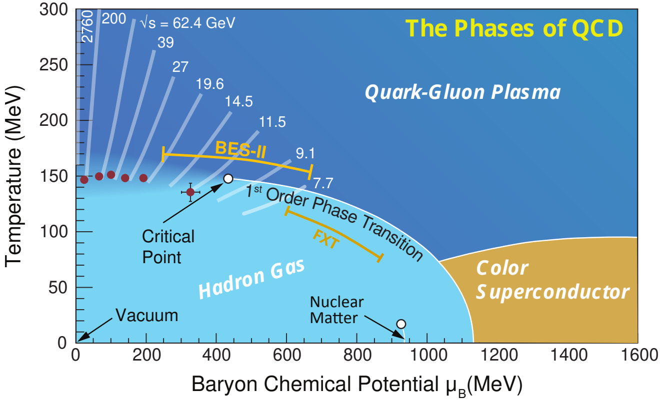
# QCD EOS, critical point, and hydrodynamic fluctuations

M. Stephanov

# QCD critical point

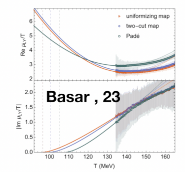
Where on the QCD phase boundary is the CP?




Motivation for BES at RHIC and BEST topical collaboration.

# Latest theory developments on locating CP

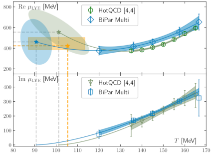
From Maneesha Pradeep's talk at CPOD 2024:




**G. Basar, Fri, 11:40 am**

$$(\mu_{BC}, T_c) \approx (580, 100)\ \mathrm{MeV}$$

Extrapolations of Lee-Yang edge singularities to real axis




**C. Schmidt, Tue, 11:00 am**

**Clarke et al., 24**

$$(\mu_{BC}, T_c) = (422^{+80}_{-35}, 105^{+8}_{-18})\ \mathrm{MeV}$$

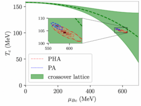
**Bayesian Holography + Lattice input at $\mu = 0$**

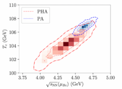
Hippert et al, e-Print: 2309.00579 [nucl-th]

*Predict CEP (95% confidence level):*

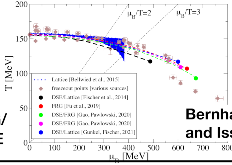
$$T_c = 101 - 108\ \mathrm{MeV} \quad \mu_c = 560 - 625\ \mathrm{MeV}$$

$$\sqrt{s} = 4.0 - 4.8\ \mathrm{GeV}$$




**J. Noronha, Tue, 11:40 am**




**C. Fischer, Tue, 11:20 am**

**FRG/ DSE**

**Bernhardt, Fischer and Isserstedt, 23**

$$(\mu_{BC}, T_c) = (495 - 654, 108 - 119)\ \mathrm{MeV}$$

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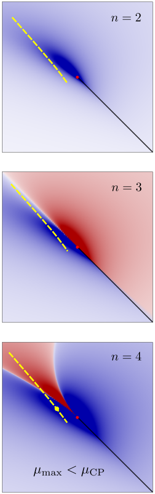
*(Ratti's talk)*

# Theory vs BES-II data

(universal EOS) critical $\chi_n$:

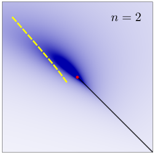


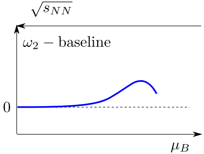

Bzdak et al review 1906.00936

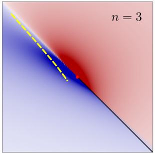
# Theory vs BES-II data

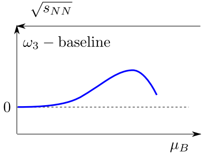
(universal EOS) critical $\chi_n$:

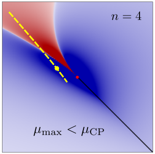
(irreducible correlations) $\mathrm{FC}_n[N_p] \sim \chi_n$ (Pradeep, MS 2211.09142), $\omega_n \equiv \mathrm{FC}_n/\mathrm{FC}_1$

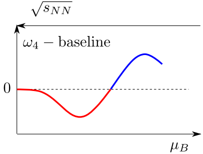
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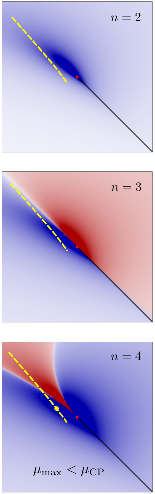


