Entropy production and dissipation in spin hydrodynamics

A relativistic quantum-statistical approach

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Phys.Lett.B 850 (2024) 138533

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Motivations

- There is a growing interest in spin hydrodynamics,

[1] K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, "Fate of spin polarization in a relativistic fluid: An entropy-current analysis," *Phys. Lett. B* **795** (2019) 100–106, arXiv:1901.06615 [hep-th].

[2] K. Fukushima and S. Pu, "Spin hydrodynamics and symmetric energy-momentum tensors – A current induced by the spin vorticity –," *Phys. Lett. B* **817** (2021) 136346, arXiv:2010.01608 [hep-th].

[3] A. D. Gallegos, U. Gürsoy, and A. Yarom, "Hydrodynamics of spin currents," *SciPost Phys.* **11** (2021) 041, arXiv:2101.04759 [hep-th].

[4] D. She, A. Huang, D. Hou, and J. Liao, "Relativistic viscous hydrodynamics with angular momentum," *Sci. Bull.* **67** (2022) 2265–2268, arXiv:2105.04060 [nucl-th].

[5] M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, and H.-U. Yee, "Relativistic spin hydrodynamics with torsion and linear response theory for spin relaxation," *JHEP* **11** (2021) 150, arXiv:2107.14231 [hep-th].

[6] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke, "Relativistic second-order dissipative spin hydrodynamics from the method of moments," *Phys. Rev. D* **106** no. 9, (2022) 096014, arXiv:2203.04766 [nucl-th].

[7] R. Biswas, A. Daher, A. Das, W. Florkowski, and R. Ryblewski, "Relativistic second-order spin hydrodynamics: An entropy-current analysis," *Phys. Rev. D* **108** no. 1, (2023) 014024, arXiv:2304.01009 [nucl-th].

- Spin hydrodynamics involves spin. Therefore quantum methods cannot be avoided.

Remarks

- Spin hydrodynamics suffers from the pseudo-gauge non-invariance,

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\nabla_\lambda\left(\widehat{\Phi}^{\lambda\mu\nu} - \widehat{\Phi}^{\mu\lambda\nu} - \widehat{\Phi}^{\nu\lambda\mu}\right),$$
$$\widehat{S}'^{\mu\lambda\nu} = \widehat{S}^{\mu\lambda\nu} - \widehat{\Phi}^{\mu\lambda\nu}.$$

- While spin hydrodynamics holds theoretical promise, no experimental evidence has yet emerged to demonstrate its relevance for QGP description.

Machinery

One of the main goals of spin hydro is to determine the constituent equations of $T^{\mu\nu}$ and $\mathcal{S}^{\lambda\mu\nu}$. For this purpose, an educated guess of the thermodynamic relation is assumed:

$$Ts + \mu n = \rho + p - \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\mathrm{d}p = s\,\mathrm{d}T + n\,\mathrm{d}\mu + \frac{1}{2}S^{\mu\nu}\mathrm{d}\omega_{\mu\nu}$$

where $\mathcal{S}^{\mu\nu} = u_\lambda \mathcal{S}^{\lambda\mu\nu}$ and ω is Spin potential.

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The goal is to just use QFT and quantum density operators to derive the entropy current, the entropy production rate, whence the constitutive equations.

Quantum-statistical Framework for Relativistic Fluid

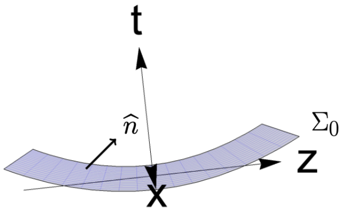
Local equilibrium is achieved at initial hypersurface Σ_0, where entropy is maximum provided that the mean vales of energy, momentum, particle number, and spin densities are their actual values:

