4D-*T*ExS: A new 4D lattice-QCD equation of state with extended density coverage

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in collaboration with
Ahmed Abuali, Szabolcs Borsányi, Micheal Kahangirwe, Paolo Parotto, Attila Pásztor, Claudia Ratti, Hitansh Shah and Seth Trabulsi

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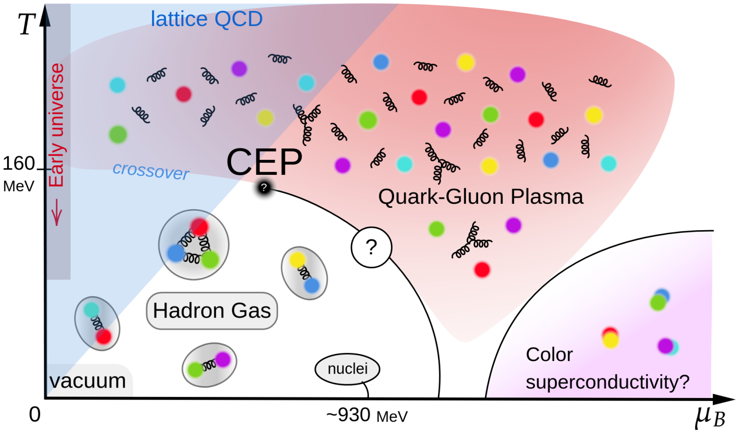
1. Introduction
   - Thermodynamics of QCD
   - Lattice QCD
2. T'-Expansion Scheme
3. 4D-TExS
4. Conclusion & Outlooks

Phase diagram of nuclear matter

Understanding thermodynamics of nuclear matter, or how matter in atomic nuclei behaves under extreme conditions like in the early Universe ⇔ studying the phase diagram of nuclear matter.



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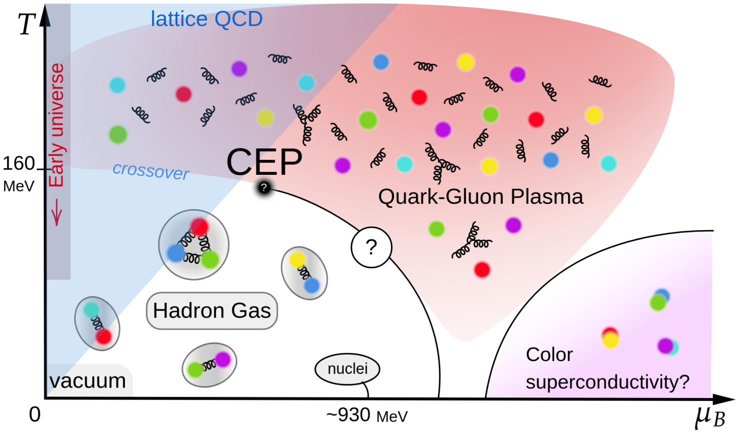

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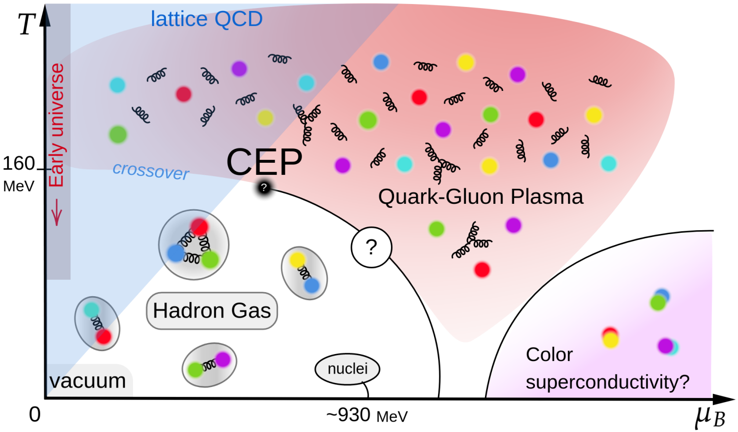
- 1st order phase transition + critical endpoint (CEP) predicted by by extrapolation from the chiral limit and several models (PNJL, fRG, holography...)