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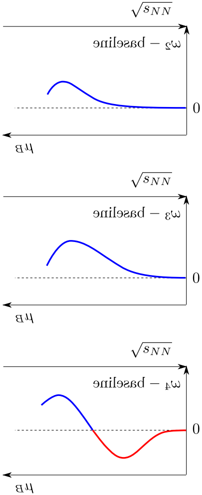


Bzdak et al review 1906.00936

Expected signatures: bump in $\omega_2$ and $\omega_3$, dip then bump in $\omega_4$

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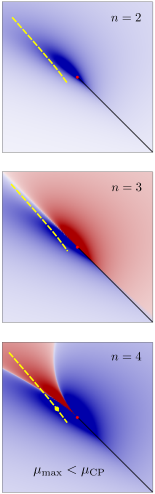


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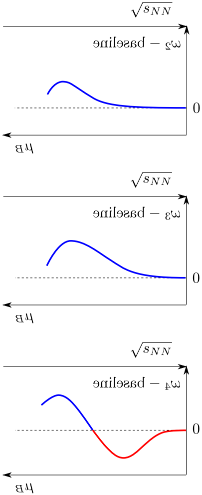


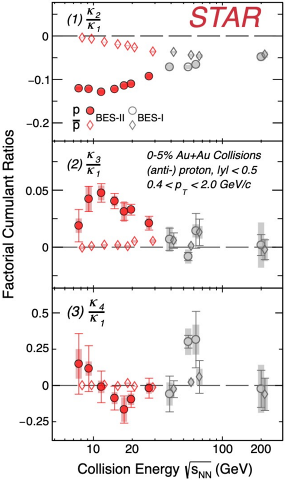


Bzdak et al review 1906.00936

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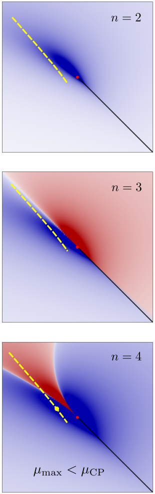


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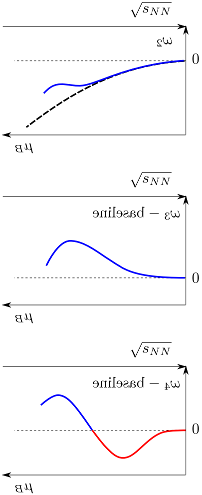


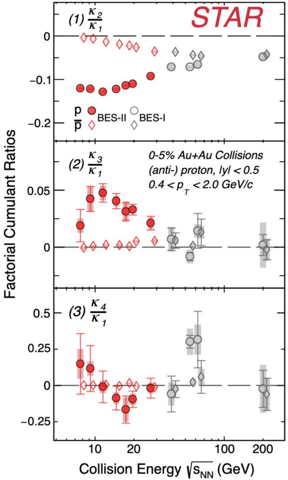


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# BEST Framework

The goal of BES theory: connect observables to QCD phase diagram.

*BEST framework: An et al (40+ authors, 100+ pp, 369 refs) 2108.13867*

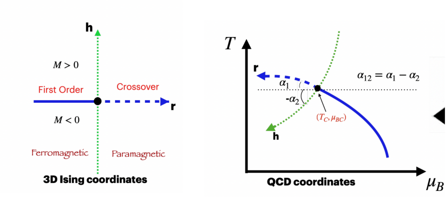
*BES theory review: Du, Sorensen, MS 2402.10183*

- Lattice EOS + CP $\rightarrow$ parametric EOS
- EOS $\rightarrow$ Hydrodynamics with (non-gaussian) fluctuations.
- Freezeout, including fluctuations. *reviewed in 2403.03255*
- Comparison with experiment. Bayesian analysis (ML).
  Determine/constrain EOS, critical point parameters.

# Parametric EOS (now with $T'$-expansion)

From Maneesha Pradeep's review talk at CPOD 2024:

$$P_{\mathrm{QCD}}(\mu,T) \;=\; P_{\mathrm{BG}}(\mu,T) \,+\, A\,G(r(\mu,T),h(\mu,T))$$

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**Kahangirwe et al., 24**

Independent & non-universal parameters

$$\mu_C \,,\; \alpha_{12} \,,\; \rho \,,\; w$$

Weakly constrained in the chiral limit

**MP, Stephanov, 19**

**Range of Validity improved**

$$0 \le \mu_B \le 700\,\mathrm{MeV}\,,\; 25\,\mathrm{MeV} \le T \le 800\,\mathrm{MeV}$$