$$S = -\,\mathrm{Tr}(\widehat{\rho}\log\widehat{\rho})$$

$$F[\widehat{\rho}] = -Tr[\widehat{\rho}\log\widehat{\rho}] - \int d\Sigma_0\, n_\mu\, (T^{\mu\nu}_{\mathrm{LE}} - T^{\mu\nu})\,\beta_\nu(x) - \int d\Sigma_0\, n_\mu\, (j^{\mu}_{\mathrm{LE}} - j^{\mu})\,\zeta(x) - \int d\Sigma_0\, n_\mu \left(S^{\mu\lambda\nu}_{\mathrm{LE}} - S^{\mu\lambda\nu}\right)\Omega_{\lambda\nu}(x)$$

$$T^{\mu\nu}_{\mathrm{LE}} \sim Tr\left[\widehat{\rho}\,\widehat{T}^{\mu\nu}\right]$$

$$T^{\mu\nu} \equiv \text{Actual Value}$$



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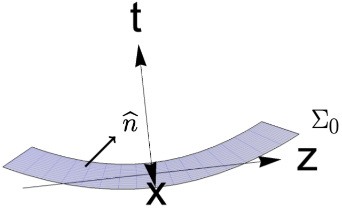

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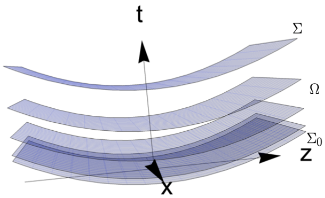
$$\widehat{\rho}_{\mathrm{LE}} = \frac{1}{Z}\exp\left[-\int_{\Sigma_0} \mathrm{d}\,\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{S}^{\mu\lambda\nu}\right)\right]$$

The Lagrange multipliers are obtained by solving the constraint equations at Σ_0. Their evolution is determined by solving the conservation equations:

- $\beta^\mu \to u^\mu = \beta^\mu/\sqrt{\beta^2} \quad T = 1/\sqrt{\beta^2}$
- $\zeta = \mu/T$
- $\Omega_{\mu\nu} = \omega_{\mu\nu}/T$
- Thermal Shear: $\xi_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu\right)$ Thermal Vorticity: $\varpi_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu \beta_\mu - \nabla_\mu \beta_\nu\right)$

At global equilibrium,

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \,, \quad \text{with} \;\; b, \varpi = \text{const} \,, \qquad \Omega = \varpi \,, \qquad \zeta = \text{const}$$



Using Gauss and Divergence theorems:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[\underbrace{-\int_{\Sigma} \mathrm{d}\,\Sigma_{\mu}\left(\widehat{T}^{\mu\nu}\beta_{\nu} - \zeta\widehat{j}^{\mu} - \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu\lambda\nu}\right)}_{\widehat{\rho}_{\mathrm{LE}}(t)\ at\ \Sigma} + \underbrace{\int_{\Omega} d\Omega\ \widehat{T}^{\mu\nu}_{S}\xi_{\mu\nu} + \widehat{T}^{\mu\nu}_{A}(\Omega_{\mu\nu} - \varpi_{\mu\nu}) - \frac{1}{2}\widehat{\mathcal{S}}^{\mu\lambda\nu}\nabla_{\mu}\Omega_{\lambda\nu}}_{\text{Dissipative Corrections}}\right]$$

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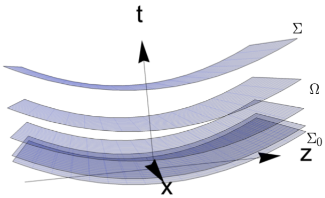