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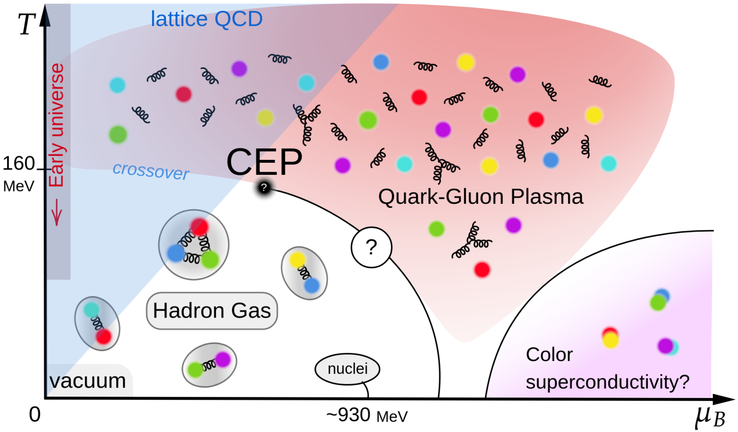
- colour superconductivity expected at low-T / high-μ_B



Determining thermodynamics of nuclear matter from Taylor expansion

Among the different ways to calculate the EoS of nuclear matter, lattice QCD is the most accurate way to get thermodynamics directly from QCD first principles.

[1] D'Elia *et al.*, PRD 95, 094503 (2017) / Bazavov *et al.*, PRD 101, 074502 (2020) / Borsányi *et al.*, arXiv:2312.07528

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$$\frac{P(T,\hat{\mu}_B,\hat{\mu}_S,\hat{\mu}_Q)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{BSQ}(T)\, \hat{\mu}_B^i\, \hat{\mu}_Q^j\, \hat{\mu}_S^k \qquad \left(\text{with } \hat{\mu}_i = \frac{\mu}{T}\right)$$

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with Taylor coefficients χ^{BQS}_{ijk} (susceptibilities), computed on the lattice at zero chemical potentials:

$$\chi^{BQS}_{ijk}(T) = \left.\frac{\partial^{i+j+k}(P/T^4)}{\partial\hat{\mu}_B^i \hat{\mu}_S^j \hat{\mu}_Q^k}\right|_{\hat{\mu}_B,\hat{\mu}_S,\hat{\mu}_Q=0}$$

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4D-Taylor EoS built from continuum extrapolated diagonal + off-diagonal $\chi_{2/4}^{BQS}(T)$

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Despite recent results for χ_6 and χ_8 [1]:

- still limited to $\mu_i/T \lesssim 2.5$ for $i = B,Q,S$
- lack of convergence due to large errors on high order terms which dominate at high $\hat{\mu}$
- expansion achieved at $T=$const, missing out the curved behaviour of pseudo-critical line

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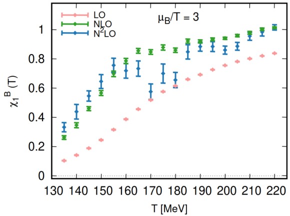

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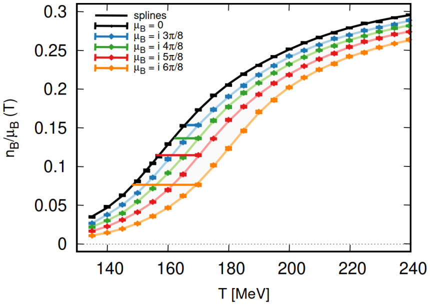
A novel expansion scheme for lattice QCD EoS at finite μ_B



Simulations at $\mathrm{Im}(\hat{\mu}_B)$: T-dependence of normalised baryon density $\left(\chi_1^B = n_B/T^3\right)$ at finite $\hat{\mu}_B$ appears to be shifted from the value at $\hat{\mu}_B = 0$.

