling parameters w, ρ

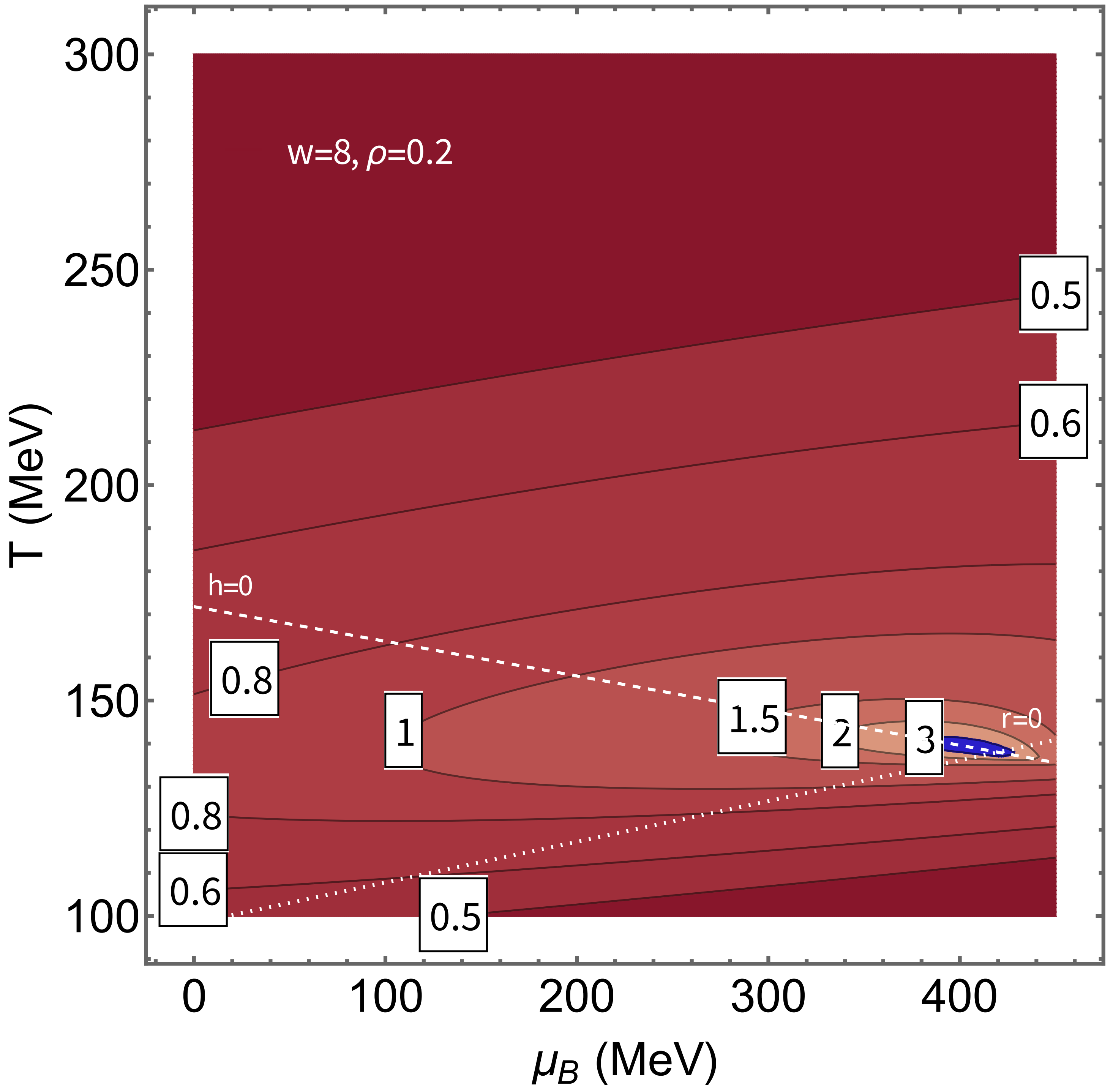
$$\frac{T - T_C}{T_C} = w\ (t\ \rho\ sin\alpha_1 + h\ sin\alpha_2)$$

$$\frac{\mu_B - \mu_{B,C}}{T_C} = -\,w\ (t\ \rho\ cos\alpha_1 + h\ cos\alpha_2)$$

Choice of original BEST EoS



Choice as of QM23



M. Pradeep and M. Stephanov, PRD (2019)

Coupling Constraint from MaxEnt

- Now we can also constrain the coupling by writing the critical fluctuations from the maximum entropy method in terms of the cumulants from the BEST EoS

$$\omega_p^k = \frac{C_k}{C_1}_{\text{crit}} \approx \frac{T^3}{n_p}\left(\frac{X^T \bar{H}^{-1} P}{w \sin(\alpha_1 - \alpha_2)}\right)^k \kappa_k(\mu, T)$$