The new construction is causal and stable for a larger range of $\rho$ and w

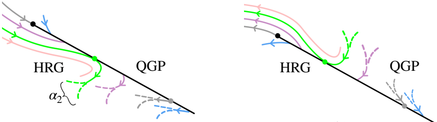
**M. Kahangirwe, Wed, 12:10 pm**

*Kahangirwe et al 2402.08636, PRD (also Jahan's and Karthein's talks)*

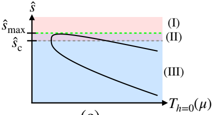
# Critical point and non-trivial hydro trajectories

*Pradeep, Sogabe, MS, Yee 2402.09519, PRC*

- $\hat{s} \equiv s/n$ is non-monotonic along coexistence (1st order) line
- non-trivial deformation of trajectories




(a) (b)

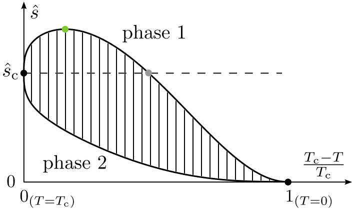



(c)

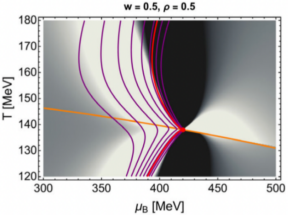
| | (a) | (b) |
|---|---|---|
| (I) | Crossover | |
| (II) | HRG → HRG | QGP → QGP |
| (III) | QGP → HRG | |

(d)

depending on $(\partial P/\partial T)_n$ at CP




explains “lensing”, “cusp”




**Critical lensing~Dore et al,22, Nonaka&Asakawa, 05**

# Deterministic approach to non-Gaussian fluctuations

*non-Gaussian* fluctuations are non-trivial and sensitive signatures of the critical point

- *Infinite* hierarchy of coupled equations *An et al 2009.10742, PRL*

  for connected hydro correlators $H_n \equiv \langle \underbrace{\delta\breve{\psi} \ldots \delta\breve{\psi}}_{n} \rangle^{\text{connected}}$:

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi, H, H_3, H_4, \ldots];$$

$$\partial_t H = \mathsf{F}[\psi, H, H_3, H_4, \ldots];$$

$$\partial_t H_3 = \mathsf{F}_3[\psi, H, H_3, H_4, \ldots];$$

$$\vdots$$

# Controlled perturbation theory

*An et al* 2009.10742*, PRL*

- *Small* fluctuations are *almost* Gaussian
- Introduce expansion parameter $\varepsilon$, so that $\delta\breve{\psi} \sim \sqrt{\varepsilon}$.

  Then $H_n \equiv \varepsilon^{n-1}$ and to leading order in $\varepsilon$:

$$\partial_t \psi = -\nabla \cdot (\text{Flux}[\psi] + \mathcal{O}(\varepsilon));$$

$$\partial_t H = -2\Gamma(H - \bar{H}[\psi]) + \mathcal{O}(\varepsilon^2);$$

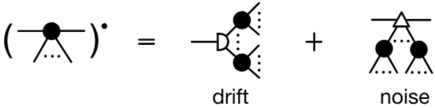
$$\vdots$$

$$\partial_t H_n = -n\Gamma(H_n - \bar{H}_n[\psi, H, \ldots, H^{n-1}]) + \mathcal{O}(\varepsilon^n);$$

  To leading order, the equations are iterative and "linear".
- In hydrodynamics the small parameter is $(q/\Lambda)^3$, i.e., fluctuation wavelength $1/q \gg$ size of hydro cell $1/\Lambda$ (UV cutoff).

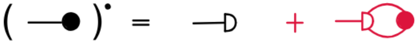
*An et al 2009.10742, 2212.14029, An's talk at CPOD 2024*

- Leading order in $\varepsilon$ $\Leftrightarrow$ tree diagrams.




all *combinatorial* configurations of trees

- Loops describe feedback of fluctuations (renormalization and long-time tails).