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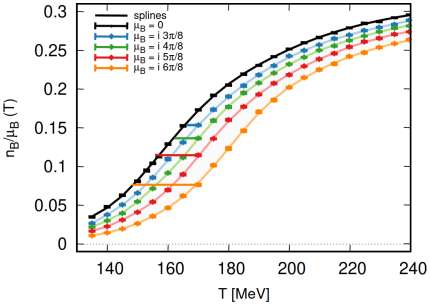

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For the 0/0 limit, we have: $\frac{\chi_1^B(T,\hat{\mu}_B)\to 0}{\hat{\mu}_B \to 0} \to \frac{\partial \chi_1^B}{\partial \hat{\mu}_B} = \chi_2^B$

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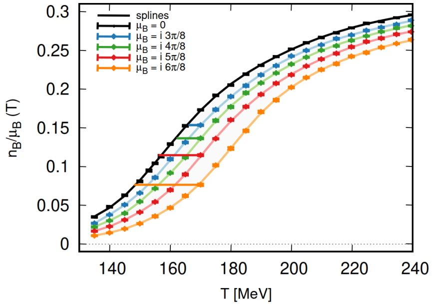

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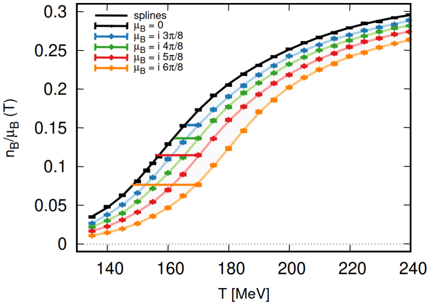

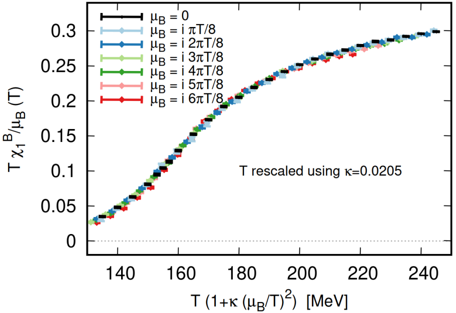

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with $\mathbf{T'(T,\hat{\mu}_B) = T\left(1 + \kappa_2\,.\,\hat{\mu}_B^2 + \kappa_4\,.\,\hat{\mu}_B^4 + \dots\right)}$

captures the finite $\hat{\mu}_B$ dependence of the expansion

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2D equation of state from T'-Expansion Scheme

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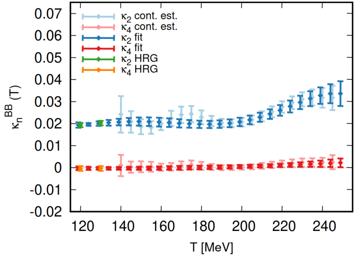

New *TExS EoS* based on coefficients $\kappa_{2/4}^{BB}(T)$ evaluated directly from lattice QCD simulations at $\mu_B = 0$

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- $\kappa_4^{BB}(T,0) = \frac{1}{360T \times \chi_2^{\prime B}(T)^3}\left(3\chi_2^{\prime B}(T)\times\chi_6^B(T) - 5\chi_2^{\prime\prime B}(T)\times\chi_4^B(T)^2\right)$

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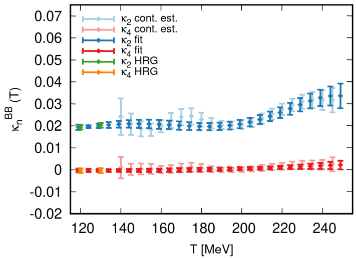

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$\Rightarrow$ Clear separation of scales between $\kappa_2(T)$ and $\kappa_4(T)$

$\Rightarrow$ $\kappa_4(T)$ is almost 0 $\rightarrow$ faster convergence

$\Rightarrow$ $\kappa_{2/4}(T)$ has a smooth T-dependence

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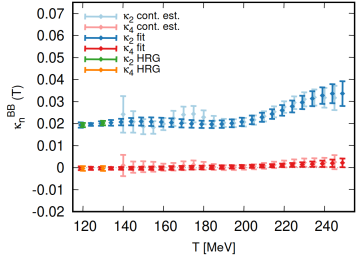
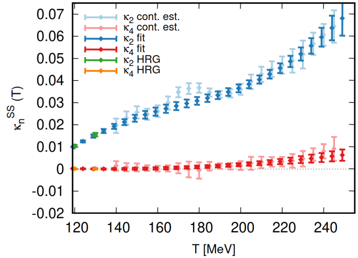
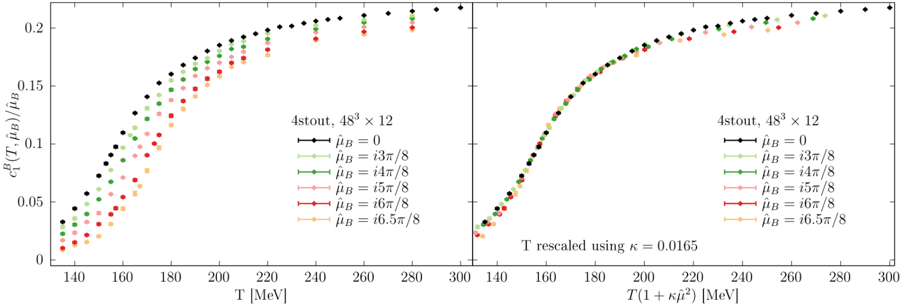