and comparing to the Athanasiou *et al* approach

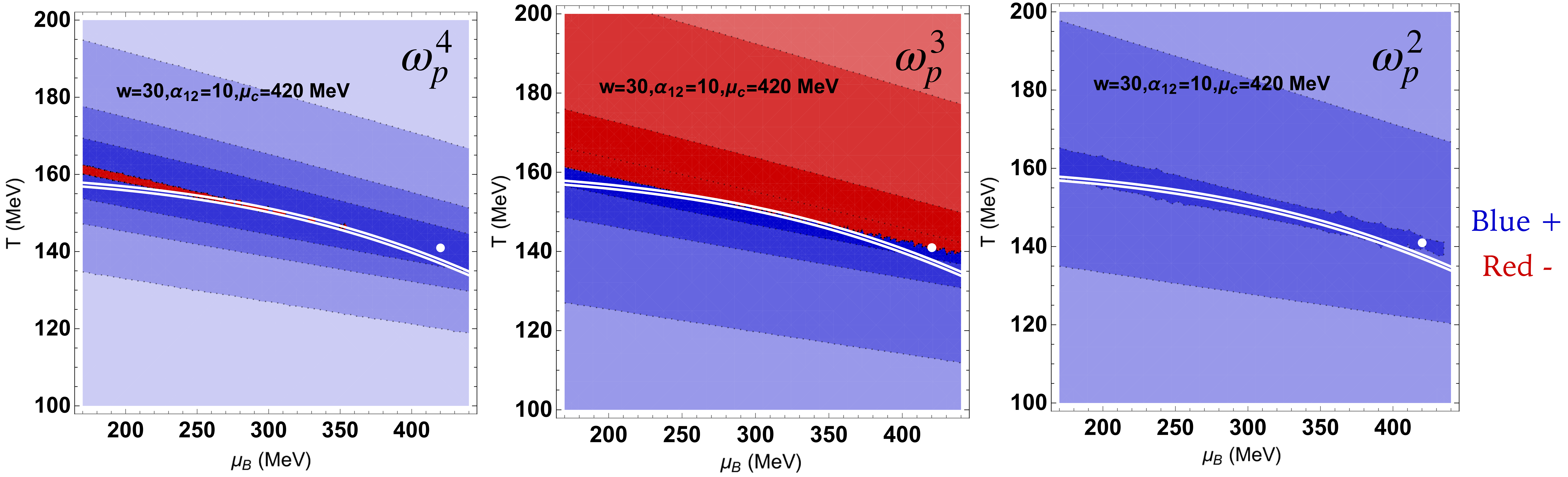
$$\tilde{\omega}_A^k = \frac{\hat{\Delta}\left\langle \delta N_A^k \right\rangle}{\langle N_A \rangle} = \frac{g_A^k}{n_A}\left(\int_q \frac{d_A f_A'}{T_f \gamma_q^A}\right)^k T_f^{3-2k} \frac{(h_0^2 f_0^{5-\eta})^{k/2}}{M_0 h_0 f_0^3} \kappa_k$$

- These expressions match for all k allowing for the determination of g_A in terms of the equation of state

M. Pradeep and M. Stephanov, PRL (2023)

Equilibrium Critical Normalized p Fluctuations

- Updated equilibrium estimates for factorial cumulants $\omega_p^k = \dfrac{C_k}{C_1}_{\text{crit}}$ with universal inputs



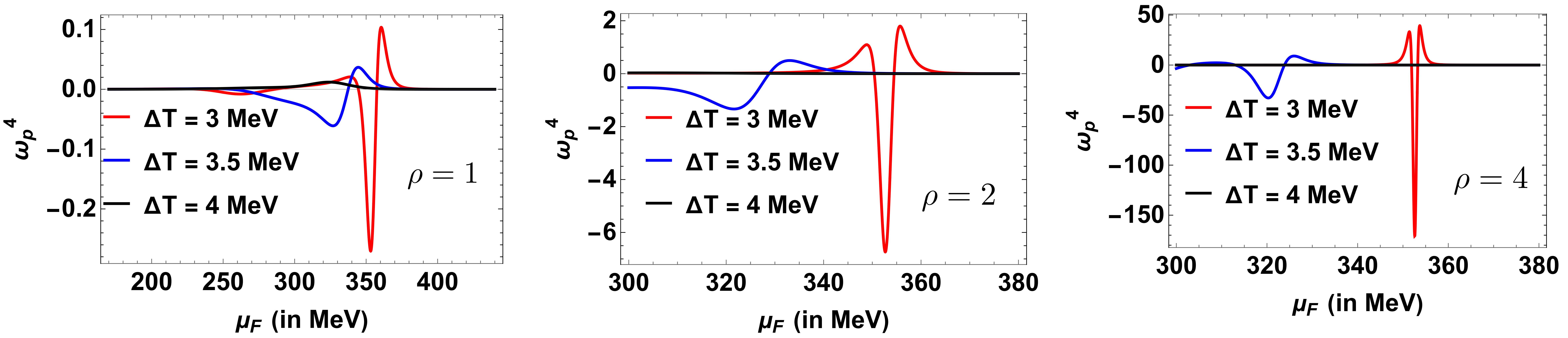
Example parameter choice:

$\mu_{B,C} = 420\,\text{MeV}$ $T_C = 141\,\text{MeV}$ $w = 30,\ \rho = 1$ $\alpha_1 = 4.6^o$ $\alpha_2 = -5.4^o$

J.M. Karthein, M. Pradeep, K. Rajagopal, M. Stephanov, Y. Yin, to appear

Equilibrium Critical Normalized p Fluctuations

- Behavior along freeze-out trajectories for various values of ρ scaling parameter
  - Very sensitive to the freeze-out line and non-universal choices
  - Features (dips & peaks) depend on freeze-out location relative to transition line



Freeze-out line from Andronic et al 2018 with variable additive constant: $T_c - T_f(\mu_c) = \Delta T$

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Conclusions

- Improvements of equilibrium results on fluctuations made possible by groundwork laid with BEST EoS
  - Additional inputs from universality considered including small angle difference and coupling constrained by maximum entropy approach
  - Future work: constrain parameters given new precise BES-II data
    - $\Delta\alpha, \mu_{B,c} w, \rho$ parameters most strongly constrained
- These equilibrium results form the basis of better out-of-equilibrium estimates
  - Future work: estimate dynamical effects & predict behavior at lower energies