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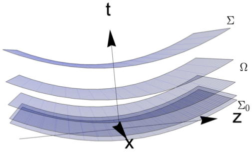
This implies that dissipation in spin hydrodynamics occurs when:

$$\xi \neq 0 \quad \Omega \neq \varpi \quad \nabla\Omega \neq 0$$

Entropy Current and Entropy-Gauge Transformation

Near local equilibrium at the hypersuface Σ, the entropy is defined as:

$$
\begin{aligned}
S &= -\operatorname{Tr}\left[\widehat{\rho}_{\mathrm{LE}}(t)\log\widehat{\rho}_{\mathrm{LE}}(t)\right] \\
&= \log Z_{\mathrm{LE}} + \int_{\Sigma} \mathrm{d}\Sigma_{\mu}\left[\operatorname{Tr}\left(\widehat{\rho}_{\mathrm{LE}}\widehat{T}^{\mu\nu}\right)\beta_{\nu} - \zeta\operatorname{Tr}\left(\widehat{\rho}_{\mathrm{LE}}\widehat{j}^{\mu}\right) - \frac{1}{2}\Omega_{\lambda\nu}\operatorname{Tr}\left(\widehat{\rho}_{\mathrm{LE}}\widehat{S}^{\mu\lambda\nu}\right)\right]
\end{aligned}
$$



Can we define an entropy current out of S ? In other words, is it possible to show that $\log Z_{\mathrm{LE}}$ is an extensive quantity?

$$\log Z_{\rm LE} \sim \int_\Sigma {\rm d}\Sigma_\mu \, \phi^\mu$$

[F. Becattini, D. Rindori PhysRevD.99.125011]

where ϕ^μ is defined as thermodynamic potential vector field:

$$\phi^\mu(x) = \int_0^{T(x)} \frac{{\rm d}T'}{T'^2} \left(T^{\mu\nu}_{\rm LE}(x)[T',\mu,\omega] u_\nu(x) - \mu(x) j^\mu_{\rm LE}(x)[T',\mu,\omega] - \frac{1}{2}\omega_{\lambda\nu}(x)\mathcal{S}^{\mu\lambda\nu}_{\rm LE}(x)[T',\mu,\omega] \right)$$

For a fluid at global equilibrium with vanishing thermal vorticity $\varpi_{\mu\nu} = 0$:

$$\phi^\mu = p\,\beta^\mu$$

where "p" is the hydrostatic pressure.

Therefore, entropy current exists:

$$S = \int_\Sigma \mathrm{d}\Sigma_\mu \, \phi^\mu + T^{\mu\nu}_{\mathrm{LE}} \beta_\nu - \zeta j^\mu_{\mathrm{LE}} - \frac{1}{2} \Omega_{\lambda\nu} \mathcal{S}^{\mu\lambda\nu}_{\mathrm{LE}}$$

$$s^\mu_{LE} = \phi^\mu + T^{\mu\nu}_{\mathrm{LE}} \beta_\nu - \zeta j^\mu_{\mathrm{LE}} - \frac{1}{2} \Omega_{\lambda\nu} \mathcal{S}^{\mu\lambda\nu}_{\mathrm{LE}} .$$

In quantum theory, we only have the total entropy, not the entropy current. We need to construct an entropy current through an integral. However, this introduces ambiguities, as several fields can lead to the same integral. However if $s^\mu - s^\mu_{LE} \perp n^\mu$,

$$s^\mu = \phi^\mu + T^{\mu\nu} \beta_\nu - \zeta j^\mu - \frac{1}{2} \Omega_{\lambda\nu} \mathcal{S}^{\mu\lambda\nu} \qquad \phi^\mu = \int_0^T \frac{\mathrm{d}T'}{T'^2} \left(T^{\mu\nu}[T'] u_\nu - \mu j^\mu[T'] - \frac{1}{2} \omega_{\lambda\nu} \mathcal{S}^{\mu\lambda\nu}[T'] \right)$$

Even though the forms s^μ and ϕ^μ are objective, they are not *unique*. It is quite clear that a transformation,

$$\phi^\mu \to \phi^{\mu'} = \phi^\mu + \partial_\lambda A^{\lambda\mu} \implies s^\mu \to s^{\mu'} = s^\mu + \partial_\lambda A^{\lambda\mu}$$

$$s.t\ S = \int_\Sigma d\Sigma_\mu\ s^\mu = \int_\Sigma d\Sigma_\mu\ s^{\mu'}$$

$(A^{\lambda\mu}$ is arbitrary anti-symmetric tensor)

Therefore, just like $T^{\mu\nu}$ and $\mathcal{S}^{\mu\lambda\nu}$ are not invariant due to pseudo-gauge, the entropy current s^μ is not uniquely defined and can be changed, henceforth defined as entropy-gauge transformations.