Applying Stefan-Boltzmann limit normalisation



To ensure that our main identity holds when $T \to \infty$,
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$\overline{\chi}_1^B(\hat{\mu}_B)$ and $\overline{\chi}_2^B(0)$:

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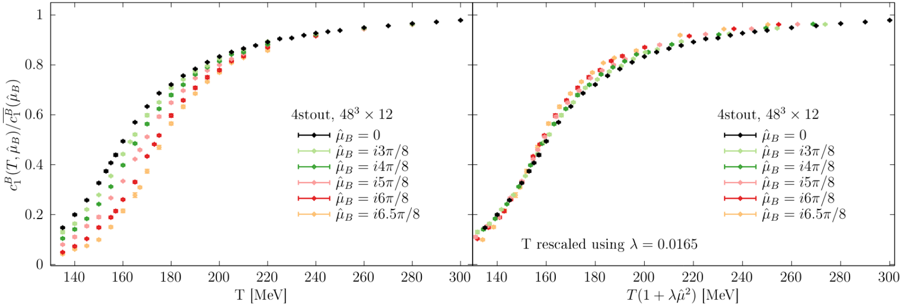

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This leads to redefine:

$$T'(T,\mu_B) = T\left(1 + \lambda_2^{BB}(T)\hat{\mu}_B^2 + \ldots\right)$$

with the new expansion coef. embedding the S.B. limit:

$$\lambda_2^{BB}(T) = \frac{1}{6T\chi_2'^B(T)} \times \left(\chi_4^B(T) - \frac{\overline{\chi}_4^B(0)}{\overline{\chi}_2^B(0)}\chi_2^B(T)\right)$$

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1 Introduction

2 T'-Expansion Scheme

3 4D-TExS
- Motivation
- Extending TExS to multiple conserved charges B, Q and S
- Preliminary results for thermodynamics

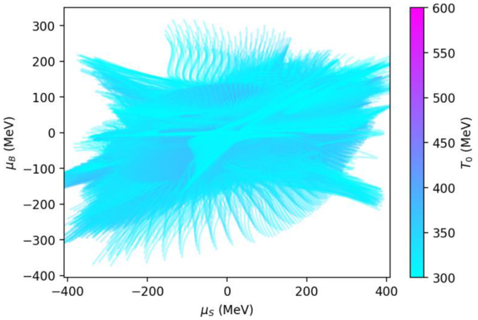
4 Conclusion & Outlooks

Why do we need a 4D EoS with extended coverage?

- **Hydrodynamics simulation for HIC becomes more accurate and realistic:**
  need to go beyond usual criteria of strangeness neutrality ($\langle n_S \rangle = 0$) and global charge conservation ($n_Q = 0.4 n_B$)

  → offer an EoS with 3 independent (μ_B, μ_Q, μ_S) which goes beyond the limit of Taylor ($\hat{\mu}_i \lesssim 2.5$) and is better suited for simulations at lower collision energies

Almaalol, talk at QM 2023



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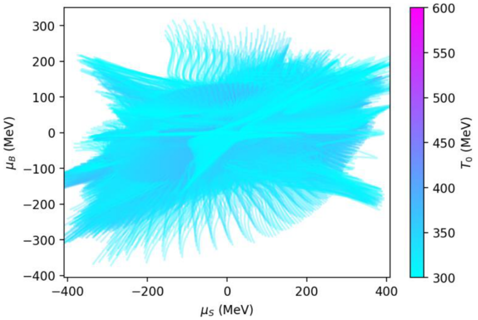