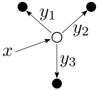
1-pt equation including leading loop

*conventional hydro equations*

*one loop (renormalization & long-time tails)*

# Generalizing Wigner transform

*An et al 2009.10742, PRL*



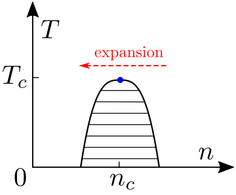

- Definition:

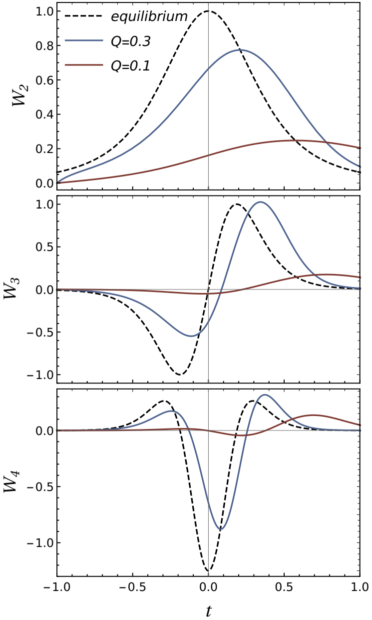
$$W_n(\boldsymbol{x};\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n) \equiv \int d\boldsymbol{y}_1^3 \ldots \int d\boldsymbol{y}_n^3 \, H_n\left(\boldsymbol{x}+\boldsymbol{y}_1,\ldots,\boldsymbol{x}+\boldsymbol{y}_n\right) \delta^{(3)}\left(\frac{\boldsymbol{y}_1+\ldots+\boldsymbol{y}_n}{n}\right) e^{-i(\boldsymbol{q}_1\cdot\boldsymbol{y}_1+\ldots+\boldsymbol{q}_n\cdot\boldsymbol{y}_n)};$$

- $W_n$'s quantify magnitude and non-gaussianity of fluctuation harmonics with wave-vectors $\boldsymbol{q}_i$.

# Example: expansion through a critical region

*An et al 2009.10742, PRL*




- Two main features:
  - Lag, "memory".
  - Smaller $Q$ – slower evolution. Conservation laws.

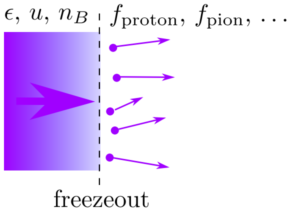
# Freezeout of fluctuations

- Freezeout: translation of correlators of hydrodynamic fluctuations ($\psi = \epsilon, n_B, u$)

  $\langle \delta\psi \ldots \delta\psi \rangle = H_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$

  to particle correlators

  $\langle \delta f \ldots \delta f \rangle = G_n(\boldsymbol{x}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{p}_n)$.




- *Conservation laws* relate $\boldsymbol{p}$ integrals of $G_n$ to $H_n$.

- But the $\boldsymbol{p}$ dependence in $G_n$ is not constrained.

  There are $\infty$ many possibilities/solutions ($G_n$) satisfying conservation laws.

# Maximum entropy freezeout

*Pradeep, MS, 2211.09142, PRL*

- There is a unique solution which maximizes the entropy!
  - for $n = 1$ equivalent to Cooper-Frye
  - for critical fluctuations similar to the $\sigma$ field coupling
  - but applies much more generally
  - model independent, i.e., determined by QCD EOS

$$\underbrace{\hat{\Delta} G_{ABC}}_{\text{irreducible particle correlations (FC)}} = \underbrace{\hat{\Delta} H_{abc}}_{\text{hydrodynamic correlations}} \underbrace{(\bar{H}^{-1} P \bar{G})^a_A (\bar{H}^{-1} P \bar{G})^b_B (\bar{H}^{-1} P \bar{G})^c_C}_{\text{kinematic factors}}$$

- Work in progress – implement in a hydro model and estimate *nonequilibrium* expectations for multiplicity cumulants in BES

*Karthein, Pradeep, MS, Rajagopal, Yin; Karthein's talk*

# Summary

- BES-II data is in.

  Qualitatively agrees with non-monotonic expectations from CP, not only in $n = 4$ factorial cumulant, but $n = 3$ and $n = 2$.

- To produce such signatures the CP has to be at $\mu_B > 420$ MeV.

  Agreement with recent theory estimates by different approaches.

- To convert these qualitative statements into quantitative ones, i.e., provide constraints on the QCD EOS from BES-II data more work is needed and is underway.

# More

# Factorial Cumulants are better experimental measures

Three reasons:

- Normal cumulants (NC) measure non-gaussianity;
  Factorial cumulants (FC) measure non-poissonianity,
  (irreducible particle correlations).

  NCs are for densities (continuous);
  FCs are for multiplicities (discrete).

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- Acceptance dependence:

  FCs are powers of $\Delta y$ for small $\Delta y$; NCs are polynomials.

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- Maximum Entropy freezeout *(Pradeep, MS 2211.09142)*:

  FCs of multiplicities are directly related to hydrodynamic correlators (or susceptibilities in thermodynamics).