Entropy Production Rate

- Entropy production is invariant under entropy-gauge transformation, i.e.,

$$\partial_\mu s^\mu = \partial_\mu s^{\mu'}$$

- Using the entropy current $s^\mu = \phi^\mu + T^{\mu\nu}\beta_\nu - \frac{1}{2}\Omega_{\lambda\nu}S^{\mu\lambda\nu}$, we obtain:

$$\partial_\mu s^\mu = \left(T_S^{\mu\nu} - T_{S(\mathrm{LE})}^{\mu\nu}\right)\xi_{\mu\nu} - \left(j^\mu - j_{\mathrm{LE}}^\mu\right)\partial_\mu\zeta + \left(T_A^{\mu\nu} - T_{A(\mathrm{LE})}^{\mu\nu}\right)\left(\Omega_{\mu\nu} - \varpi_{\mu\nu}\right) - \frac{1}{2}\left(\mathcal{S}^{\mu\lambda\nu} - \mathcal{S}_{\mathrm{LE}}^{\mu\lambda\nu}\right)\partial_\mu\Omega_{\lambda\nu}$$

$\varpi_{\mu\nu}$: is the thermal vorticity

1. This formula is a generalization of what was obtained *C. Van Weert* without spin:

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2. We stress that the formula is exact and not an approximation at some order of a gradient expansion.

3. A novel feature is apparently the simultaneous appearance of the last two terms of the right hand side [D.she et al. Sci.Bull.67(2022)2265-2268].

Entropy Current and Pseudo-Gauge Transformation

Q: Does the change in the entropy current induced by a pseudo-gauge transformations boils down to an entropy-gauge transformation?

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\nabla_\lambda \left(\widehat{\Phi}^{\lambda\mu\nu} - \widehat{\Phi}^{\mu\lambda\nu} - \widehat{\Phi}^{\nu\lambda\mu}\right),$$
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$$\widehat{S}'^{\mu\lambda\nu} = \widehat{S}^{\mu\lambda\nu} - \widehat{\Phi}^{\mu\lambda\nu}.$$

A: If this was the case, then entropy production rate would be invariant under a pseudo -gauge transformation.

Applying pseudo-gauge transformation to the entropy current (for $\Omega = \varpi$), we get

$$\phi'^{\mu} = \phi^{\mu} + \int_0^T \frac{\mathrm{d}T'}{T'} \left[\nabla_\lambda A^{\lambda\mu} - \Phi^{\lambda\mu\nu}\xi_{\lambda\nu}\right],$$
$$s'^{\mu} = s^{\mu} + \int_0^T \frac{\mathrm{d}T'}{T'} \left[\nabla_\lambda A^{\lambda\mu} - \Phi^{\lambda\mu\nu}\xi_{\lambda\nu}\right] + \nabla_\lambda A^{\lambda\mu} - \Phi^{\lambda\mu\nu}\xi_{\lambda\nu}.$$

The last term on the right-hand side cannot be written as a total derivative of an anti-symmetric tensor like in the case of entropy-gauge.

Therefore, the divergence of the entropy current is, in general, not invariant under a pseudo-gauge transformation.

Conclusions and Outlooks

- We used first-principle quantum-statistical method to derive the entropy current and the entropy production rate.
- Established the notion of entropy-gauge transformation.
- Entropy production rate is, in general, not invariant under pseudo-gauge transformation.

The first next step is to:

- Calculate various dissipative currents.

Polish National Science Centre Grants No. 2018/30/E/ST2/00432