- **Entering a new era for astrophysics with the observation of NS mergers:**
  merger simulations also employs hydrodynamics which need an EoS going to finite μ_B and finite μ_I (related to μ_Q)

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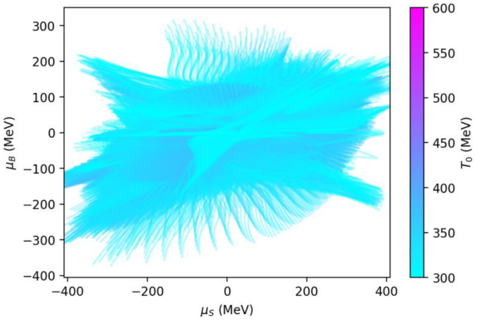

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⇒ **Why not generalising the T'-Expansion Scheme to several conserved charges..?**

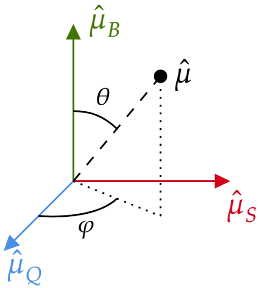
Construction of the new scheme - Basics

One can chose to project the $(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S)$ Cartesian coordinate system into a spherical one using $(\hat{\mu}, \theta, \varphi)$, following the relations:

$$\hat{\mu}_B = \hat{\mu}\,.\,\cos(\theta) \qquad \hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_Q^2 + \hat{\mu}_S^2}$$

$$\hat{\mu}_Q = \hat{\mu}\,.\,\sin(\theta)\cos(\varphi) \iff \varphi = \arccos\left(\frac{\hat{\mu}_Q}{\sqrt{\hat{\mu}_Q^2 + \hat{\mu}_S^2}}\right)$$

$$\hat{\mu}_S = \hat{\mu}\,.\,\sin(\theta)\sin(\varphi) \qquad \theta = \arccos\left(\frac{\hat{\mu}_B}{\hat{\mu}}\right)$$



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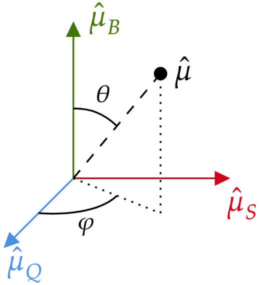
$$\begin{aligned} \hat{\mu}_B &= \hat{\mu}\,.\cos(\theta) & & & \hat{\mu} &= \sqrt{\hat{\mu}_B^2 + \hat{\mu}_Q^2 + \hat{\mu}_S^2} \\ \hat{\mu}_Q &= \hat{\mu}\,.\sin(\theta)\cos(\varphi) & &\Longleftrightarrow & \varphi &= \arccos\left(\frac{\hat{\mu}_Q}{\sqrt{\hat{\mu}_Q^2 + \hat{\mu}_S^2}}\right) \\ \hat{\mu}_S &= \hat{\mu}\,.\sin(\theta)\sin(\varphi) & & & \theta &= \arccos\left(\frac{\hat{\mu}_B}{\hat{\mu}}\right) \end{aligned}$$



Simple way to **reduce** the problem **from 4D to 2D**: a single $\hat{\mu}$ projected along a given direction in the 3D $(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S)$ space.

$\rightarrow$ All previous equations from the 2D-TExS can be used as is!

Redefinition of the lattice-based Taylor coefficient

We introduce then X_2, a "generalised 2nd order susceptibility" at $\hat{\mu} = 0$:

$$X_2^{\theta,\varphi}(T) = \left.\frac{\partial^2 P/T^4}{\partial \hat{\mu}^2}\right|_{\hat{\mu}=0}$$
$$= c_\theta^2 \cdot \chi_2^B(T) + s_\theta^2 c_\varphi^2 \cdot \chi_2^Q(T) + s_\theta^2 s_\varphi^2 \cdot \chi_2^S(T)$$
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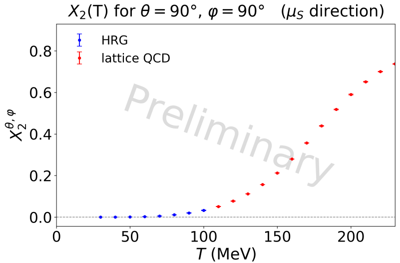

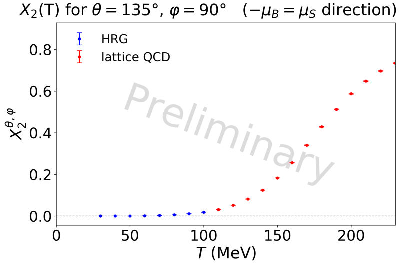

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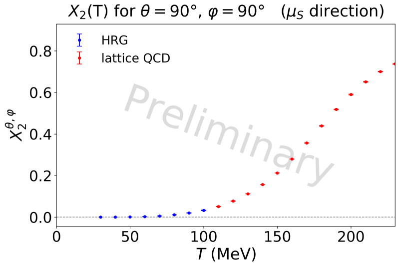

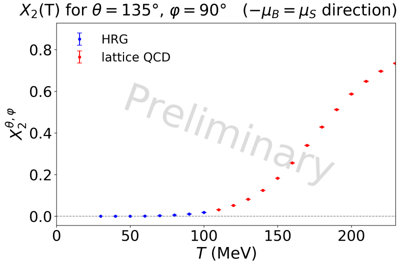

The expansion coefficient $\lambda_2^{\theta,\varphi}(T)$

From there, we can build the generalised 2nd order expansion coefficient λ_2:

$$\lambda_2^{\theta,\varphi}(T) = \frac{1}{6T}\frac{1}{X_2'^{\theta,\varphi}(T)} \times \left(X_4^{\theta,\varphi}(T) - \frac{\overline{X}_4^{\theta,\varphi}(0)}{\overline{X}_2^{\theta,\varphi}(0)} X_2^{\theta,\varphi}(T) \right)$$

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embedding the S.B. limit correction $\left(\overline{\lambda}_2 = \lim_{T\to\infty}(\lambda_2) = 0\right)$, with $\overline{X}_{2/4}^{\theta,\varphi}(0)$ being the S.B. limits for $X_{2/4}^{\theta,\varphi}(T)$ at $\hat{\mu} = 0$. We employ

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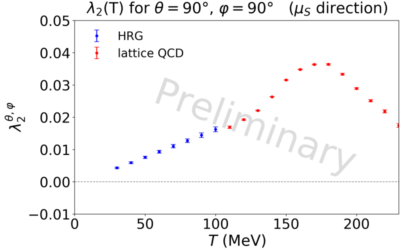

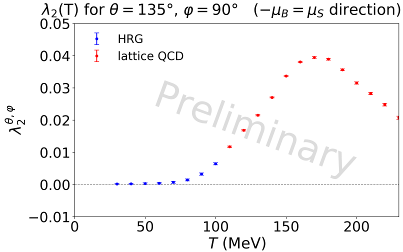

From shifted temperature T' to generalised charge density X_1

Using the previously obtained expansion coefficient $\lambda_2^{\theta,\varphi}(T)$, one can build the shifted temperature expansion $T'^{\theta,\varphi}(T,\hat{\mu})$:

$$T'^{\theta,\varphi}(T,\hat{\mu}) = T\left(1+\lambda_2^{\theta,\varphi}(T)\hat{\mu}_B^2\right)$$

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$$\boxed{X_1^{\theta,\varphi}(T,\hat{\mu}) = \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} \times X_2^{\theta,\varphi}\left(T'^{\theta,\varphi}(T,\hat{\mu}),0\right)}$$

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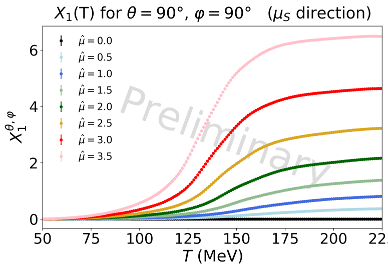

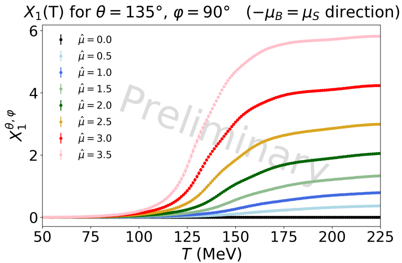

[a] Borsányi *et al.*, JHEP 01 (2012) 138
[b] Bellwied *et al.*, PRD 101 (2020) 3, 034506
[c] Borsányi *et al.*, PRL 126 (2021) 23, 232001

Pressure

We integrate $X_1^{\theta,\varphi}(T,\hat{\mu})$ to compute the pressure:

$$P^{\theta,\varphi}(T,\hat{\mu}) = P(T,0) + \int_0^{\hat{\mu}} X_1^{\theta,\varphi}(T,\hat{\mu}')d\hat{\mu}'$$

$$= P(T,\hat{\mu}_B,\hat{\mu}_Q,\hat{\mu}_S)$$

using lattice results for $P(T,0)$ with recent precision improvement from the Wuppertal-Budapest collaboration.[a]

Examples:

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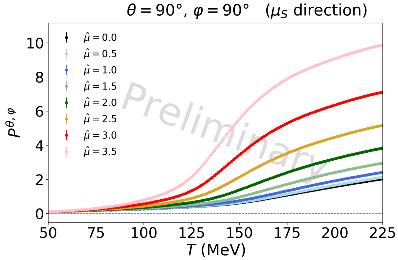

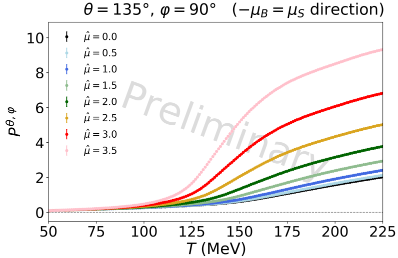

[a] P. Parotto, talk at QM 2023

Entropy density

The expression for entropy density s is given as:

$$s^{\theta,\varphi}(T,\hat{\mu}) = \left.\frac{\partial P}{\partial T}\right|_{\mu} = \frac{\partial}{\partial T}\left[\int_0^{\mu'} X_1^{\theta,\varphi}(T,\hat{\mu})d\mu'\right]_{\mu}$$

$$= s(T,0) + \int_0^{\mu'} \frac{\partial}{\partial T}\left[\frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)}\right]_{\mu} \times X_2^{\theta,\varphi}(T',0)d\mu'$$

$$+ \int_0^{\mu'} \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} \times \frac{\partial T'}{\partial T} \times \frac{\partial X_2^{\theta,\varphi}(T',0)}{\partial T'} d\mu'$$

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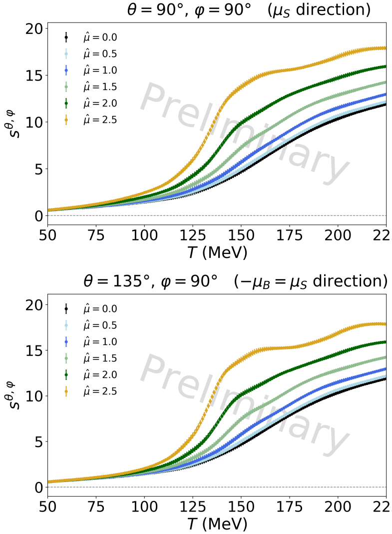


Charge densities & entropy density

One can then compute energy density ε as:

$$\varepsilon^{\theta,\varphi}(T,\hat{\mu}) = s.T - P + \sum_{i=B,Q,S} \mu_i \times n_i$$

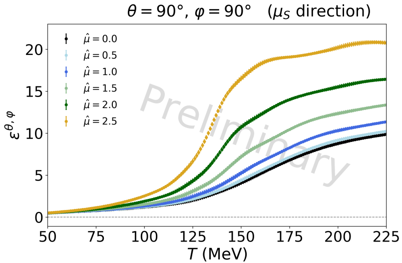
$$= s.T - P + \mu_B \times n_B + \mu_Q \times n_Q + \mu_S \times n_S$$

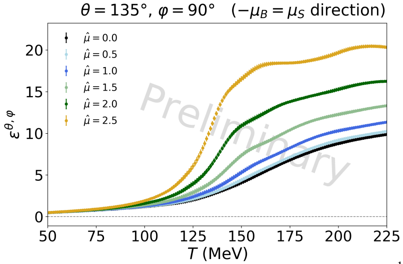
$$= s.T - P + \mu \times (c_\theta \cdot \chi_1^B + s_\theta c_\varphi \cdot \chi_1^Q + s_\theta s_\varphi \cdot \chi_1^S)$$

$$= s.T - P + \mu \times X_1^{\theta,\varphi}$$

Examples:

- for $(\theta = 90°, \varphi = 90°)$, $\varepsilon(T,\mu_B,\mu_S) = s.T - P + \mu_S n_S$
- for $(\theta = 135°, \varphi = 90°)$, $\varepsilon(T,\mu_B,\mu_S) = s.T - P + \frac{\mu_S}{\sqrt{2}} n_S - \frac{\mu_B}{\sqrt{2}} n_B$





1 Introduction

2 T'-Expansion Scheme

3 4D-TExS

4 Conclusion & Outlooks

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We present a new 4D lattice-based EoS construction using the T'-Expansion Scheme to extend the coverage from the 4D Taylor expansion ($\hat{\mu} \lesssim 2.5$) up to $\hat{\mu} \sim 3.5$.

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- We have generalized the T'-Expansion Scheme to 4D by computing $X_{2/4}^{\theta,\varphi}(T,\mu)$ from lattice data at $\hat{\mu} = 0$
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Disclaimer: error shown in the preliminary results of this talk are underestimated
→ need to complete the analysis of error consistently

Additional material

Complete formula for $X_4^{\theta,\varphi}(T,\hat{\mu})$

$$
\begin{aligned}
X_4^{\theta,\varphi}(T,\hat{\mu}) = {} & c_\theta^4 \cdot \chi_4^B(T,0) + s_\theta^4 c_\varphi^4 \cdot \chi_4^Q(T,0) + s_\theta^4 s_\varphi^4 \cdot \chi_4^S(T,0) \\
& + 4c_\theta^3 s_\theta c_\varphi \cdot \chi_{31}^{BQ}(T,0) + 4c_\theta^3 s_\theta s_\varphi \cdot \chi_{31}^{BS}(T,0) + 4s_\theta^3 c_\varphi^3 s_\varphi \cdot \chi_{31}^{QS}(T,0) \\
& + 4c_\theta s_\theta^3 c_\varphi^3 \cdot \chi_{13}^{BQ}(T,0) + 4c_\theta s_\theta^3 s_\varphi^3 \cdot \chi_{13}^{BS}(T,0) + 4s_\theta^3 c_\varphi s_\varphi^3 \cdot \chi_{13}^{QS}(T,0) \\
& + 6c_\theta^2 s_\theta^2 c_\varphi^2 \cdot \chi_{22}^{BQ}(T,0) + 6c_\theta^2 s_\theta^2 s_\varphi^2 \cdot \chi_{22}^{BS}(T,0) + 6s_\theta^4 c_\varphi^2 s_\varphi^2 \cdot \chi_{22}^{QS}(T,0) \\
& + 12c_\theta^2 s_\theta^2 c_\varphi s_\varphi \cdot \chi_{211}^{BQS}(T,0) + 12c_\theta s_\theta^3 c_\varphi^2 s_\varphi \cdot \chi_{121}^{BQS}(T,0) + 12c_\theta s_\theta^3 c_\varphi s_\varphi^2 \cdot \chi_{112}^{BQS}(T,0)
\end{aligned}
$$

Lattice QCD datasets

To compute $\lambda_2^{\theta,\varphi}(T)$: continuum extrapolated $\chi_{2/4}^{BQS}$ from the latest LT=2 (small volume) 4HEX WB data.

To compute $X_1^{\theta,\varphi}(T,\hat{\mu})$: continuum extrapolated $\chi_{2/4}^{BQS}$ from the LT=4 ($\approx \infty$ volume) 4stout WB data.

Why do we mix 2 datasets? *($\approx$ lattice phenomenology)*

$\rightarrow$ 4HEX data don't have a sufficient coverage in T yet
$\rightarrow$ 4stout has too big errors in the transition region

BUT because of the small volume used for 4HEX (LT=2), more configurations simulated

$\rightarrow$ smaller errors

Checking the convergence of the 2D-TExS

Computed thermodynamics quantities in the 2D-TExS adding $\kappa_4^{BB}(T)$ (NLO in T' expansion).

$\rightarrow$ adding κ_4 only increases the errors but we see no change in the result overall